SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

December 2010
Assessment Task 1
Year 11

Mathematics Extension

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- All necessary working should be shown in every question if full marks are to be awarded.
- All answers to be given in simplified exact form unless otherwise stated.
- Marks may not be awarded for messy or badly arranged work

Total Marks – 74

- Attempt questions 1-4
- All questions are **NOT** of equal value.
- Start each new question in a separate answer booklet.
- Hand in your answers in 4 separate bundles:

Question 1, Question 2, Question 3 and Question 4

Examiner: *R Boros*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

Start a new booklet.

Question 1 (18 marks). **Marks**

a) Express $\frac{7\pi}{18}^{c}$ in degrees. 1

b) Find the value of *k* if $(x-2)$ is a factor of $P(x) = x^4 - 3x^3 + kx^2 - 4$ 1

c) Solve for *x*, $\frac{x+1}{x-3} \geq 2$ 2

d) Find the acute angle, to the nearest degree, between the lines $y = 2x + 3$ and $x + y = 0$ 2

e) Find the general solution of $2\cos\theta - 1 = 0$, where θ is in radians. 3

f) Prove that $\tan\left(\frac{\pi}{4} + A\right) + \tan\left(\frac{\pi}{4} - A\right) = 2\sec 2A$. 3

g) (i) In how many ways can 3 men and 3 women be arranged in a circle? 1

(ii) How many of these arrangements have at least 2 women sitting next to each other? 2

h) Find the co-ordinates of the point *P* which divides the interval joining *A*(-1,-3) and *B*(3,7) externally in the ratio 5:3. 3

End of Question 1

Start a new booklet.

Question 2 (18 Marks). **Marks**

a) The roots of a cubic polynomial equation are 0, 1 and 3; and the coefficient of x^3 is 2. Find the polynomial in full expanded form. 3

b) Solve for n : $2\times {}^nC_4 = 5\times {}^nC_2$. 2

c) If α, β and γ are the roots of $x^3 - 3x + 1 = 0$, find;

(i) $\alpha+\beta+\gamma$ 1

(ii) $\alpha\beta\gamma$ 1

(iii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$ 2

d) (i) Express $\sqrt{3}\cos x - \sin x$ in the form 2

$$R\cos(x+\alpha),\ R>0 \text{ and } 0\le\alpha\le\frac{\pi}{2}.$$

(ii) Hence, find the general solution for $\sqrt{3}\cos x - \sin x = 1$. 2

e) α and β are acute angles, such that $\cos\alpha = \frac{3}{5}$ and $\sin\beta = \frac{1}{\sqrt{5}}$. Without finding the size of either angle, show that $\alpha = 2\beta$. 2

f) Using the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, solve the equation $\sin 3\theta = 2\sin\theta$, where $0\le\theta\le 2\pi$. 3

End of Question 2

Start a new booklet.

Question 3 (18 marks). **Marks**

a) (i) Show that the equation $\sin x^\circ - 3\cos x^\circ = \tan\left(\frac{x}{2}\right)^\circ$ may be written as 3

$t^3 - 3t^2 - t + 3 = 0$, where $t = \tan\left(\frac{x}{2}\right)^\circ$

(ii) Hence, find all 3 solutions of the equation $\sin x^\circ - 3\cos x^\circ = \tan\left(\frac{x}{2}\right)^\circ$, for 3

$0^\circ \le x \le 360^\circ$. Give answers to the nearest minute, if necessary.

b) Six identical yellow discs and 4 identical blue discs are placed in a straight line. How many arrangements are possible? 1

c) (i) In how many ways can a committee of 2 Australians, 2 New Zealanders and 1 Nauruan be chosen from 6 Australians, 7 New Zealanders and 3 Nauruans? 2

(ii) In how many of these ways do 2 friends, a particular Australian and a particular New Zealander, belong to the committee? 1

d) For the parabola $x^2 = 12y$:

(i) Derive the equation of the tangent at $(6t, 3t^2)$. 2

(ii) Find the equations of the two tangents that pass through the point $(5, -2)$. 3

e) Find the value of the constants p and q if $x^2 - 4x + 3$ is a factor of 3

$x^3 + px^2 - x + q$.

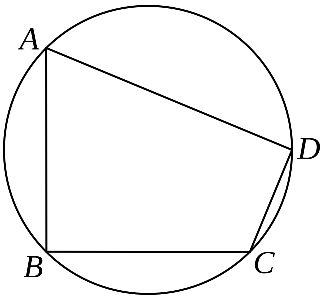
End of Question 3

Start a new booklet.

Question 4 (20 marks). **Marks**

a) In how many ways can 8 different books be arranged in a row such that 3 particular books are always together? 2

b) *ABCD* is a cyclic quadrilateral. One of its properties is that opposite angles are supplementary. Show that $\tan A + \tan B + \tan C + \tan D = 0$ 2



c) Eight people including James and Sarah are to be seated around a table. How many arrangements are possible if James and Sarah do not wish to sit next to each other. 2

d) The straight line $y = mx + b$ meets the parabola $x^2 = 4ay$ at the points $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$

(i) Find the equation of the chord PQ and, hence or otherwise, show that 2

$$pq = -\frac{b}{a}$$

(ii) Prove that $p^2 + q^2 = 4m^2 + \dfrac{2b}{a}$ 2

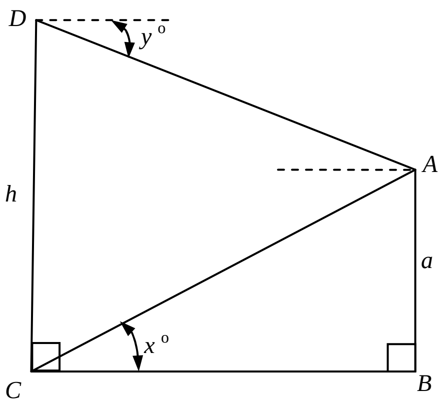
(iii) Given that the equation of the normal to the parabola at P is $x + py = 2ap + ap^3$, and that, N, the point of intersection of the normals at P and Q has the co-ordinates $\left[-apq(p+q),\ a(2 + p^2 + pq + q^2)\right]$, express these co-ordinates in terms of *a, m* and *b*. 2

(iv) Suppose that the chord PQ is free to move while maintaining a fixed gradient. Find the locus of *N*, and show that this locus is a straight line. Verify that this line is a normal to the parabola. 2

Question 4 continues on next page

Question 4 (continued).

e)



From the foot of a tower *CD*, the angle of elevation of a building *AB*, '*a'* metres high, is x°. From the top of the tower, *D*, the angle of depression to the top of the building, *A* is y°. Show that the height, *'h'*, of the tower is given by: 6

$$h = \frac{a\sin\left(x+y\right)}{\sin x\cos y}$$

End of Question 4

End of Examination

1)a) $\frac{7\pi}{18} \times \frac{180}{\pi} = \underline{70^\circ}$

(b) $P(x) = x^4 - 3x^3 - kx^2 - 4$

$P(2) = (2)^4 - 3(2)^3 - k(2)^2 - 4 = 0$

$4k - 12 = 0$

$\underline{k = 3}$

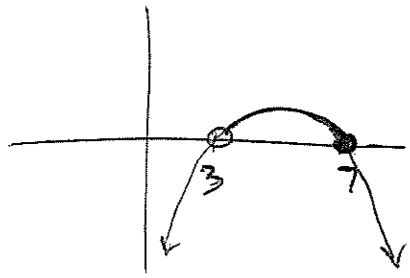
(c) $\frac{x+1}{x-3} \geqslant 2 \qquad x \neq 3.$

$(x+1)(x-3) \geqslant 2(x-3)^2$

$(x+1)(x-3) - 2(x-3)^2 \geqslant 0$

$(x-3)[x+1-2(x-3)] \geqslant 0$

$(x-3)(-x+7) \geqslant 0$



$\underline{3 < x \leq 7}$

(d) $y = 2x + 3 \qquad x + y = 0$

$m_1 = 2 \qquad y = -x$

$m_2 = -1$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

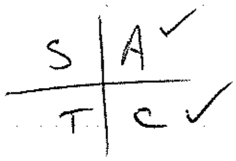
$= \left|\frac{(2) - (-1)}{1 + (2)(-1)}\right|$

$= 3$

$\underline{\theta = 72^\circ}$ (to nearest degree)

(e) $2\cos\theta - 1 = 0$

$\cos\theta = \frac{1}{2}$



$\cos\alpha = \frac{1}{2}$

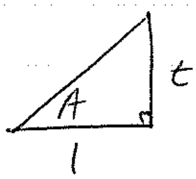
$\alpha = \frac{\pi}{3}$

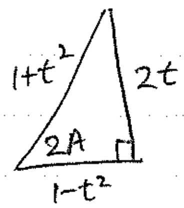
$\theta = 2\pi n \pm \frac{\pi}{3}$ where n is an integer.

(f) $LHS = \tan\left(\frac{\pi}{4} + A\right) + \tan\left(\frac{\pi}{4} - A\right)$

$= \dfrac{\tan\frac{\pi}{4} + \tan A}{1 - \tan\frac{\pi}{4}\tan A} + \dfrac{\tan\frac{\pi}{4} - \tan A}{1 + \tan\frac{\pi}{4}\tan A}$

$\tan\frac{\pi}{4} = 1$ let $\tan A = t$





$= \dfrac{1+t}{1-t} + \dfrac{1-t}{1+t}$

$= \dfrac{(1+t)^2 + (1-t)^2}{1-t^2}$

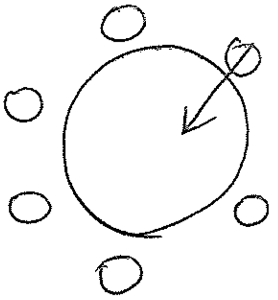
$= \dfrac{1 + 2t + t^2 + 1 - 2t + t^2}{1-t^2}$

$= \dfrac{2(1+t^2)}{1-t^2}$

$= 2\sec 2A$

$= RHS.$

(g) i)



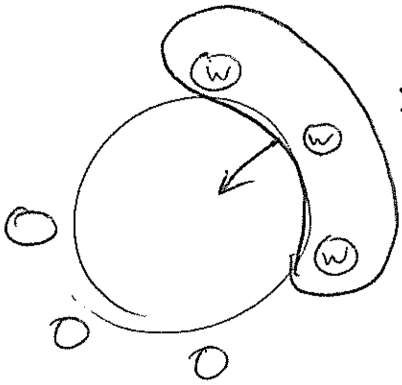
$1 \times 5! = \underline{120}$

(ii) Separated $1 \times 3! \times 2! = 12$

At least 2: $120 - 12 = \underline{108}$

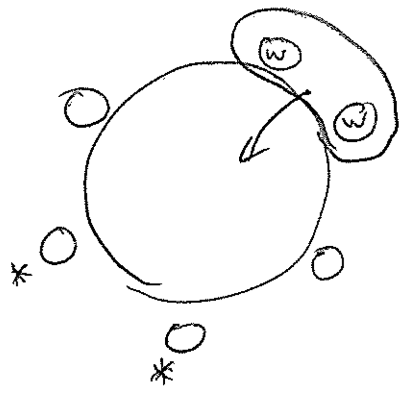
OR

3 women together:



$3! \times 1 \times 3! = 36$

2 women together:



$\underset{^3P_2}{\underline{3 \times 2}} \times 1 \times 2 \times 3! = 72$

* possibilities for other woman.

At least 2: $36 + 72 = \underline{108}$

(h) $A(-1,-3)$ $B(3,7)$

$5 : -3$

$P\left(\frac{-3(-1)+5(3)}{5+(-3)}, \frac{-3(-3)+5(7)}{5+(-3)}\right)$

$P(9, 22)$.

EXTENSION 1 SOLN'S

QUESTION 2.

a) $P(x) = 2x(x-1)(x-3)$

$= (2x^2 - 2x)(x-3)$

$= 2x^3 - 8x^2 + 6x$

OR.

$\alpha + \beta + \gamma = 0 + 1 + 3$

$= 4$

$4 = -\frac{b}{a}$

But $a = 2$

$\therefore \boxed{b = -8}$

$\alpha\beta + \beta\gamma + \alpha\gamma = 0 + 0 + 3$

$= 3$

$\therefore 3 = \frac{c}{a}$

$3 = \frac{c}{2}$

$\boxed{c = 6}$

$\alpha\beta\gamma = 0 \times 1 \times 3$

$= 0$

$\therefore 0 = -\frac{d}{a}$

$\therefore \boxed{d = 0}$

Given $a = 2$

$\therefore P(x) = 2x^3 - 8x^2 + 6x.$

b) $2 \times {}^nC_4 = 5 \times {}^nC_2.$

$2 \times \frac{n!}{4!(n-4)!} = 5 \times \frac{n!}{2!(n-2)!}$

$2.2!(n-2)! = 5.4!(n-4)!$

$\frac{(n-2)!}{(n-4)!} = \frac{5.4!}{2.2!}$

$(n-3)(n-2) = \frac{5 \times 4 \times 3 \times 2}{2 \times 2}$

$n^2 - 5n + 6 = 30$

$n^2 - 5n - 24 = 0$

$(n-8)(n+3) = 0$

$n = 8, -3 \quad n > 0$

$\therefore \boxed{n = 8}$

c) (i) $\alpha + \beta + \gamma = -\frac{b}{a}$

$= 0$

(ii) $\alpha\beta\gamma = -\frac{d}{a}$

$= -1$

(iii) $\frac{\alpha\gamma + \beta\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{-3}{-1}$

$= 3.$

d)(i) $\sqrt{3}\cos x - \sin x = R\cos(x + \alpha)$

$R = \sqrt{3+1}$

$= 2$

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$

$\therefore R\cos(x + \alpha) = 2\cos\left(x + \frac{\pi}{6}\right)$

d) (ii) $2\cos\left(x+\frac{\pi}{6}\right)=1$

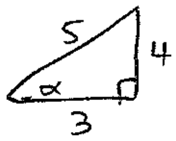
$$\cos\left(x+\frac{\pi}{6}\right)=\frac{1}{2}$$

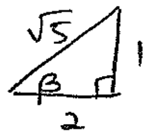
$$x+\frac{\pi}{6}=2n\pi\pm\frac{\pi}{3}$$

$$x=2n\pi\pm\frac{\pi}{3}-\frac{\pi}{6}$$

OR $x=2n\pi+\frac{\pi}{6},\ 2n\pi-\frac{\pi}{2}$

e)





$\cos\alpha=\frac{3}{5}$ $\quad\sin\beta=\frac{1}{\sqrt{5}}$

$$\sin 2\beta=2\sin\beta\cos\beta$$
$$=2\times\frac{1}{\sqrt{5}}\times\frac{2}{\sqrt{5}}$$
$$=\frac{4}{5}$$

$$\sin\alpha=\frac{4}{5}$$

$$\sin\alpha=\sin 2\beta$$

$$\therefore \alpha=2\beta.$$

f) $\sin 3\theta=3\sin\theta-4\sin^3\theta$

But $\sin 3\theta=2\sin\theta$

$$\therefore 3\sin\theta-4\sin^3\theta=2\sin\theta$$

$$\therefore \sin\theta-4\sin^3\theta=0$$

$$\sin\theta(1-4\sin^2\theta)=0$$

$\sin\theta=0$, $\quad 1-4\sin^2\theta=0$

$\therefore \theta=0,\pi,2\pi$ $\quad \sin^2\theta=\frac{1}{4}$

$\sin\theta=\pm\frac{1}{2}$

$\sin\theta=\frac{1}{2}$ $\quad \theta=\frac{\pi}{6},\frac{5\pi}{6}$

$\sin\theta=-\frac{1}{2}$ $\quad \theta=\frac{7\pi}{6},\frac{11\pi}{6}$

Marking Scale Comments

2 a) 2 marks for $2x^3 - 4x^2 + 3x$

2 marks for not fully expanding.

1 mark for each coefficient.

b) 1 mark for $2 \times \frac{n!}{4!(n-4)!} = 5 \times \frac{n!}{2!(n-2)!}$

$1\frac{1}{2}$ marks if they almost got the answer of $n = 8$ because only made a small error

c) (i) (ii) } no half marks, 1 mark only.

(iii) $\frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ + wrong answer – 1 mark only.

If carried on with previous wrong answer, should not have been penalized again – 2 marks.

d) (i) $\alpha = 30°$ instead of $\alpha = \frac{\pi}{6}$ – lost ½ mark.

$\alpha = -\frac{\pi}{6}$ – ½ mark.

(ii) 2 marks – $2n\pi \pm \frac{\pi}{3} - \frac{\pi}{6}$

– $2n\pi + \frac{\pi}{6}$, $2n\pi - \frac{\pi}{2}$

1 mark – $2n\pi \pm \frac{\pi}{6}$

1 mark – $(x + \frac{\pi}{6}) = 2n\pi \pm \frac{\pi}{3}$

½ mark – $2n\pi \pm \alpha$

2010 Extension Mathematics Task 1:
Solutions— Question 3

3. (a) (i) Show that the equation $\sin x^\circ - 3\cos x^\circ = \tan\left(\frac{x}{2}\right)^\circ$ may be written as $t^3 - 3t^2 - t + 3 = 0$, where $t = \tan\left(\frac{x}{2}\right)^\circ$. [3]

Solution:
$$\begin{aligned} \frac{2t}{1+t^2} - \frac{3(1-t^2)}{1+t^2} &= t, \\ 2t - 3 + 3t^2 &= t + t^3, \\ \textit{i.e. } t^3 - 3t^2 - t + 3 &= 0. \end{aligned}$$

(ii) Hence find all three solutions of the equation $\sin x^\circ - 3\cos x^\circ = \tan\left(\frac{x}{2}\right)^\circ$, for $0^\circ \leqslant x^\circ \leqslant 360^\circ$. Give answers to the nearest minute if necessary. [3]

Solution: Method 1
$$\begin{aligned} t^2(t-3) - (t-3) &= 0, \\ (t^2-1)(t-3) &= 0, \\ (t+1)(t-1)(t-3) &= 0, \\ \text{so } t &= 3, \pm 1. \\ \frac{x^\circ}{2} &\approx 71^\circ 34',\ 45^\circ,\ 135^\circ, \\ x^\circ &\approx 143^\circ 8',\ 90^\circ,\ 270^\circ. \end{aligned}$$

Solution: Method 2
$$\begin{aligned} \text{Put } P(t) &= t^3 - 3t^2 - t + 3 \quad = 0, \\ P(3) &= 27 - 3\times 9 - 3 + 3 = 0. \end{aligned}$$
$\therefore$ $(t-3)$ is a factor.

	1	−3	−1	3
3		3	0	−3
	1	0	−1	0

$$\begin{aligned} \therefore\ P(t) &= (t-3)(t^2-1), \\ &= (t-3)(t-1)(t+1), \\ \text{so } t &= 3, \pm 1. \\ \frac{x^\circ}{2} &\approx 71^\circ 34',\ 45^\circ,\ 135^\circ, \\ x^\circ &\approx 143^\circ 8',\ 90^\circ,\ 270^\circ. \end{aligned}$$

(b) Six identical yellow discs and four identical blue discs are placed in a straight line. How many arrangements are possible? [1]

Solution: $\dfrac{10!}{6!4!} = 210.$

(c) (i) In how many ways can a committee of 2 Australians, 2 New Zealanders, and one Nauruan be chosen from 6 Australians, 7 New Zealanders, and 3 Nauruans? [2]

Solution: $\binom{6}{2}\binom{7}{2}\binom{3}{1} = 945.$

(ii) In how many of these ways do two friends, a particular Australian and a particular New Zealander, belong to the committee? [1]

Solution: $\binom{5}{1}\binom{6}{1}\binom{3}{1} = 90.$

(d) For the parabola $x^2 = 12y$:

(i) Derive the equation of the tangent at $\left(6t,\, 3t^2\right)$. [2]

Solution:
$$\begin{aligned} \frac{dy}{dx} &= \frac{2x}{12}, \\ &= \frac{x}{6}. \end{aligned}$$
$$\begin{aligned} \therefore \text{ Tangent is } \quad y - 3t^2 &= \frac{6t}{6}(x - 6t), \\ y - 3t^2 &= tx - 6t^2, \\ \textit{i.e. } y &= tx - 3t^2. \end{aligned}$$

(ii) Find the equations of the tangents that pass through the point $(5, -2)$. [3]

Solution: As they pass through $(5, -2)$,
$$\begin{aligned} -2 &= 5t - 3t^2, \\ 3t^2 - 5t - 2 &= 0, \\ (3t + 1)(t - 2) &= 0, \\ \therefore\ t &= 2, \quad -1/3. \\ \therefore\ \text{When } t &= 2, \text{ tangent is} \\ y &= 2x - 12, \\ \text{or } 2x - y - 12 &= 0. \\ \text{When } t &= -1/3, \text{ tangent is} \\ y &= \frac{-x}{3} - \frac{1}{3}, \\ \text{or } x + 3y + 1 &= 0. \end{aligned}$$

(e) Find the value of the constants p and q if x^2-4x+3 is a factor of x^3+px^2-x+q. [3]

Solution: Method 1

$$\begin{aligned}
x^2-4x+3 &= (x-3)(x-1).\\
P(x) &= x^3+px^2-x+q,\\
P(3) &= 27+9p-3+q = 0,\\
&\qquad 9p+q+24 = 0 \ldots\ldots\ldots\ldots \boxed{1}\\
P(1) &= 1+p-1+q = 0,\\
&\qquad p+q = 0 \ldots\ldots\ldots\ldots \boxed{2}\\
\boxed{1}-\boxed{2}: &\qquad 8p+24 = 0,\\
&\qquad p = -3,\\
&\qquad q = 3.
\end{aligned}$$

Solution: Method 2

$$\begin{aligned}
x^3+px^2-x+q &= (x-\alpha)(x^2-4x+3),\\
&= (x-\alpha)(x-3)(x-1) \text{ i.e. roots } \alpha,\ 3,\ 1.\\
\Sigma\alpha: \quad \alpha+3+1 &= -p,\\
\alpha+4 &= -p.\\
\Sigma\alpha\beta: \quad 3\alpha+\alpha+3 &= -1,\\
4\alpha &= -4,\\
\alpha &= -1.\\
\Sigma\alpha\beta\gamma: \quad -1.3.1 &= -q,\\
q &= 3,\\
-1+4 &= -p,\\
p &= -3.
\end{aligned}$$

Solution: Method 3

$$\begin{array}{r|llllllll}
 & x & + & (p+4) & & & & & \\
\hline
x^2-4x+3 & x^3 & + & px^2 & - & x & + & q \\
 & x^3 & - & 4x^2 & + & 3x & & \\
\hline
 & & & (p+4)x^2 & - & 4x & + & q \\
 & & & (p+4)x^2 & - & 4(p+4)x & + & 3(p+4) \\
\hline
 & & & & & 4px+12x & + & q-3(p+4)
\end{array}$$

$$\begin{aligned}
\text{Thus } x^3+px^2-x+q &= \big(x^2-4x+3\big)\big(x+(p+4)\big),\\
\text{equating coefficients, } 3(p+4) &= q.\\
\text{Also, } 4px+12x+q-3(p+4) &= 0,\\
\text{i.e. } 4(px+3)x+3(p+4)-3(p+4) &= 0,\\
4(p+3)x &= 0,\\
p &= -3,\\
q &= 3(-3+4),\\
&= 3.
\end{aligned}$$

Question 4 (Yr 11 Ext 1)

(a) $6! \times 3! = 4320$ [2]

(b) $\tan A + \tan B + \tan C + \tan D$

$= \tan A + \tan B + \tan(180^\circ - A) + \tan(180 - B)$

$= \tan A + \tan B - \tan A - \tan B$

$= 0$ [2]

(c) Total Arrangements $= 7!$

Total with J & S together $= 6! \times 2$

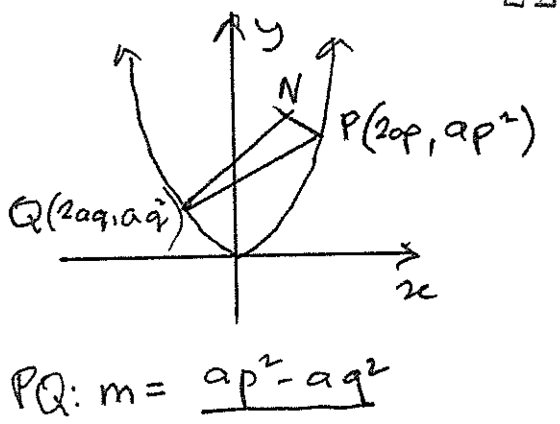
$\therefore$ Arrangements w/o J & S together $= 7! - 2 \times 6!$

$= 6!(7-2)$

$= 5 \times 6!$

$= 3600$ [2]

(d)



(i) For PQ: $m = \dfrac{ap^2 - aq^2}{2ap - 2aq}$

$= \dfrac{p+q}{2}$

$\therefore y - ap^2 = \dfrac{p+q}{2}(x - 2ap)$

$2y - 2ap^2 = (p+q)x - 2ap^2 - 2apq$

$y = \left(\dfrac{p+q}{2}\right)x - apq$ — ①

or $(p+q)x - 2y - 2apq = 0$

Now ① may also be written $y = mx + b$

$\therefore b = -apq$

So $pq = -\dfrac{b}{a}$ [2]

(ii) $\dfrac{p+q}{2} = m$

$\therefore \dfrac{p^2 + 2pq + q^2}{4} = m^2$

So $p^2 + q^2 + 2\left(-\dfrac{b}{a}\right) = 4m^2$

$\therefore p^2 + q^2 = 4m^2 + \dfrac{2b}{a}$

QED [2]

(iii) $N[-apq(p+q), a(2 + p^2 + pq + q^2)]$

$x = -a \times \dfrac{-b}{a}(p+q)$

$= b(p+q)$

$= b \times 2m$

$\therefore x = 2bm$

$y = a(2 + p^2 + pq + q^2)$

$= a\left(2 + 4m^2 + \dfrac{2b}{a} - \dfrac{b}{a}\right)$

$\therefore y = 2a + 4am^2 + b$

[2]

(iv) Locus of N;

$x = 2bm$; $y = 2a + 4am^2 + b$

$\therefore y = 2a + 4am^2 + \frac{x}{2m}$

$2my = 4am + 8am^3 + x$

$x - 2my = -4am - 8am^3$

$x + (-2m)y = 2a(-2m) + a(-2m)^3$

Compare

$x + py = 2ap + ap^3$

clearly this is:

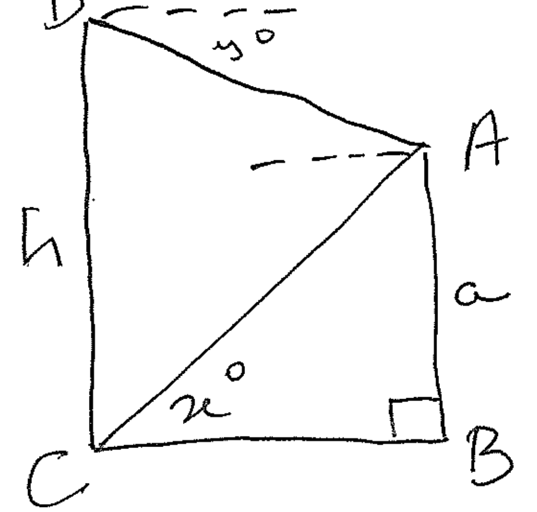
(i) a straight line

(ii) a normal at the point where $t = -2m$.

ie at $(-4am, 4am^2)$

[2]

(e)



$\frac{\sin(x+y)}{h} = \frac{\sin(90-y)}{AC}$ — Sine Rule

But $\sin(90-y) = \cos y$

$\sin x = \frac{a}{AC}$

$\therefore \frac{\sin(x+y)}{h} = \cos y \cdot \frac{\sin x}{a}$

$\therefore h = \frac{a \sin(x+y)}{\sin x \cos y}$

QED

[6]