Sydney Girls High School

November 2012
MATHEMATICS
EXTENSION 1

YEAR 12
ASSESSMENT TASK 1 for HSC 2013

Time Allowed: 60 minutes
+ 5 minutes Reading Time

Topics: Locus, Integration, Mathematical Induction, Division of an interval in a given ratio, Other Inequalities

General Instructions:

- There are Seven (7) Questions which are not of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- Start each Question on a new page.
- Write on one side of paper only.
- Diagrams are NOT to scale.
- Board-approved calculators may be used.
- Write your student number clearly at the top of each question and clearly number each question.
- Note that: $\int x^n dx = \frac{1}{n+1}x^{n+1}; \quad n \neq -1$

Total: 50 marks

Student Name:_____   Teacher Name:_____

QUESTION 1 (7 Marks)

a) Find a primitive function of each of the following:

   i. $5x^3 - 4$ 1

   ii. $\dfrac{7 - x^4}{x^2}$ 2

   iii. $(8x + 3)^5$ 2

b) Find the equation of the parabola, with vertical axis, vertex at the origin and passing through the point (6,3).   2

QUESTION 2 (7 Marks)

a) Given the equation of a parabola is $(x + 4)^2 = -2(y - 8)$, find:
   i. the vertex of the parabola.   1
   ii. the focal length.   1

b) Find the equation of the parabola with focus (2,0) and directrix $x = -2$.   2

c) A wasteland bordering a river bank and a straight road was fenced off and used as a recreational park. Perpendicular distances from the road to the river bank are shown on the diagram below.

Use Simpson's rule, with 5 function values, to approximate the area of the recreational park.   3
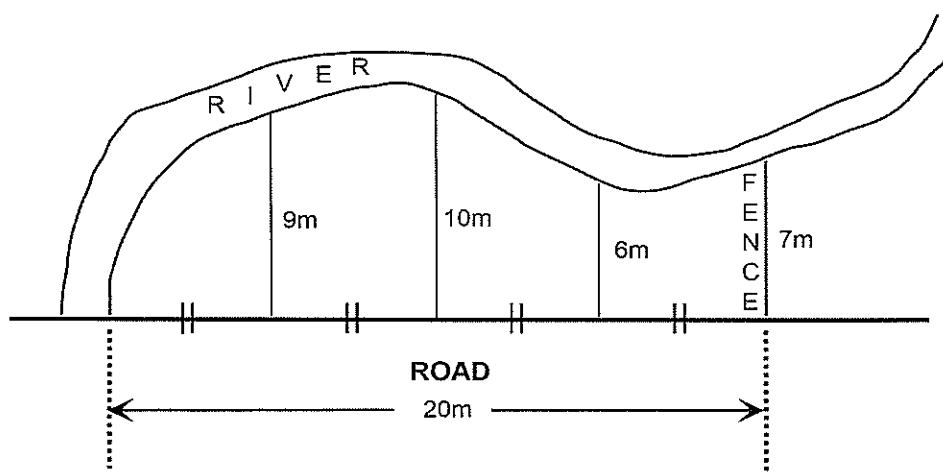
MARKS

a) Solve the following: $\dfrac{x}{x-4} \le -3$.

3

b) i. Find the $x-$ coordinates of the points of intersection of the line $2x+y=0$ and the parabola $y=x(x+4)$.

1

ii. Find the area bounded by $2x+y=0$ and $y=x(x+4)$.

3

MARKS

a) Find in what ratio A divides BC, given that the points A(-1,3), B(0,1) and C(1,-1) lie in one line.

3

b) The point P(k,7) lies on the parabola $x^2 = 20y$.
Find the distance from P to the focus.

2

c) Given the graph of $y = f(x)$ below, find $\displaystyle\int_{-8}^{4} f(x)\,dx$.

2
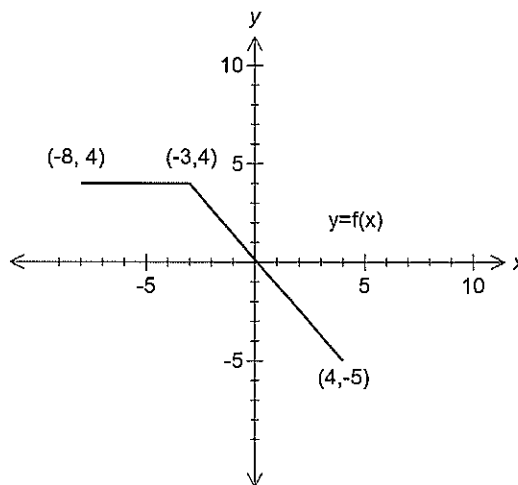
MARKS

a)   i. Draw a diagram showing the area bounded by $y = 2x^3$, $y = 2$ and the
       y – axis.   1

       ii. The area is now rotated around the x – axis. Calculate the exact volume
       of the solid of revolution formed.   3

b) The vertex of a parabola is V(-1,4) and its directrix is $x = 3$.

       i. Find the coordinates of the focus.   1

       ii. Find the equation of the parabola.   1

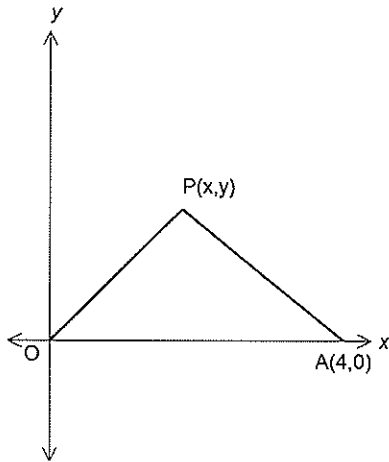   iii. Hence, sketch the parabola, showing the main features.   1

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MARKS

a) Prove by mathematical induction that $1^2 + 3^2 + ... + (2n-1)^2 = \dfrac{1}{3}n(2n-1)(2n+1)$
   for $n \geq 1$ 4

b) Evaluate $\displaystyle\int_{\frac{-8}{3}}^{-1} \dfrac{2}{\sqrt{1-3x}}\, dx$ 3

a)

i. A point $P(x, y)$ moves such that $\angle OPA$ is a right angle. Show that the locus of P is given by: $x^2 + y^2 - 4x = 0$. 2

ii. The locus of P is a circle, find its centre and radius. 1

b) A hollow vessel is formed when the parabola $9x^2 = 4y$ is rotated about the y-axis. The distance across the open end of the vessel is 1.5 metres.

i. Show that, when the depth of water in the vessel is h metres, the volume of water is $\dfrac{2\pi h^2}{9}$ m^3. 3

ii. Find the depth of the water in the vessel when it is half full. 2

End of Paper

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Extension One Task One: Feedback December 2012

| Question	Comment
1	a) For indefinite integrals, don't forget to write " $+C$ "! b) Students who used a simple diagram and wrote the formula $x^2 = 4ay$ were more successful than those who did not.
2 P. L.	a) Focal length is $\frac{1}{2}$ not $a = -\frac{1}{2}$ b) Students who used a diagram were more successful than those who did not. Although $x = -2$ was confused with $y = -2$ c) Tabular method (or variation) worked better than others
4.	b) k can be found by substitution c) Better to use area formulae rather than calculus
3.	a) Note that $x = 4$ is NOT a solution to the original inequation and must be excluded. b)(ii) Most efficient solution uses $\int (f(x) - g(x)) dx$ where $y = f(x)$ is the upper curve.
5	a) Alternate solution: $V =$ cylinder $-$ area under the curve. b) Sketching the parabola should include any intercepts; in this case $x = -2$.
6	a) Better solutions factorised the LHS expression in step 3. b) Most common mistake was the integration step (forgetting to divide by -3)
7	b ii) Many students incorrectly interpreted the vessel being half full as meaning the height was $\frac{h}{2}$ rather than the volume being $V/2$, which lead to no marks being awarded.

Question 1

a) i) $\int (5x^3 - 4)\, dx = \dfrac{5x^4}{4} - 4x + C$ 　　1

ii) $\int \left(\dfrac{7 - x^4}{x^2}\right) dx = \int \left(7x^{-2} - x^2\right) dx$

$= \dfrac{7x^{-1}}{-1} - \dfrac{x^3}{3} + C$

$= -\dfrac{7}{x} - \dfrac{x^3}{3} + C$ 　　$\dfrac{}{2}$

iii) $\int (8x + 3)^5\, dx = \dfrac{(8x+3)^6}{8 \times 6} + C$

$= \dfrac{(8x+3)^6}{48} + C$ 　　$\dfrac{}{2}$

b)
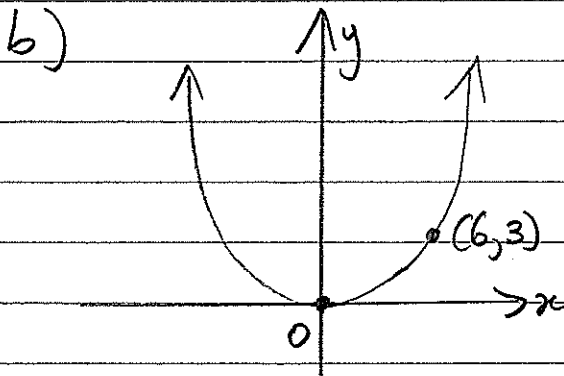
$x^2 = 4ay$

Sub in $(6, 3)$

$36 = 4a(3)$

$36 = 12a$

$a = 3$

$\therefore x^2 = 12y$ 　　$\dfrac{}{2}$
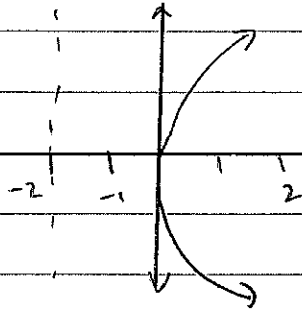
Question 2.

a) $(x+4)^2 = -2(y-8)$

i) Vertex $(-4, 8)$ ✓

ii) focal length $4a = -2$

$a = -\frac{1}{2}$ ②

ie $\frac{1}{2}$ unit ✓

b)

$a = 2$ ✓

$y^2 = 4ax$

$y^2 = 8x$ ✓ ②

c)

x	$f(x)$	w	$wf(x)$
0	0	1	0
5	9	4	36
10	10	2	20
15	6	4	24
20	7	1	7

$\sum wf(x) = 87$

$A \doteqdot \frac{5}{3}(87)$

$= 145 m^2$ ③

1	2	3	4	5	6	7	8	9	10
1					2	3		1	1

MARKS

a) Solve the following: $\dfrac{x}{x-4} \le -3$.
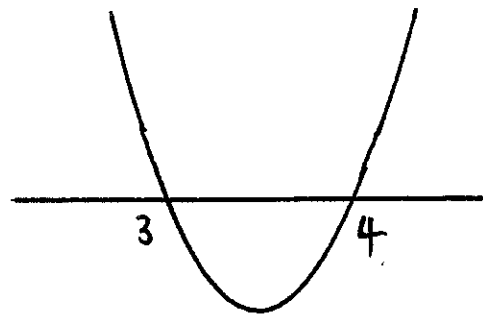
3

b) i. Find the points of intersection of the line $2x + y = 0$ and the parabola
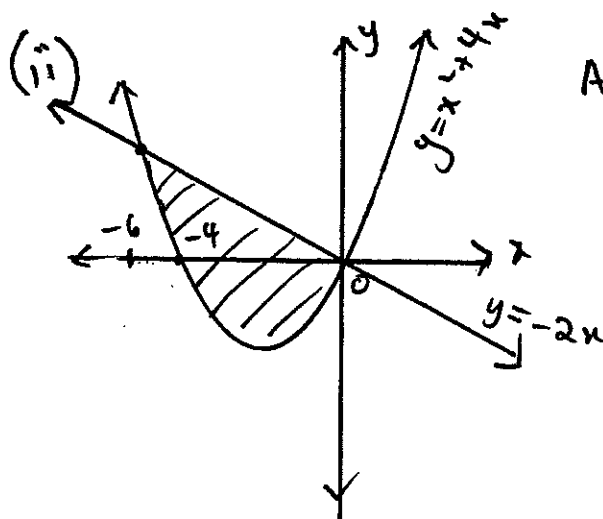 $y = x(x+4)$.

1

    ii. Find the area bounded by $2x + y = 0$ and $y = x(x+4)$.
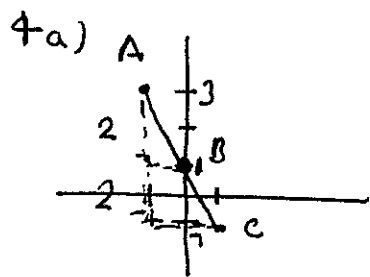
3

(a) $x\,(x-4) \le -3\,(x-4)^2$

$(x-4)\,(x + 3\,(x-4)) \le 0$

$(x-4)\,(4x - 12) \le 0$

$\therefore \; 3 \le x \le 4$

but $\; x \ne 4 \qquad 3 \le x < 4$

(b)   (i) $2x + x\,(x+4) = 0$

$x^2 + 6x = 0 \qquad x\,(x+6) = 0$

$\therefore \; x = 6 \qquad or \qquad x = -6$

(ii)

$A = \displaystyle\int_{-6}^{0} \left(-2x - (x^2 + 4x)\right) dx$

$= \displaystyle\int_{-6}^{0} \left(-x^2 - 6x\right) dx$

$= \left[-\dfrac{x^3}{3} - 3x^2\right]_{-6}^{0}$

$= 0 - (72 - 108)$

$A = 36 \; units^2$

4a)

A divides BC
externally 2:4
i.e. 1:2

b) $4a = 20$
$a = 5$

focus is $(0, 5)$

$k^2 = 20 \times 7$
$= 140$
$k = \sqrt{140}$

$PS = \sqrt{(\sqrt{140})^2 + (7-5)^2}$
$= \sqrt{140 + 4}$
$= \sqrt{144}$
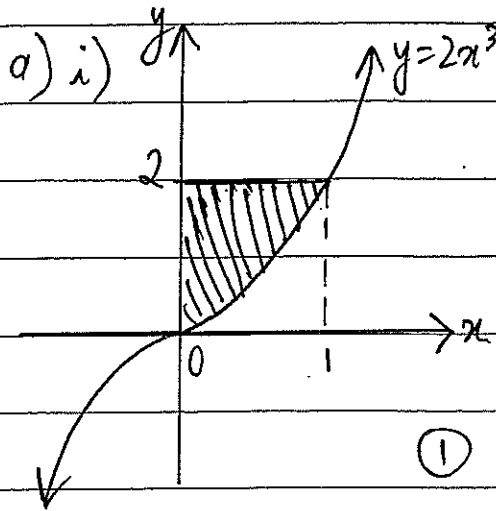$= 12$

c) $\int_{-1}^{4} f(x)\, dx$

$= 5 \times 4 + \frac{1}{2} \times 3 \times 4 - \frac{1}{2} \times 4 \times 5$

$= 16$

Question 5: (7 Marks)

a) i)

$y = 2x^3$

①

ii) $V = \pi \int_0^1 y^2 \, dx$

$= \pi \int_0^1 2^2 \, dx - \pi \int_0^1 (2x^3)^2 \, dx$

$= \pi \int_0^1 (4 - 4x^6) \, dx$

$= \pi \left[4x - \frac{4x^7}{7} \right]_0^1$

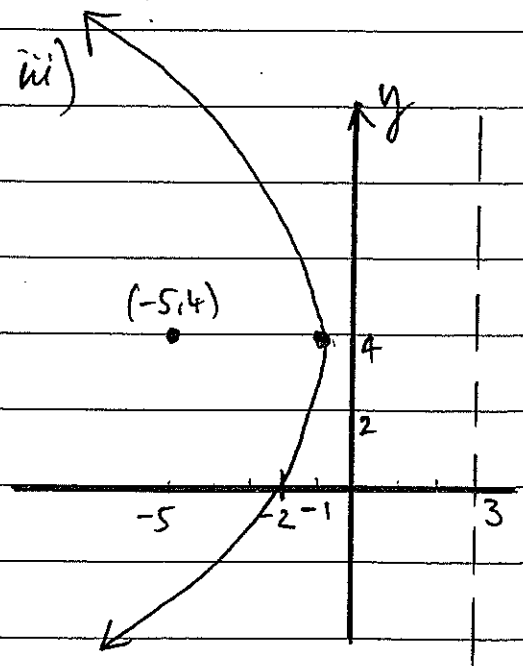$= \pi \left(4 - \frac{4}{7} \right)$

$= \frac{24\pi}{7} \text{ units}^3.$

③

b) i) $a = 4$

$\therefore \text{focus} = (-5, 4)$ ①

ii) $(y-h)^2 = -4a(x-k)$

$\therefore (y-4)^2 = -16(x+1)$ ①

iii)

$(-5,4)$

①

Question 6:

a) $1^2 + 3^2 + \ldots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$

Prove true for n = 1:

$$LHS = 1^2 \qquad\qquad RHS = \frac{1}{3} \times 1(2-1)(2+1)$$
$$= 1 \qquad\qquad\qquad\qquad\qquad = 1$$

LHS = RHS

∴ Proven true for $n = 1$

Assume true for n = k:

$$1^2 + 3^2 + \ldots + (2k-1)^2 = \frac{1}{3}k(2k-1)(2k+1)$$

Prove true for n = k + 1:

R.T.P $1^2 + 3^2 + \ldots + (2k-1)^2 + (2k+1)^2 = \frac{1}{3}(k+1)(2k+1)(2k+3)$

$$LHS = 1^2 + 3^2 + \ldots + (2k-1)^2 + (2k+1)^2$$

$$= \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2 \text{ by assumption}$$

$$= \frac{1}{3}(2k+1)\left[k(2k-1) + 3(2k+1)\right]$$

$$= \frac{1}{3}(2k+1)\left[2k^2 + 5k + 3\right]$$

$$= \frac{1}{3}(2k+1)(k+1)(2k+3)$$

$$= RHS$$

∴ proven true for $n = k+1$

∴ proven true by mathematical induction for $n \geq 1$

$\cancel{4}$

b) $\displaystyle\int_{-\frac{8}{3}}^{-1} \frac{2}{\sqrt{1-3x}}dx = \int_{-\frac{8}{3}}^{-1} 2(1-3x)^{-\frac{1}{2}}dx$

$$= 2\left[\frac{(1-3x)^{\frac{1}{2}}}{-3 \times \frac{1}{2}}\right]_{-\frac{8}{3}}^{-1}$$

$$= -\frac{4}{3}\left[\sqrt{1-3x}\right]_{-\frac{8}{3}}^{-1}$$

$$= -\frac{4}{3}\left(\sqrt{1-3(-1)} - \sqrt{1-3\left(-\frac{8}{3}\right)}\right)$$

$$= -\frac{4}{3}\left(\sqrt{4} - \sqrt{9}\right)$$

$$= -\frac{4}{3}(2-3)$$

$$= \frac{4}{3}$$

$\cancel{3}$

Question 7:

a.

i. $m_{OP} = \dfrac{y}{x}$

$m_{AP} = \dfrac{y}{x-4}$

$m_{OP} \times m_{AP} = -1$

$\dfrac{y}{x} \times \dfrac{y}{x-4} = -1$

$y^2 = -x(x-4)$

$y^2 + x^2 - 4x = 0$

$\diagup 2$

ii. $y^2 + x^2 - 4x + 4 = 4$

$y^2 + (x-2)^2 = 4$

centre $(2,0)$ radius $= 2$

$\diagup 1$

b.

i. $V = \pi \displaystyle\int_o^h x^2 \, dy$

$= \pi \displaystyle\int_o^h \dfrac{4y}{9} \, dy$

$= \pi \left[\dfrac{2y^2}{9} \right]_0^h$

$= \dfrac{2\pi h^2}{9}$

$\diagup 3$

ii. When vessel is full:

$x = \dfrac{3}{4}$

$y = \dfrac{9}{4} \times \left(\dfrac{3}{4} \right)^2$

$= \dfrac{81}{64}$

$V = \dfrac{2\pi}{9} \times \left(\dfrac{81}{64} \right)^2$

$= \dfrac{729\pi}{2048}$

When vessel is half full:

$V = \dfrac{729\pi}{4096}$

$\dfrac{729\pi}{4096} = \dfrac{2\pi h^2}{9}$

$h^2 = \dfrac{6561}{8192}$

$h = 0.89$

$\diagup 2$