SYDNEY TECHNICAL HIGH SCHOOL

HSC ASSESSMENT TASK 1

MATHEMATICS
EXTENSION 1

FEBRUARY 2004

Time Allowed: 70 minutes

General Instructions:

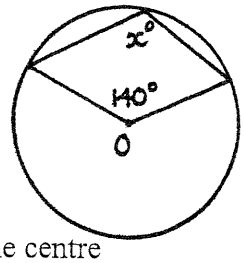
- Start each question on a new page
- Board approved calculators may be used
- Marks indicated are approximate only
- All necessary working should be shown
- Marks may not be awarded for messy or poorly arranged work.

Name:____________________________ Class:____________________________

Q1	Q2	Q3	Q4	Q5	Q6	Total
/9	/9	/9	/7	/8	/9	/51

Question 1

(a)



0 is the centre

Find x **1**

Do not give reasons

(b) Evaluate $\sum_{n=2}^{6} 2^n + 2n$ **2**

(c) i. Find the gradient of the tangent to the parabola $x^2 = 4ay$ at the point $P(2ap, ap^2)$ **1**

ii. Q is the point $(2aq, aq^2)$ and O is the origin. Show that if OQ is parallel to the tangent then $q = 2p$. **2**

iii. If M is the midpoint of PQ, find the equation of the locus of M as P and Q vary along the parabola such that OQ remains parallel to the tangent at P. **3**

Question 2

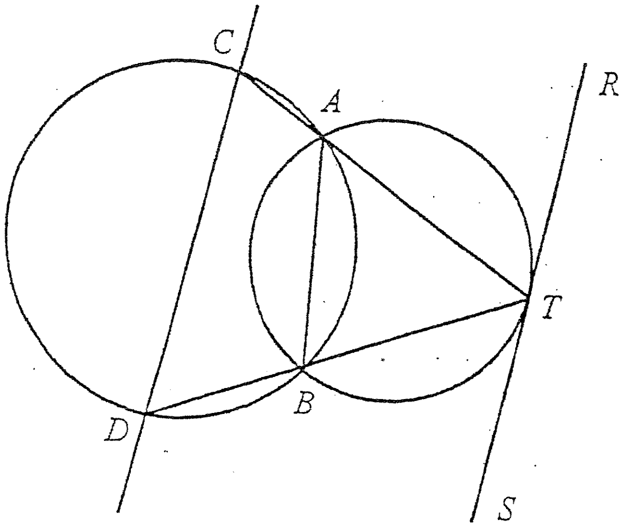
(a) Kerry deposits \$1500 into a superannuation fund on January 1st 2001. He makes further deposits of \$1500 on the first of each month up to and including December 1st 2010. The fund pays compound interest at a monthly rate of 0.75%. In each of the following questions give your answer to the nearest dollar.

i. How much is the first \$1500 deposit worth on December 31st 2010? **1**

ii. Form a geometric series and hence determine the total amount in the fund on December 31st 2010. **2**

iii. If each deposit was increased to \$1600, what difference does it make to the total amount in the fund on December 31st 2010? **2**

(b)



In the diagram, two unequal circles intersect at A and B. The line RS is tangential to the smaller circle at T. The lines TA and TB meet the larger circle at C and D respectively.

i. Copy the diagram

ii. Explain why $\angle BAT = \angle BDC$ **1**

iii. Prove $RS \parallel CD$ **3**

Question 3

(a) i. Expand $(n+1)^3$ **1**

ii. Use the method of proof by induction to show that **4**

$1 + 7 + 19 + \dots\dots\dots + (3n^2 - 3n + 1) = n^3$ where n is a positive integer.

(b) Three numbers whose product is 216 are in geometric progression. If 1, 4 and 8 are subtracted from them respectively the results are in arithmetic progression. Find the numbers. **4**

Question 4

A caterer organises parties for groups of up to 200. She calculates the cost price of a party by allowing \$22 per head for the first 10 guests, \$21 per head for the next 10 guests, and so on, allowing one dollar less per head for each subsequent group of 10 guests or part thereof.

i Show that the cost price, in dollars, for each guest in the n^{th} group of 10 guests, or part thereof, is given by **1**

$$T_n = 23 - n$$

where T_n is the n^{th} term of an arithmetic series.

ii. Find the increase in the cost price of a party if 4 more persons are added to a guest list of 85. **2**

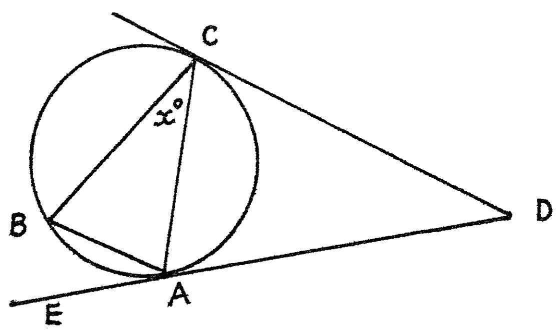
iii. Determine the cost price of a party attended by 115 people. **2**

iv. If the caterer wishes to make a 25% profit on the cost price, calculate the average charge per head for a party of 115 guests. **2**

Question 5

(a) Use mathematical induction to show that if x is a positive integer then $(1+x)^n - 1$ is divisible by x for all positive integers $n \geq 1$ **4**

(b)



AD and CD are tangents to a circle. B is a point on the circle such that $\angle CBA$ and $\angle CDA$ are equal and are double $\angle BCA$.

Let $\angle BCA = x°$

i. Without adding any constructions find the value of x. Give reasons **2**

ii. Hence, prove that BC is a diameter of the circle. **2**

Question 6

A fund is set up with a single investment of \$2000 to provide an annual prize of \$150. The fund accrues interest at 5% pa paid yearly. After the interest is added the prize money is withdrawn from the fund.

i. Find the value of the fund immediately after the first prize has been awarded. **1**

ii. Show that the value of the fund after n years is given by **4**

$A_n = 3000 - 1000(1.05)^n$

iii. (a) In which year are there insufficient funds to award the full prize? **2**

(b) For this final year, what is the maximum value of the prize that can be awarded. **2**

Question 1

(a) $x = 110$ — (1)

(b) $\sum_{n=2}^{6} 2^n + 2n$

$= 2^2 + 2.2 + 2^3 + 2.3 + \ldots + 2^6 + 2.6$ (1)

$= 164$ — (1)

(c) i. $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$

$= \frac{x}{2a}$

$m_{tangent} = \frac{2ap}{2a} = p$ (1)

ii. $m_{OQ} = \frac{aq^2 - 0}{2aq - 0}$

$= \frac{q}{2}$ — (1)

OQ // tangent so

$p = \frac{q}{2}$ — (1)

$\therefore 2p = q$

iii. $M\left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$

$M\left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$

$x = a(p+q) \quad y = \frac{a(p^2+q^2)}{2}$ (1)

using $2p = q$

$x = a(p + 2p)$

$x = 3ap$

$p = \frac{x}{3a}$ — (1)

$y = \frac{a(p^2 + (2p)^2)}{2}$

$2y = 5ap^2$

$2y = 5a \times \left(\frac{x}{3a}\right)^2$

$2y = \frac{5ax^2}{9a^2}$

$\therefore 18ay = 5x^2$ — (1)

Question 2

(a) i. $A_1 = 1500(1.0075)^{120}$

$= \$3677$ — (1)

ii. $A_2 = 1500(1.0075)^{119}$

$\vdots$

$A_{120} = 1500(1.0075)^1$

(1)

$A = 1500(1.0075) + 1500(1.0075 \ldots$

$+ \ldots + 1500(1.0075) \ldots$

$= 1500(1.0075 + \ldots + 1.0075^{1\ldots}$

GP with $a = 1.0075$
$r = 1.0075$
$n = 120$

$= 1500 \times \frac{1.0075(1.0075^{120} - 1)}{1.0075 - 1}$

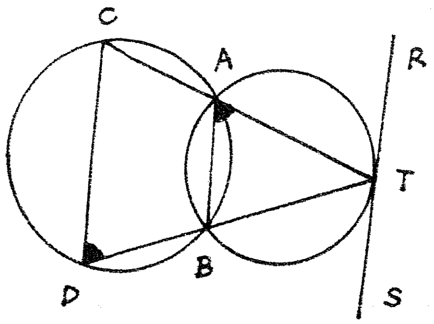
$= \$292\,448$ — (1)

iii. $A = 1600 \times \frac{1.0075(1.0075^{120} - 1)}{1.0075 - 1}$

$= 311\,945$ — (1)

$\therefore$ difference is $\$19\,497$ — (1)

(b) i.



(1)

ii. The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle

iii. $\angle BAT = \angle BDC$ (above)

(1) $\angle BAT = \angle BTS$ (alternate segment theorem)

(1) $\therefore \angle BDC = \angle BTS$

$\therefore RS \parallel CD$ (since alternate angles are equal) (1)

Question 3

(a) i. $(n+1)^3 = (n+1)(n^2+2n+1)$

$= \underline{\underline{n^3 + 3n^2 + 3n + 1}}$ (1)

ii. $1 + 7 + \ldots + (3n^2 - 3n + 1) = n^3$

Step 1 : show true for $n=1$

LHS = 1

RHS = 1^3

= 1

$\therefore$ true for $n=1$ (1)

Step 2 : assume true for $n=k$

ie $S_k = k^3$ (1)

Step 3 : hence show true for $n=$

ie show $S_{k+1} = (k+1)^3$ (1)

$S_{k+1} = S_k + T_{k+1}$ (1)

$= k^3 + [3(k+1)^2 - 3(k+1) + 1]$

$= k^3 + 3k^2 + 6k + 3 - 3k - 3 +$

$= k^3 + 3k^2 + 3k + 1$

$= (k+1)^3$

Step 4 : since true for $n=1$, then from step 3 true for $n = 1+1 = 2$, $n = 2+1 = 3$ .. and so on for all positive integers.

(b) let the numbers be

$a, \ ar, \ ar^2 \leftarrow GP$

$a \times ar \times ar^2 = 216$

$(ar)^3 = 216$ — (1)

$ar = 6$

$\therefore$ numbers are

$a, \ 6, \ 6r$

$a-1, \ 6-4, \ 6r-8 \leftarrow AP$

$a-1, \ 2, \ 6r-8$

(1)

$2-(a-1) = 6r-8-2$

$3-a = 6r-10$

$a+6r = 13$

using $r = \frac{6}{a}$

(1)

$a + 6 \cdot \frac{6}{a} = 13$

$a^2 + 36 = 13a$

$a^2 - 13a + 36 = 0$

$(a-9)(a-4) = 0$

$a = 9$ or $a = 4$

$r = \frac{2}{3}$ $\quad r = \frac{3}{2}$ (1)

$\therefore$ the numbers are 9, 6, 4

OR 4, 6, 9

Question 4

i. 22, 21, 20, ...

$T_n = a + (n-1)d$

$= 22 + (n-1).-1$ — (1)

$= 22 - n + 1$

$\therefore T_n = 23 - n$

ii. for 85 guests $n = 9$

$T_9 = 23 - 9$ (1)

$= 14$ (per guest)

$\therefore$ increase in cost $= 14 \times 4$

$= \$56$ (1)

iii. for the 111th – 120th guest $n = 12$

$T_{12} = 23 - 12$

$= 11$ (per guest) (1)

Cost $= 10 \times 22 + 10 \times 21 + \ldots$

$+ 10 \times 12 + 5 \times 11$

$= 10[22 + 21 + \ldots + 12] + 55$

$= 10 \times \frac{11}{2}[22 + 12] + 55$

$= \$1925$ — (1)

iv. $125\% \times 1925 = 2406 \cdot 25$ (1)

$\frac{2406 \cdot 25}{115} = \$20 \cdot 93$ (1)

Question 5

(a) $(1+x)^n - 1$ divisible by x

Step 1 : show true for $n=1$

$(1+x)^1 - 1 = x$ (1)

which is divisible by x

Step 2 : assume true for $n = k$

$(1+x)^k - 1 = Mx$ (M is some integer)

(1)

Step 3 : hence show true for $n = k+1$

ie show $(1+x)^{k+1} - 1$ is div. by

$(1+x)^{k+1} - 1$ — (1)

$= (1+x)^k (1+x) - 1$

$= (Mx+1)(1+x) - 1$

$= Mx + Mx^2 + 1 + x - 1$

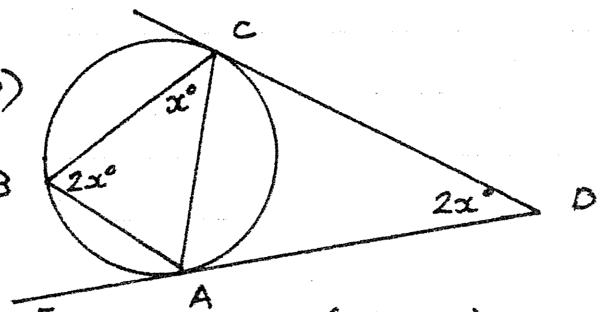
$= Mx + Mx^2 + x$

$= x(M + Mx + 1)$ — (1)

which is div by x

Step 4 : .....

(b)



$\angle CDA = 2x$ (given)

i. $\angle CAD = 2x$ (alternate segment theorem)

$\angle DCA = 2x$ (" ")

(1) $6x = 180$ (angle sum of $\triangle$)

$\therefore x = 30$ (1)

ii. $\angle BCA = 30$

$\angle CBA = 60$ — (1)

$\therefore \angle CAB = 90^\circ$ (angle sum of $\triangle$)

$\therefore$ BC is diameter — (1)

(angle in semi circle is 90°)

Question 6

i. $A_1 = 2000 + 0.05 \times 2000 - 150$

$= 2000(1.05) - 150$

$= \$1950$ — (1)

ii. $A_2 = A_1 + 0.05 \times A_1 - 150$

$= A_1(1.05) - 150$

$= [2000(1.05) - 150]1.05 - 150$

$= 2000(1.05)^2 - 150(1 + 1.05)$ — (1)

$A_n = 2000(1.05)^n - 150(1 + 1.05 + \ldots + 1.05^{n-1})$ — (1)

GP with $a = 1$, $r = 1.05$, $n = n$

$= 2000(1.05)^n - 150 \times \frac{1(1.05^n - 1)}{1.05 - 1}$ — (1)

$= 2000(1.05)^n - \frac{150(1.05^n - 1)}{0.05}$

$= 2000(1.05)^n - 3000(1.05^n - 1)$ — (1)

$= 2000(1.05)^n - 3000(1.05)^n + 3000$

$\therefore A_n = 3000 - 1000(1.05)^n$

iii. (a) Prize can be awarded

$A_n \geq 0$

$3000 - 1000(1.05)^n \geq 0$ — (1)

$1000(1.05)^n \leq 3000$

$1.05^n \leq 3$

$n \leq 22.5$

$n = 22$

$(A_{22} = 74.74)$

$\therefore$ insufficient funds in 22nd year — (1)

(b) $74.74 \times 1.05 = \$78.47$ — (2)

(−1 if interest not added)