SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

NOVEMBER 2008

First Assessment

Mathematics Extension 1

General Instructions

- Reading time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Unless otherwise stated, all answers should be given in simplest exact form.

Total Marks – 64

- Attempt questions 1 – 4
- Questions are not of equal value.

Examiner: *A. Fuller*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Total marks 64

Attempt questions 1 to 4

Answer each **Question** in a **Separate** writing booklet

(Use a SEPARATE writing booklet)

Question 1 (16 marks)

(a) If one root of $x^2 + 2x - c = 0$ is $x = 1$. What is the value of c? [1]

(b) If $f(x) = x^2 - 3x - 5$, what are the values of k such that $f(k) = k$? [2]

(c) A and B are the points $(-5, 12)$ and $(4, 9)$ respectively. Find the coordinates of the point P that divides AB externally in the ratio $5 : 2$. [2]

(d) Find the size of the acute angle between the lines $x + 2y = 5$ and $3x - y = 1$ to the nearest minute. [3]

(e) Given that $x^3 - 3x^2 + 2x - 4 = (x^2 + 2) \times Q(x) + R(x)$, where the degree of $R(x) \leq 1$, find the quotient $Q(x)$ and the remainder $R(x)$. [2]

(f) Solve the following inequalities algebraically, and in each case graph the solution on the number line: [6]

(i) $\dfrac{1}{x-2} \leq 4$

(ii) $\dfrac{1}{x} < |x|$.

(Use a SEPARATE writing booklet)

Question 2 (16 marks)

(a) (i) Write down the expansion for $\cos(x - y)$. [3]

(ii) Hence, prove that $\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{6} + \sqrt{2}}{4}$.

(b) The roots of $x^3 - x^2 - 5x + 2 = 0$ are α, β and γ. [4]

Write down the values of:

(i) $\alpha\beta\gamma$

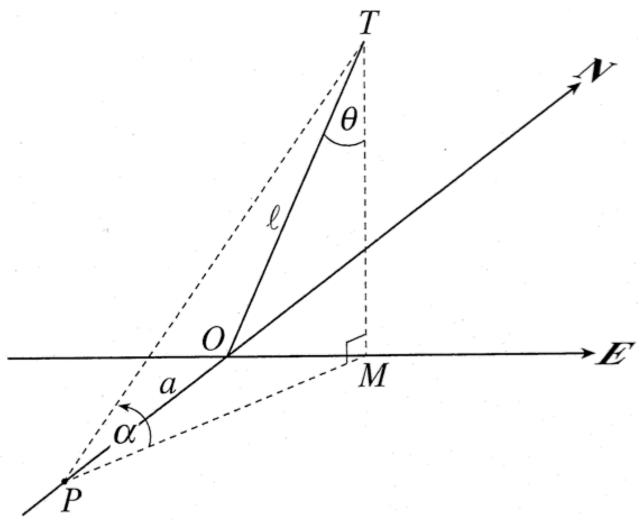
(ii) $\beta\gamma + \gamma\alpha + \alpha\beta$

(iii) $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$

(iv) $\frac{\beta + \gamma}{\alpha} + \frac{\gamma + \alpha}{\beta} + \frac{\alpha + \beta}{\gamma}$.

(c) Find the general solution to $\sin 2x = \sqrt{3} \cos 2x$. [3]

(d) 6



A pole, OT, of length l m, stands on horizontal ground. The pole leans towards the east, making an angle of θ with the vertical.

From P, a m south of O, the elevation of T is α.

(i) Find expressions, in terms of l and θ, for OM and MT.

(ii) Prove that $PM = l\cos\theta\cot\alpha$.

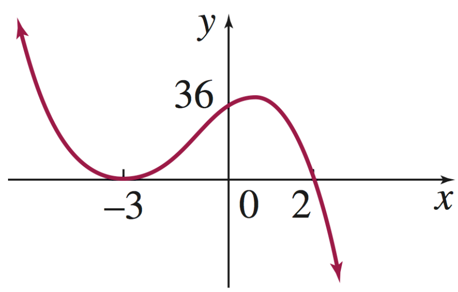
(iii) Prove that $l^2 = \dfrac{a^2}{\cos^2\theta\cot^2\alpha - \sin^2\theta}$.

(iv) Find the length of the pole, to the nearest metre, if $a = 25$, $\theta = 20^\circ$ and $\alpha = 24^\circ$.

(Use a SEPARATE writing booklet)

Question 3 (15 marks)

(a) Find the equation of the cubic curve below. **2**



(b) (i) Write $\sin x + \sqrt{3}\cos x$ in the form $A\cos(x - \alpha)$. **4**

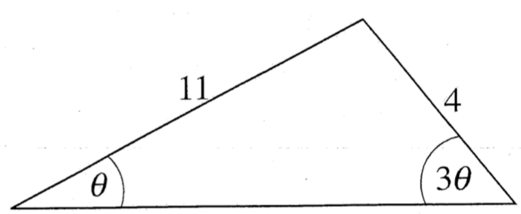
(ii) Hence, or otherwise. Solve $\sin x + \sqrt{3}\cos x = 1$ for $0 \le x \le 2\pi$.

(c) Prove the identity $\dfrac{\sin 2\theta}{1 + \cos 2\theta} = \tan\theta$. **2**

(d) Find the sum of $3^n + 3^{n-1} + ... + 3^{-2n}$. **2**

(e) (i) Show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. **5**

(ii) Find θ to the nearest minute.



(Use a SEPARATE writing booklet)

Question 4 (17 marks)

(a) $P(2at, at^2)$ is a variable point on the parabola $x^2 = 4ay$, whose focus is S. $Q(x, y)$ divides the interval from P to S in the ratio $t^2 : 1$. 5

(i) Find the coordinates of Q in terms of a and t.

(ii) Verify that $\dfrac{y}{x} = t$

(iii) Prove that, as P moves on the parabola, Q moves on a circle, and state its centre and radius.

(b) On January 1st 2008, Sue Bright invested \$1000 in a superannuation scheme. On January 1st of each of the subsequent 14 years, she will make further investments, increasing them by 5% each year to account for inflation. The scheme pays 10% per annum interest, calculated annually, and she will withdraw her funds when the scheme reaches maturity on January 1st, 2023. 4

Find:

(i) the value of her first investment when it is withdrawn.

(ii) the value of her last investment when it is withdrawn.

(iii) to the nearest dollar, the amount she will withdraw on January 1st, 2023.

(c) (i) If $pq = 32$ and $p + q = 12$. Form a quadratic equation with integer coefficients with roots p and q. 8

(ii) The equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four real roots $\alpha, \frac{1}{\alpha}, \beta$ and $\frac{1}{\beta}$. Prove that

(α) $d = 1$

(β) $a = c$

(γ) $b = 2 + (\alpha + \frac{1}{\alpha})(\beta + \frac{1}{\beta})$

(iii) Using parts (i) and (ii), or otherwise, solve the equation $x^4 - 12x^3 + 34x^2 - 12x + 1 = 0$, given that its roots consist of two pairs of reciprocal real numbers.

End of paper

QUESTION 1. (EXT 1.)

(a) $1^2 + 2\times 1 - c = 0$

$3 - c = 0$

$\boxed{c = 3}$

(b) $f(k) = k$

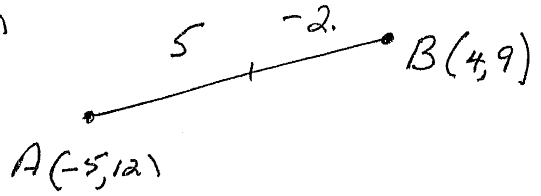
$\therefore\ k^2 - 3k - 5 = k.$

$k^2 - 4k - 5 = 0$

$(k-5)(k+1) = 0$

$\boxed{k = -1, 5}$

(c)



$$P = \left(\frac{5\times 4 + -2\times -5}{5 + -2}, \frac{5\times 9 + -2\times 12}{5 + -2} \right)$$

$$= \left(\frac{20+10}{3}, \frac{45-24}{3} \right)$$

$$= \boxed{(10, 7)}$$

(d) $x + 2y = 5$

$2y = -x + 5$

$y = -\frac{1}{2}x + \frac{5}{2}$

$\therefore m_1 = -\frac{1}{2}$

$3x - y = 1$

$y = 3x - 1$

$\therefore m_2 = 3$

Now

$$\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$= \left| \frac{-\frac{1}{2} - 3}{1 + -\frac{1}{2}\times 3} \right|$$

$$= \left| \frac{-\frac{7}{2}}{-\frac{1}{2}} \right|$$

$$= 7$$

$$\theta = \tan^{-1} 7 = \boxed{81^\circ 52'}$$

(e)

$$\begin{array}{r} x - 3 \\ x^2 + 2 \overline{) x^3 - 3x^2 + 2x - 4} \\ \underline{x^3 \qquad + 2x} \\ -3x^2 \qquad - 4 \\ \underline{-3x^2 \qquad - 6} \\ 2 \end{array}$$

$\therefore$ $\boxed{Q(x) = x - 3 \quad R(x) = 2.}$

(f)(i) $\frac{1}{x-2} \leq 4$

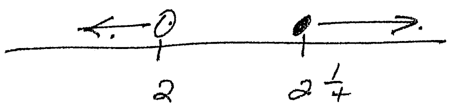
$\frac{1}{x-2} - 4 \leq 0$

$\frac{1-4(x-2)}{x-2} \leq 0$

$\frac{1-4x+8}{x-2} \leq 0$

$(9-4x)(x-2) \leq 0$ (Multiply both sides by $(x-2)^2$)

$\therefore \quad x \geq 2\frac{1}{4}, \quad x < 2 \quad (x \neq 2)$



(ii) Now $|x| = x$ for $x > 0$.

$= -x$ for $x < 0$ (NB $x \neq 0$)

Consider case $x > 0$: $\frac{1}{x} < x$

$1 < x^2$

$x^2 - 1 > 0 \Rightarrow (x-1)(x+1) > 0$

$\therefore x > 1, x < -1 \quad \therefore \boxed{x > 1}$

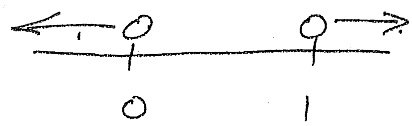
Consider case $x < 0$: $\frac{1}{x} < -x$

$1 > -x^2$

$x^2 + 1 > 0$

All x. $(x < 0) \quad \therefore \boxed{x < 0}$

Hence $\boxed{x < 0, \; x > 1}$



SHS Ext1 Ass1 Nov08

Question 2

(a)(i) $\cos(x-y) = \cos x\cos y + \sin x \sin y$ [1]

(ii) $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$

$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$

$= \frac{1}{2}\times\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$

$= \frac{1+\sqrt{3}}{2\sqrt{2}}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$ [2]

= RHS QED

(b) $x^3 - x^2 - 5x + 2 = 0$

(i) $\alpha\beta\gamma = -2$ [1]

(ii) $\beta\gamma + \gamma\alpha + \alpha\beta = -5$ [1]

(iii) $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma} = \frac{\gamma+\beta+\alpha}{\alpha\beta\gamma}$

$= \frac{1}{-2}$

$= -\frac{1}{2}$ [1]

(iv) $\frac{\beta+\gamma}{\alpha} + \frac{\gamma+\alpha}{\beta} + \frac{\alpha+\beta}{\gamma}$

$= \frac{\beta\gamma(\beta+\gamma)+\alpha\gamma(\alpha+\gamma)+\alpha\beta(\alpha+\beta)}{\alpha\beta\gamma}$

$= \frac{\beta\gamma(\alpha+\beta+\gamma)+\alpha\gamma(\alpha+\beta+\gamma)+\alpha\beta(\alpha+\beta+\gamma)-3\alpha\beta\gamma}{\alpha\beta\gamma}$

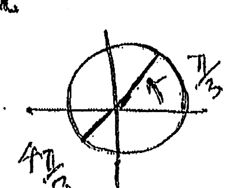
$= \frac{(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma)-3\alpha\beta\gamma}{\alpha\beta\gamma}$

$= \frac{1\times -5}{-2} - 3$

$= -\frac{1}{2}$ [1]

(c) $\sin 2x = \sqrt{3}\cos 2x$

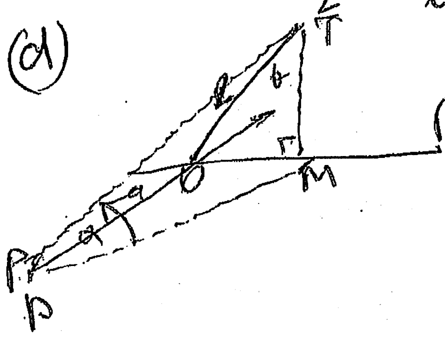
$\therefore \tan 2x = \sqrt{3}$

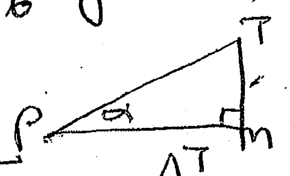


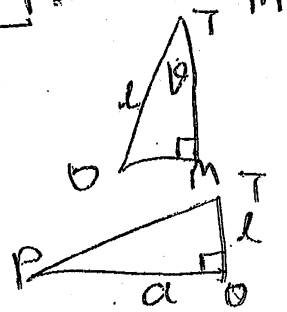
$\therefore 2x = n\pi + \frac{\pi}{3}$ [3]

$x = \frac{n\pi}{2} + \frac{\pi}{6}$ for $n = 0, \pm1, \ldots$

(d)







(i) $OM = l\sin\theta$ [1]

Now $MT = l\cos\theta$

(ii) $PM = \frac{MT}{\tan\alpha}$ $\qquad \cos\theta = \frac{MT}{l}$

$= MT\cot\alpha \qquad MT = l\cos\theta$

$= l\cos\theta\cot\alpha$ [1]

Question 2 (Contd)

(iii) In $\triangle POT$, $PT^2 = a^2 + l^2$

In $\triangle PMT$, $PT^2 = MP^2 + MT^2$

$\therefore a^2 + l^2 = MP^2 + MT^2$

$= l^2\cos^2\theta\cot^2\alpha + l^2\cos^2\theta$

$= l^2\cos^2\theta\cot^2\alpha + l^2(1 - \sin^2\theta)$

$\therefore a^2 = l^2(\cos^2\theta\cot^2\alpha - \sin^2\theta)$

$\therefore l^2 = \dfrac{a^2}{\cos^2\theta\cot^2\alpha - \sin^2\theta}$ [2]

As req'd

(iv)

$l = \sqrt{\dfrac{25^2}{\cos^2 20 \cot^2 24 - \sin^2 20}}$

$\doteqdot 12$ (12.00372...)

[2]

Question (3).

(a) $y = a(x+3)^2(x-2)$ [1]

When $x=0$, $y=36$

$-18a = 36$

(2) $\therefore a = -2$. [1]

$\therefore y = -2(x+3)^2(x-2)$

(b)

$\sin x + \sqrt{3}\cos x = A\cos(x-\alpha)$

$A\cos\alpha = \sqrt{3}$

$A\sin\alpha = 1$ [1]

$\therefore \tan\alpha = \frac{1}{\sqrt{3}}$, $\alpha = \frac{\pi}{6}$.

$A^2(\cos^2\alpha + \sin^2\alpha) = 4$

(4) $A = 2$ [1]

$\therefore 2\cos(x - \frac{\pi}{6}) = 1$

$\cos(x - \frac{\pi}{6}) = \frac{1}{2}$. [1]

$x - \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$.

$x = \frac{\pi}{2}, \frac{11\pi}{6}$. [1]

(c) $\frac{\sin 2\theta}{1+\cos 2\theta}$

$= \frac{2\sin\theta\cos\theta}{(\sin^2\theta + \cos^2\theta) + (\cos^2\theta - \sin^2\theta)}$ [1]

$= \frac{2\sin\theta\cos\theta}{2\cos^2\theta}$ [1]

(2) $= \tan\theta$.

(d) $3^n + 3^{n-1} + \dots + 3^{-2n}$

$a = 3^n$ [1]

$3n+1$ terms $r = 3^{-1}$

$\therefore S_n = \frac{a(1-r^{3n})}{1-r}$ [1]

(2) $= \frac{3^{n+1}(1-(\frac{1}{3})^{3n+1})}{2/3}$.

$= \frac{3^{n+2}}{2}[1 - 3^{-(3n+1)}]$

or

$= \frac{1}{2}(3^{2+n} - 3^{1-2n})$

$= \frac{3^{1-2n}}{2}(3^{3n+1} - 1)$.

(e) $\sin(2\theta + \theta)$ [2]

$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$

$= 2\sin\theta\cos^2\theta + (1 - 2\sin^2\theta)\sin\theta$

$= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$.

(ii) $\frac{11}{\sin 3\theta} = \frac{4}{\sin\theta}$ [1]

$11\sin\theta = 12\sin\theta - 16\sin^3\theta$

$16\sin^3\theta - \sin\theta = 0$ [3]

$\sin\theta(16\sin^2\theta - 1) = 0$ [1]

$\sin^2\theta = \frac{1}{16}$

$\sin\theta = \pm\frac{1}{4}$

$\theta = 14°29'$, [1]

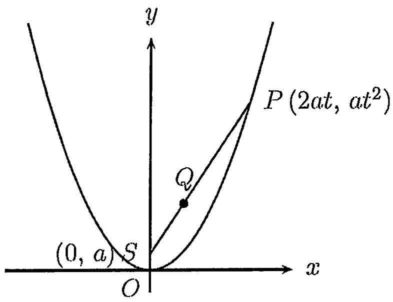
(5)

2008 Extension Mathematics Task 1:
Solutions— Question 4

4. (a) $P(2at, at^2)$ is a variable point on the parabola $x^2 = 4ay$, whose focus is S. $Q(x, y)$ divides the interval from P to S in the ratio $t^2 : 1$. 5

(i) Find the coordinates of Q in terms of a and t.

Solution:



$$x_Q = \frac{t^2 \times 0 + 1 \times 2at}{t^2+1}, \quad y_Q = \frac{t^2 \times a + 1 \times at^2}{t^2+1},$$
$$= \frac{2at}{t^2+1}. \quad = \frac{2at^2}{t^2+1}.$$

(ii) Verify that $\dfrac{y}{x} = t$.

Solution: $$\frac{y}{x} = \frac{2at^2}{t^2+1} \times \frac{t^2+1}{2at}, = t.$$

(iii) Prove that, as P moves on the parabola, Q moves on a circle, and state its centre and radius.

Solution:
$$\begin{aligned} x &= \frac{2a\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)^2 + 1}, \\ &= \frac{2axy}{y^2 + x^2}, \\ x^2 + y^2 &= 2ay, \\ x^2 + y^2 - 2ay + a^2 &= 0 + a^2, \\ x^2 + (y-a)^2 &= a^2. \end{aligned}$$
$\therefore$ Q lies on a circle, centre $(0, a)$, radius a.

(b) On January 1st 2008, Sue Bright invested \$1000 in a superannuation scheme. On January 1st of each of the subsequent 14 years, she will make further investments, increasing them by 5% each year to account for inflation. The scheme pays 10% per annum interest, calculated annually, and she will withdraw her funds when the scheme reaches maturity on January 1st, 2023. [4]

Find:

(i) the value of her first investment when it is withdrawn.

Solution: $\text{Value of } 1^{\text{st}} \text{ investment} = \$1000 \times 1.1^{15}, = \$4177.25.$

(ii) the value of her last investment when it is withdrawn.

Solution: $\text{Value of last investment} = \$1000 \times 1.05^{14} \times 1.1, = \$2177.92.$

(iii) to the nearest dollar, the amount she will withdraw on January 1st, 2023.

Solution: Total value of investments,

$$\begin{aligned} V &= \$1000\,(1.1^{15} + 1.1^{14} \times 1.05 + 1.1^{13} \times 1.05^2 + \cdots + 1.1 \times 1.05^{14}), \\ &= \$1000\left(1.1^{15} + 1.1^{15} \times \tfrac{21}{22} + 1.1^{15} \times \left(\tfrac{21}{22}\right)^2 + \cdots + 1.1 \times \left(\tfrac{21}{22}\right)^{14}\right), \\ &= \$1000 \times 1.1^{15}\left(1 + \tfrac{21}{22} + \left(\tfrac{21}{22}\right)^2 + \cdots + \left(\tfrac{21}{22}\right)^{14}\right), \\ &= \$1000 \times 1.1^{15} \times \frac{\left(\left(\frac{21}{22}\right)^{15} - 1\right)}{\frac{21}{22} - 1} \\ &= \$46\,163 \text{ (nearest dollar).} \end{aligned}$$

(c) (i) If $pq = 32$ and $p+q = 12$, form a quadratic equation with integer coefficients with roots p and q. [8]

Solution: $x^2 - 12x + 32 = 0.$

(ii) The equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four real roots $\alpha, \frac{1}{\alpha}, \beta$ and $\frac{1}{\beta}$. Prove that

(α) $d = 1$

Solution:
$$\alpha \times \tfrac{1}{\alpha} \times \beta \times \tfrac{1}{\beta} = \frac{d}{1},$$
$$\text{i.e. } d = 1.$$

(β) $a = c$

Solution:
$$\begin{aligned} \alpha + \tfrac{1}{\alpha} + \beta + \tfrac{1}{\beta} &= -a. \\ \alpha \times \tfrac{1}{\alpha} \times \beta + \alpha \times \tfrac{1}{\alpha} \times \tfrac{1}{\beta} + \alpha \times \beta \times \tfrac{1}{\beta} + \tfrac{1}{\alpha} \times \beta \times \tfrac{1}{\beta} &= -c, \\ \beta + \tfrac{1}{\beta} + \alpha + \tfrac{1}{\alpha} &= -c. \\ \therefore\ a &= c. \end{aligned}$$

(γ) $b = 2 + (\alpha + \frac{1}{\alpha})(\beta + \frac{1}{\beta})$.

Solution:
$$\begin{aligned} \alpha \times \tfrac{1}{\alpha} + \alpha\beta + \tfrac{\alpha}{\beta} + \tfrac{\beta}{\alpha} + \tfrac{1}{\alpha \times \beta} + \tfrac{\beta}{\beta} &= b, \\ 2 + \alpha\beta + \tfrac{\alpha}{\beta} + \tfrac{\beta}{\alpha} + \tfrac{1}{\alpha\beta} &= b, \\ 2 + \alpha\left(\beta + \tfrac{1}{\beta}\right) + \tfrac{1}{\alpha}\left(\beta + \tfrac{1}{\beta}\right) &= b, \\ 2 + \left(\alpha + \tfrac{1}{\alpha}\right)\left(\beta + \tfrac{1}{\beta}\right) &= b. \end{aligned}$$

(iii) Using parts (i) and (ii), or otherwise, solve the equation
$$x^4 - 12x^3 + 34x^2 - 12x + 1 = 0,$$
given that its roots consist of two pairs of reciprocal real numbers.

Solution: From part(ii), $2 + (\alpha + \frac{1}{\alpha})\left(\beta + \frac{1}{\beta}\right) = 34,$
$$\left(\alpha + \tfrac{1}{\alpha}\right)\left(\beta + \tfrac{1}{\beta}\right) = 32.$$
now, factorising part(i), $(x - 8)(x - 4) = 0,$

i.e., $\left(\alpha + \frac{1}{\alpha}\right) = 8,$ $\qquad \left(\beta + \frac{1}{\beta}\right) = 4,$

$\alpha^2 - 8\alpha + 1 = 0,$ $\qquad \beta^2 - 4\beta + 1 = 0,$

$$\alpha = \frac{8 \pm \sqrt{64 - 4}}{2}, \qquad \beta = \frac{4 \pm \sqrt{16 - 4}}{2},$$
$$= 4 \pm \sqrt{15} \qquad = 2 \pm \sqrt{3}$$

Checking reciprocals, $\dfrac{1}{4 + \sqrt{15}} \times \dfrac{4 - \sqrt{15}}{4 - \sqrt{15}} = \dfrac{4 - \sqrt{15}}{16 - 15},$
$$= 4 - \sqrt{15},$$
and, $\dfrac{1}{2 + \sqrt{3}} \times \dfrac{2 - \sqrt{3}}{2 - \sqrt{3}} = \dfrac{2 - \sqrt{3}}{4 - 3},$
$$= 2 - \sqrt{3},$$

i.e. The roots are $4 \pm \sqrt{15},\ 2 \pm \sqrt{3}$.