SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

November 2013

Assessment Task 1
Year 11

Mathematics Extension

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Section is to be returned in a separate bundle.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question if full marks are to be awarded.
- Answer in simplest exact form unless otherwise instructed.

Total Marks – 70

Section A (10 marks)

- Answer questions 1-10 on the Multiple Choice answer sheet provided.

Section B (60 Marks)

- For Questions 11-14, start a new answer booklet for each question.

Examiner: *J. Chen*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}dx = \ln x,\ x > 0$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

SECTION A [10 marks]

Attempt Questions 1-10

Use the multiple-choice answer sheet for Questions 1-10. **Marks**

1. Consider the parametric equations $x = t + 2$ and $y = 4t^2$, find the Cartesian equation. **1**

   (A) $y = 4(x+2)^2$

   (B) $y = 4(x-2)^2$

   (C) $y = 4\sqrt{x+2}$

   (D) $y = 4\sqrt{x-2}$

2. What is the solution to the inequality $\frac{2}{x+1} \geq 1$? **1**

   (A) $-1 < x \leq 1$

   (B) $-1 \leq x \leq 1$

   (C) $x < -1 \quad \text{and} \quad x \geq 1$

   (D) $x \leq -1 \quad \text{and} \quad x \geq 1$

3. What is the exact value of $\sin 75°$? **1**

   (A) $\frac{\sqrt{6}-\sqrt{2}}{4}$

   (B) $\frac{\sqrt{2}-\sqrt{6}}{4}$

   (C) $\frac{1-\sqrt{3}}{2\sqrt{2}}$

   (D) $\frac{1+\sqrt{3}}{2\sqrt{2}}$

4. What is the acute angle to the nearest degree between the lines $2x + 3y = 0$ and $y - 3x + 5 = 0$? **1**

   (A) 38°

   (B) 67°

   (C) 74°

   (D) 75°

5. What are the coordinates of the point which divides the interval with end points $(-3, 5)$ and $(7, -5)$ internally in the ratio $2: 3$? **1**

(A) $(-23, 25)$

(B) $(1, 1)$

(C) $(3, -1)$

(D) $(-27, -25)$

6. $1 - \frac{\sin x \cos x}{\tan(90° - x)} =$ **1**

(A) 0

(B) $\sin^2 x$

(C) $\cos^2 x$

(D) $\frac{1}{\cos^2 x}$

7. If k is a constant such that $x^3 - kx^2 + kx - 4$ is divisible by $x - k$, then $k =$ **1**

(A) -2

(B) 0

(C) 2

(D) -2 or 2

8. Solve the equation $\frac{1}{\sin\theta + 1} = -1$, where $0° \leq \theta \leq 90°$. **1**

(A) 0°

(B) 30°

(C) 90°

(D) No solutions

9. If $a < 0 < b$, which of the following must always be true? **1**

(A) $a + 1 < b$

(B) $\frac{1}{a} < \frac{1}{b}$

(C) $a^2 > b^2$

(D) $-3a < -2b$

10. A student committee consists of 20 boys and 15 girls. A team of 8 students is selected from the committee to participate in an activity. Find the probability that the team chosen consists of at least 3 boys and at least 3 girls, correct to 3 decimal places. **1**

(A) 0.022

(B) 0.275

(C) 0.281

(D) 0.726

End of Multiple Choice Section

SECTION B [60 marks]

Attempt Questions 11-14

Answer each question in a SEPARATE writing booklet.

Question 11 [15 marks] – Use a SEPARATE writing booklet. **Marks**

(a) Differentiate the following with respect to x:

$$\sin(x°)$$ **2**

(b) How many numbers greater than 500 can be formed with the digits $2, 3, 5, 6$ and 8 if no digit is repeated? **2**

(c) Let $f(x) = 2x^3 + kx^2 - (2k-1)x - 6$, and $x + 1$ is a factor of $f(x)$.

(i) Find the value of k. **1**

(ii) Let $g(x) = x^2 - 15x - 16$. Find the quadratic factor of $f(x) - g(x)$. **2**

(d) Find the gradients of the 2 lines which make angles of $45°$ with the line whose equation is $3x - 4y + 5 = 0$. **2**

(e) If $x = \sqrt{5}\cos 2\theta$ and $y = 3\cos^2\theta$, find an equation independent of θ. **2**

(f) The equation of the chord of contact from $P(x_0, y_0)$ is $xx_0 = 2a(y + y_0)$. [DO NOT PROVE THIS] **2**

Find the coordinates of the point which gives a chord of contact of $x - 3y - 15 = 0$ to the parabola $x^2 = 12y$.

(g) Find the exact value of $\sin 15° \cos 15°$. **2**

End of Question 11

Question 12 [15 marks] – Use a SEPARATE writing booklet. **Marks**

(a) $P(2ap, ap^2)$ is a variable point on the parabola $x^2 = 4ay$. The tangent to the parabola at P meets the x-axis at A and the y-axis at B.

(i) Find the equation of the tangent at P. **1**

(ii) Find the coordinates of A and B. **1**

(iii) If M is the midpoint of A and B, describe the locus of M. **2**

(b) Solve $\sin x + \cos x + 1 = 0$ for $0° \leq x \leq 360°$ by using the substitution $t = \tan\frac{x}{2}$. **3**

(c) Find the primitives of the following functions with respect to x.

(i) $y = \sec^2(3x - 2)$ **1**

(ii) $y = \frac{x}{3x^2-5}$ **2**

(d) A bag contains five cards numbered 1, 2, 3, 4 and 5 respectively. Mary repeats drawing one card at a time randomly from the bag without replacement until the number on the card drawn is 4. Find the total number of ways. **2**

(e)

(i) Sketch the curve $y = x \ln x$, showing stationary points and intercepts. **2**

(ii) Hence, or otherwise, find the value(s) of k for which $x \ln x = k$ has 1 solution. **1**

End of Question 12

Question 13 [15 marks] – Use a SEPARATE writing booklet. **Marks**

(a) If α, β and γ are the roots of $x^3 - 2x^2 + 5x - 4 = 0$, find the value of

(i) $\alpha + \beta + \gamma$ **1**

(ii) $\alpha\beta\gamma$ **1**

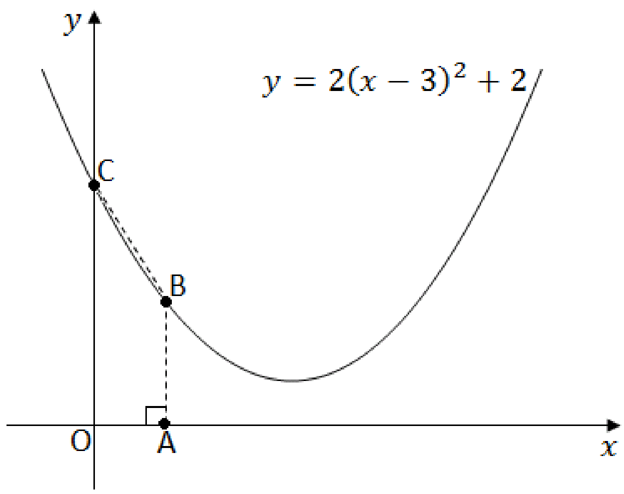
(iii) $\alpha^2 + \beta^2 + \gamma^2$ **2**

(iv) $\alpha^3 + \beta^3 + \gamma^3$ **2**

(b)

(i) Show that $2x - 3$ is a factor of $8x^3 - 24x^2 + 40x - 33$. **1**

(ii) The diagram below shows the graph of $y = 2(x-3)^2 + 2$. B is a variable point on the graph in the first quadrant. A is the foot of the perpendicular from B to the x-axis. C is the point of intersection of the graph $y = 2(x-3)^2 + 2$ and the y-axis.



(α) Let $(2m, 0)$ be the coordinates of A. Express the area of the trapezium $OABC$ in terms of m. **2**

(β) If the area of the trapezium $OABC$ is 33, find the coordinates of B. **2**

(c) Find the general solution of the trigonometric equation $\sin 3\theta = \sin\theta$, express your answer in radians. **4**

End of Question 13

Question 14 [15 marks] – Use a SEPARATE writing booklet.

Marks

(a)

(i) Write $2\sin x - \cos x$ in the form $R\sin(x - \alpha)$, correct α to the nearest degree. **2**

(ii) Hence, solve $2\sin x - \cos x = 1$ for $0 \leq x \leq 360°$. **1**

(b) There are 5 different maths classes in year 12. To form a graduate dinner committee of 15 members, 3 representatives are nominated by each class. From the committee, 4 members are randomly selected.

(i) Find the number of ways if the 4 selected members are nominated by 2 different classes. **2**

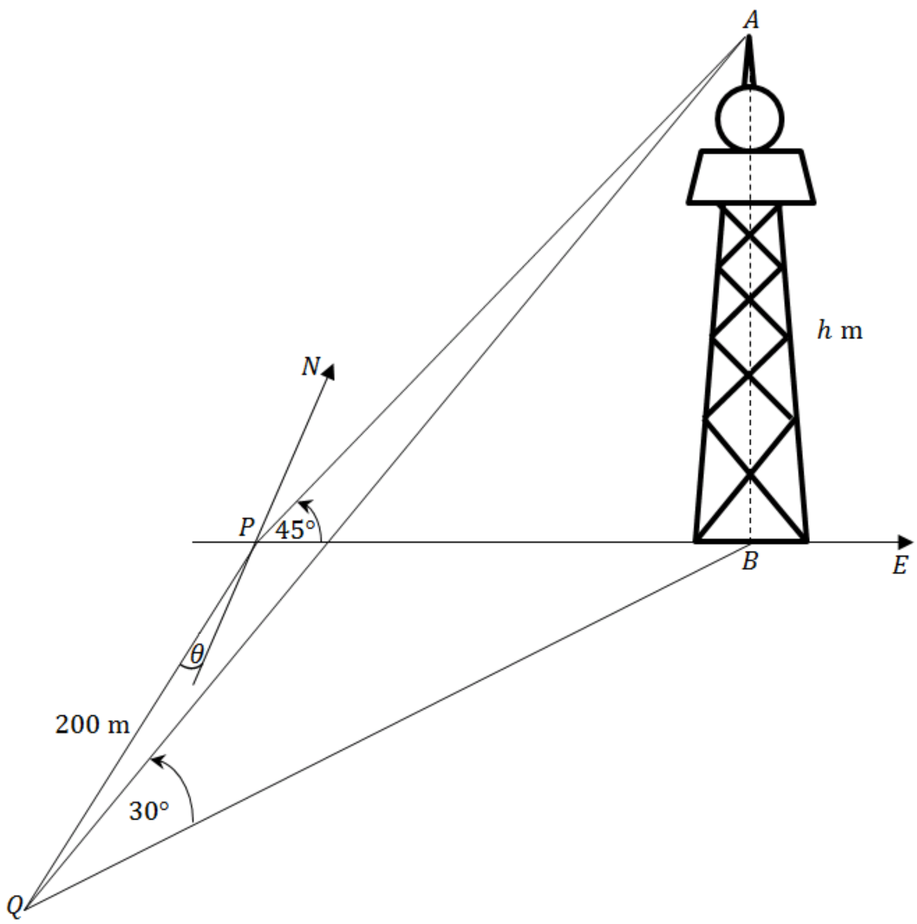
(ii) Find the probability that the 4 selected members are nominated by at least 3 different classes. **2**

(c) For $0 \leq \theta \leq \frac{\pi}{2}$, find the greatest value of **3**

$$\frac{\cos^2\left(\frac{\pi}{2} - \theta\right) - 2\cos^2\theta}{20}$$

Question 14 continues on next page

(d) The diagram below shows a tower AB with height h m. The angle of elevation of A from the observers P and Q are 45° and 30° respectively. The bearing of Q from P is $S\theta W$ and the bearing of AB from Q is $N(30° + \theta)E$. The distance between P and Q is 200 m.



(i) Show that $h^2 - (200 \sin \theta)h - 20000 = 0$. **2**

(ii) Show that $\theta = 30°$. **2**

(iii) Hence, find the value of h. **1**

End of Question 14
End of Exam

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Student Number: Answers

Mathematics Extension
Task 1 2013

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ☒ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

A ☒ B ☒ (correct) C ○ D ○

Section I: **Multiple choice answer sheet.**

Completely colour the cell representing your answer. Use black pen.

1. B
2. A
3. D
4. D
5. B
6. C
7. D
8. D
9. B
10. D

2013 Ext 1, Task 1 ✓ = ½ m

Q11.

(a) $y = \sin(x^\circ)$

$= \sin\left(x \times \frac{\pi}{180}\right)$ ✓✓

$y' = \frac{\pi}{180}\cos\left(\frac{\pi}{180}x\right)$ or $\frac{\pi}{180}\cos(x^\circ)$ ✓✓

(2)

(b) No's > 500 with 2, 3, 5, 6, 8 – no repetitions

3 digits $\boxed{3} \times \underline{4} \times \underline{3} = 36$

↑ 5, 6 or 8

or 4 digits $\boxed{5} \times \boxed{4} \times \boxed{3} \times \boxed{2} = 120$

or 5 digits $\underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1} = 120$

1½

Total ways = 276 ways ½

(2)

(c) $f(x) = 2x^3 + kx^2 - (2k-1)x - 6$

(i) $f(-1) = 0 \Rightarrow 0 = -2 + k + 2k - 1 - 6$

$3k = 9$

$k = 3$ ✓✓ (1)

(ii) $g(x) = x^2 - 15x - 16$.

Then $f(x) - g(x) = 2x^3 + 3x^2 - 5x - 6 - (x^2 - 15x$

$= 2x^3 + 2x^2 + 10x + 10$

$= 2(x^3 + x^2 + 5x + 5)$ ✓

Try $x = -1 \Rightarrow -1 + 1 - 5 + 5$ ✓ ✓ ∴ $(x+1)$ is a factor

$x = -5 \Rightarrow -125 + 25 - 25 + 5$ ✗

11(c) (ii) (cont)

$$\begin{array}{r} x^2 + 5 \\ x+1 \overline{) x^3 + x^2 + 5x + 5} \\ \underline{x^3 + x^2 \quad\quad\quad} \\ 0 + 5x + 5 \\ \underline{5x + 5} \\ 0 \end{array}$$ ✓

Then $f(x) - g(x) = 2(x+1)(x^2+5)$ ✓

$\therefore$ Quadratic factor is x^2+5 ✓ (2)

(d) $\alpha = 45^\circ$. Line $3x - 4y + 5 = 0$

$\Rightarrow y = \frac{3}{4}x + \frac{5}{4}$

$\Rightarrow m_1 = \frac{3}{4}$. ✓

Then let lines which are 45° to given line be in form $y = m_2 x + b$

$\therefore \tan 45^\circ = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$ ✓

$1 = \left| \frac{\frac{3}{4} - m_2}{1 + \frac{3}{4} m_2} \right|$ (2)

$\Rightarrow \frac{\frac{3}{4} - m_2}{1 + \frac{3}{4}m_2} = 1$ or $\frac{\frac{3}{4} - m_2}{1 + \frac{3}{4}m_2} = -1$

$\Rightarrow \frac{3}{4} - m_2 = 1 + \frac{3}{4}m_2$ or $\frac{3}{4} - m_2 = -1 - \frac{3}{4}m$

$\Rightarrow m_2 = -\frac{1}{7}$ ✓ or $\Rightarrow m_2 = 7$ ✓

11(e) $x = \sqrt{5\cos 2\theta}$ (1) and $y = 3\cos^2\theta$ (2)

From (2): $y = 3\left(\frac{1}{2}(\cos 2\theta + 1)\right)$ ✓

$\Rightarrow y = \frac{3}{2}\cos 2\theta + \frac{3}{2}$ (3) ✓

From (1) $x^2 = 5\cos 2\theta$

$\Rightarrow \cos 2\theta = \frac{x^2}{5}$ (4) ✓ (2)

Sub (4) into (3) $\Rightarrow$ $y = \frac{3x^2}{10} + \frac{3}{2}$ ✓ or $10y = 3x^2 +$

(f) $x x_0 = 2a(y + y_0)$

$a = 3$ $\Rightarrow x x_0 = 6(y + y_0)$ (1) ✓

Now $x - 3y - 15 = 0$ ✓

$\Rightarrow x = 3(y + 5)$

$\times 2$ $\Rightarrow 2x = 6(y + 5)$ ✓ (2) (2)

Since (1) ≡ (2) $\Rightarrow x_0 = 2,\ y_0 = 5$

$\Rightarrow$ Point is $(2, 5)$ ✓

(g) $\sin 15^\circ \cos 15^\circ$ Now $\sin x \cos x = \frac{1}{2}\sin$

$= \frac{1}{2}\sin 30^\circ$ ✓✓

$= \frac{1}{2} \times \frac{1}{2}$ (2)

$= \frac{1}{4}$ ✓✓

12)a)i) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$y' = \frac{x}{2a}$

when $x = 2ap$

$m_T = \frac{2ap}{2a}$

$= p$

$y - y_1 = m(x - x_1)$

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

ii) let $y = 0$

$px - ap^2 = 0$

$px = ap^2$

$x = ap$

$\therefore A$ is $(ap, 0)$

let $x = 0$

$y = -ap^2$

$\therefore B$ is $(0, -ap^2)$

iii) M is $\left(\frac{ap+0}{2}, \frac{0+(-ap^2)}{2}\right)$

$\left(\frac{ap}{2}, -\frac{ap^2}{2}\right)$

$x = \frac{ap}{2}$

$p = \frac{2x}{a}$ —— ①

$y = -\frac{ap^2}{2}$ —— ②

sub ① into ②

$y = \frac{-a\left(\frac{2x}{a}\right)^2}{2}$

$y = -\dfrac{2x^2}{a}$

$x^2 = -\dfrac{ay}{2}$

$x^2 = -4\left(\dfrac{a}{8}\right)y$ in the form $x^2 = -4Ay$ where $A = \dfrac{a}{8}$

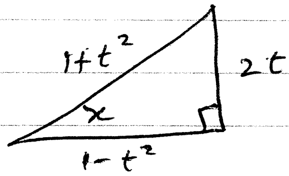
The locus of M is a parabola with vertex $(0,0)$, focus $\left(0, -\dfrac{a}{8}\right)$

b) $\sin x + \cos x + 1 = 0$

let $t = \tan\dfrac{x}{2}$

$0^\circ \leq x \leq 360^\circ$

$0^\circ \leq \dfrac{x}{2} \leq 180^\circ$



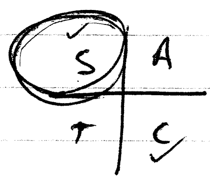
$\dfrac{2t}{1+t^2} + \dfrac{1-t^2}{1+t^2} + 1 = 0$

$2t + 1 - t^2 + 1 + t^2 = 0$

$2t = -2$

$t = -1$

$\therefore \tan\dfrac{x}{2} = -1$



$\tan\alpha = -1$

$\alpha = 45^\circ$

$\dfrac{x}{2} = 135^\circ$

$x = 270^\circ$

must also test $x = 180^\circ$

$LHS = \sin 180^\circ + \cos 180^\circ + 1$

$= (0) + (-1) + 1$

$= 0$

$= RHS$

$\therefore x = 180^\circ, 270^\circ$

c) i) $\displaystyle\int \sec^2(3x-2)\,dx = \frac{1}{3}\tan(3x-2) + C$

ii) $\int \frac{x}{3x^2-5}\,dx$

$= \frac{1}{6}\int \frac{6x}{3x^2-5}\,dx$

$= \frac{1}{6}\ln(3x^2-5) + c$

d) $\boxed{4}$ or $\_\,\boxed{4}$ or $\_\,\_\,\boxed{4}$ or $\_\,\_\,\_\,\boxed{4}$ or $\_\,\_\,\_\,\_\,\boxed{4}$

$1 + 4\times 1 + 4\times 3\times 1 + 4\times 3\times 2\times 1 + 4\times 3\times 2\times 1\times 1$

$= 65$

e) i) $y = x\ln x$

$y' = x.\frac{1}{x} + 1.\ln x$

$y' = 1 + \ln x$

$y'' = \frac{1}{x}$

let $y' = 0$ for stat. points

$1 + \ln x = 0$

$\ln x = -1$

$x = e^{-1}$

when $x = \frac{1}{e}$

$y'' = \frac{1}{(\frac{1}{e})}$

$= e > 0$

$y = \frac{1}{e}\ln(\frac{1}{e})$

$y = -\frac{1}{e}$

$\therefore$ Minimum Turning Point at $(\frac{1}{e}, -\frac{1}{e})$

Domain: $x > 0$

let $y = 0$

$0 = x \ln x$

$x = 0$ or $\ln x = 0$

but $x > 0$. $\underline{x = 1}$

as $x \to 0^+$

$y \to 0^-$

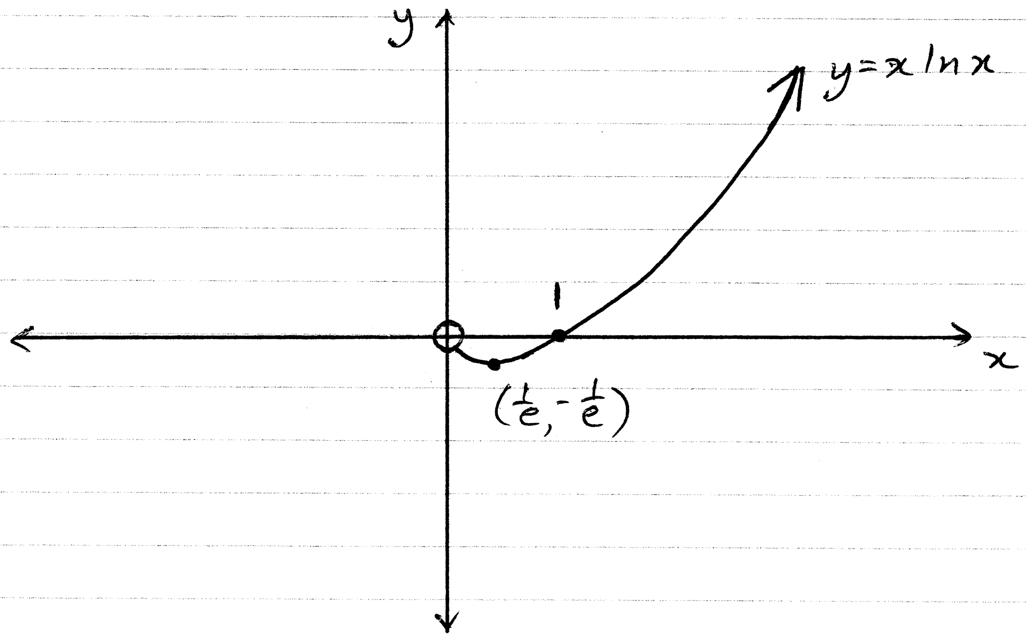
as $x \to +\infty$

$y \to +\infty$

<u>Note</u>: since $y'' = \frac{1}{x}$

and $x > 0$

$y'' > 0$ CONCAVE UP.



ii) consider where $y = k$ (a horizontal line) intersects the above curve ($y = x \ln x$) in one place.

$k = -\frac{1}{e}$, $k \geq 0$.

QUESTION 13

(a) (i) $\alpha+\beta+\gamma = 2$ 1

(ii) $\alpha\beta\gamma = 4$ 1

(iii) $\alpha\beta+\alpha\gamma+\beta\gamma = 5$ 2

$(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= 4-10$

$= -6$

(iv) $\alpha^3 - 2\alpha^2+5\alpha-4=0$
$\beta^3 - 2\beta^2+5\beta-4=0$
$\gamma^3 - 2\gamma^2+5\gamma-4=0$

$\alpha^3+\beta^3+\gamma^3 - 2(\alpha^2+\beta^2+\gamma^2)+5(\alpha+\beta+\gamma)-12=0$

$\alpha^3+\beta^3+\gamma^3+12+10-12=0.$

$\alpha^3+\beta^3+\gamma^3 = -10.$ 2

(b)(i) $(2x-3)$ $\frac{3}{2}$.

$8\left(\frac{3}{2}\right)^3 - 24\left(\frac{3}{2}\right)^2 + 40\left(\frac{3}{2}\right) - 33$

$= 27 - 54 + 60 - 33$

$= 0$ 1

(ii) (α) $A(2m, 0)$

$B(2m, 2(2m-3)^2+2)$.

$C(0, 20)$

$\text{Area} = m(2(2m-3)^2 + 2 + 20)$

$= 2m(2m-3)^2 + 22m$

$= 8m^3 - 24m^2 + 40m$. 2

(β) $33 = 8m^3 - 24m^2 + 40m$.

$8m^3 - 24m^2 + 40m - 33 = 0$.

$m = \frac{3}{2}$ is a solution from part (i).

$B(3, 2)$. 2

(c) $\sin 3\theta = \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$

$= 2\sin\theta \cos^2\theta + \cos^2\theta \sin\theta - \sin^3\theta$

$= 3\sin\theta(1 - \sin^2\theta) - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

$\therefore\ 3\sin\theta - 4\sin^3\theta = \sin\theta$

$4\sin^3\theta - 2\sin\theta = 0$

$2\sin^3\theta - \sin\theta = 0$

$\sin\theta\,(2\sin^2\theta - 1) = 0$

$\sin\theta = 0$ $\qquad\qquad$ $\sin\theta = \pm\frac{1}{\sqrt{2}}$

$\theta = n\pi$ $\qquad\qquad$ $\theta = n\pi \pm \frac{\pi}{4}$

or

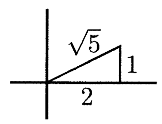
where $n \in \mathbb{Z}$. $\qquad$ $\theta = \frac{\pi n}{2} + \frac{\pi}{4}$

2013 Extension Mathematics Task 1:
Solutions— Question 14

14. (a) (i) Write $2\sin x - \cos x$ in the form $R\sin(x-\alpha)$, correct α to the nearest degree. [2]

Solution:

$$\begin{aligned} R &= \sqrt{2^2+1^2}, \\ &= \sqrt{5}. \\ \tan\alpha &= \tfrac{1}{2}, \\ \alpha &\approx 27^\circ. \\ \therefore 2\sin x - \cos x &= \sqrt{5}\sin(x-27^\circ). \end{aligned}$$



(ii) Hence, solve $2\sin x - \cos x = 1$ for $0^\circ \leqslant x \leqslant 360^\circ$. [1]

Solution:

$$\begin{aligned} \sqrt{5}\sin(x-27^\circ) &= 1, \\ x - 27^\circ &= \sin^{-1}\frac{1}{\sqrt{5}}, \\ &= 27^\circ,\ (180-27)^\circ, \\ \therefore x &= 54^\circ,\ 180^\circ. \end{aligned}$$

If using the calculated (not rounded) value of α, the result is $x = 53^\circ,\ 180^\circ$.

(b) There are 5 different maths classes in year 12. To form a graduate dinner committee of 15 members, 3 representatives are nominated by each class. From the committee, 4 members are randomly selected.

(i) Find the number of ways if the 4 selected members are nominated by 2 different classes. [2]

Solution:

$$\begin{aligned} \text{Ways of choosing classes} &= {}^5\text{C}_2, \\ &= 10. \end{aligned}$$

The possibilities are 3 & 1 or 2 & 2 or 1 & 3,

$$\begin{aligned} \text{Ways of choosing 3 \& 1} &= 1 \times 3, \\ \text{Ways of choosing 2 \& 2} &= {}^3\text{C}_2 \times {}^3\text{C}_2, \\ &= 9, \\ \text{Ways of choosing 1 \& 3} &= 3 \times 1, \\ \therefore \text{Total ways} &= 10(3+9+3), \\ &= 150. \end{aligned}$$

(ii) Find the probability that the 4 selected members are nominated by at least 3 different classes. [2]

Solution:

$$\begin{aligned} \text{Ways of not choosing 2} &= {}^{15}\text{C}_4 - 150, \\ &= 1215. \\ \therefore \text{Probability(at least 3 different)} &= \frac{1215}{{}^{15}\text{C}_4}, \\ &= \frac{81}{91}. \end{aligned}$$

(c) For $0 \leqslant \theta \leqslant \frac{\pi}{2}$, find the greatest value of **[3]**

$$\frac{\cos^2\left(\frac{\pi}{2}-\theta\right)-2\cos^2\theta}{20}.$$

Solution: $\dfrac{\cos^2\left(\frac{\pi}{2}-\theta\right)-2\cos^2\theta}{20}=\dfrac{\sin^2\theta-2\cos^2\theta}{20}.$

Clearly, as θ goes from 0 to $\frac{\pi}{2}$, $\sin\theta$ and also $\sin^2\theta$ goes from 0 to 1 and $\cos\theta$ and also $\cos^2\theta$ goes from 1 to 0. Over this domain, $\sin^2\theta$ is monotonic increasing and $\cos^2\theta$ is monotonic decreasing. So the sum, $\sin^2\theta-2\cos^2\theta$, is monotonic increasing.

$\therefore$ Maximum at $\dfrac{\pi}{2}$ is $\dfrac{1}{20}$.

Solution: Alternative method—

$$\begin{aligned}
\text{Put } f(\theta) &= \cos^2\left(\tfrac{\pi}{2}-\theta\right)-2\cos^2\theta,\\
&= \sin^2\theta-2\cos^2\theta.\\
f'(\theta) &= 2\sin\theta\cos\theta+4\sin\theta\cos\theta,\\
&= 3\sin 2\theta,\\
&= 0 \text{ when } 2\theta=0,\ \pi,\\
\textit{i.e.}\text{ when } \theta &= 0,\ \tfrac{\pi}{2} \text{ we have a stationary point.}\\
\text{Testing, } f(0) &= 0-2=-2,\\
f(\tfrac{\pi}{2}) &= 1-0=1.
\end{aligned}$$

So the maximum is $\dfrac{1}{20}$ when $\theta=\dfrac{\pi}{2}$.

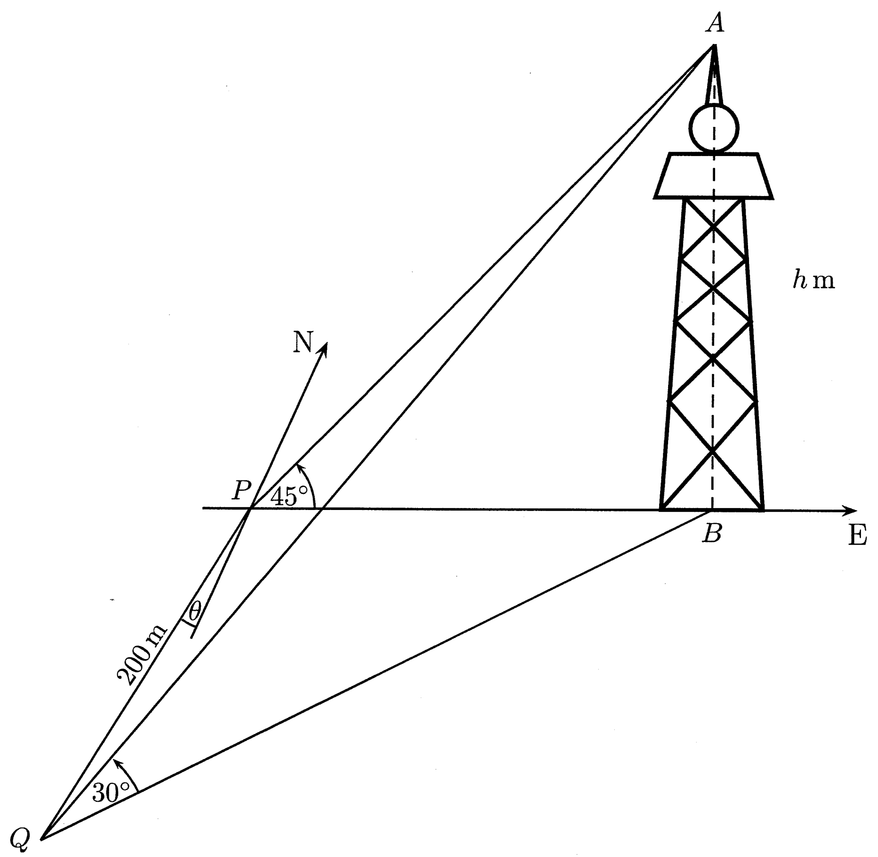
Solution: Another alternative—

$$\begin{aligned}
\cos^2\left(\tfrac{\pi}{2}-\theta\right)-2\cos^2\theta &= \sin^2\theta-2\cos^2\theta,\\
&= 1-\cos^2\theta-2\cos^2\theta,\\
&= 1-3\cos^2\theta.
\end{aligned}$$

$$\begin{array}{rcl}
0\leqslant & \theta & \leqslant \frac{\pi}{2},\\
\text{but } \cos 0\geqslant & \cos\theta & \geqslant \cos\frac{\pi}{2},\\
1\geqslant & \cos\theta & \geqslant 0,\\
1\geqslant & \cos^2\theta & \geqslant 0,\\
-3\leqslant & -3\cos^2\theta & \leqslant 0,\\
-2\leqslant & 1-3\cos^2\theta & \leqslant 1,\\
\dfrac{-2}{20}\leqslant & \dfrac{1-3\cos^2\theta}{20} & \leqslant \dfrac{1}{20},\\
-\dfrac{1}{10}\leqslant & \dfrac{1-3\cos^2\theta}{20} & \leqslant \dfrac{1}{20}.
\end{array}$$

i.e. the maximum is $\dfrac{1}{20}$ when $\theta=\dfrac{\pi}{2}$.

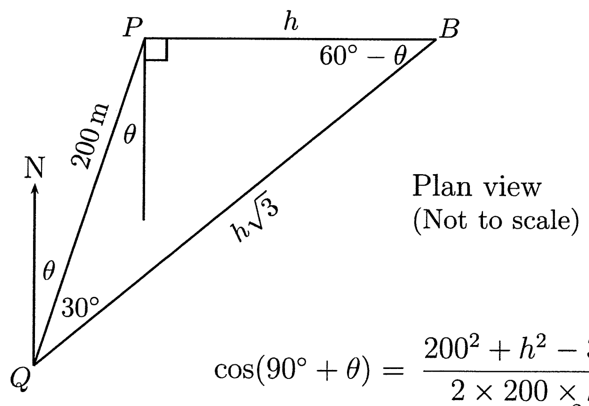
(d) The diagram below shows a tower AB with height h m. The angles of elevation of A from the observers P and Q are 45° and 30° respectively. The bearing of Q from P is S θW and the bearing of AB from Q is N$(30° + \theta)$E. The distance between P and Q is 200 m.



(i) Show that $h^2 - (200\sin\theta)h - 20\,000 = 0$. 2

Solution: $\dfrac{h}{PB} = \tan 45°$, $\quad \dfrac{h}{QB} = \tan 30°$,

$\therefore PB = h$, $\quad QB = h\sqrt{3}$.



Plan view
(Not to scale)

$$\cos(90° + \theta) = \frac{200^2 + h^2 - 3h^2}{2 \times 200 \times h},$$
$$-\sin\theta \times 400h = 40\,000 - 2h^2,$$
$$\therefore h^2 - (200\sin\theta)h - 20\,000 = 0.$$

(ii) Show that $\theta = 30°$. [2]

Solution:
$$\begin{aligned}\frac{\not{k}}{\sin 30°} &= \frac{\not{k}\sqrt{3}}{\sin(90° + \theta)},\\ 2\cos\theta &= \sqrt{3},\\ \cos\theta &= \frac{\sqrt{3}}{2},\\ \therefore \theta &= 30°.\end{aligned}$$

(iii) Hence, find the value of h. [1]

Solution: Substitute (ii) in (i)—
$$\begin{aligned}h^2 - 200 \times \tfrac{1}{2} \times h - 20\,000 &= 0,\\ h^2 - 100h + 50^2 &= 20\,000 + 2500,\\ (h - 50)^2 &= 22\,500,\\ h - 50 &= \pm 150,\\ h &= 200 \text{ or } -100.\end{aligned}$$
i.e. h is 200 m (as $h > 0$).