FILE

Name : ______________________

Teacher/ Class : ______________________

SYDNEY TECHNICAL HIGH SCHOOL

HSC ASSESSMENT TASK 1

DECEMBER 2006

MATHEMATICS - EXTENSION 1

Time Allowed : **70 minutes**

Instructions:

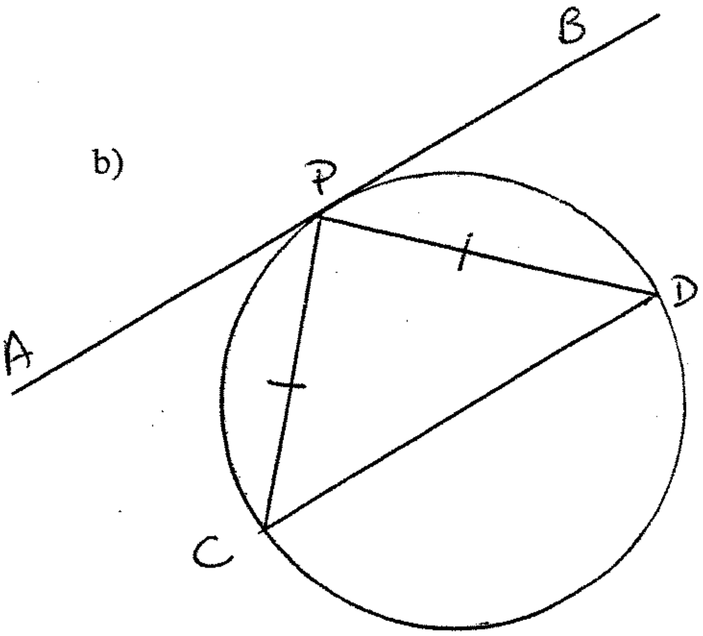
- Write your name and class at the top of each page.
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.
- Start **each** question on a new **page.**
- Diagrams unless otherwise stated are not to scale.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
/8	/5	/8	/10	/10	/9	/50

Question 1

a) The sum of an infinite geometric series is $\frac{3}{2}$. If the common ratio is halved the sum of the resulting infinite series is $\frac{12}{17}$. Find the first term and common ratio of the original series. (4 marks)

b)



PC and PD are equal chords of a circle. A tangent AB is drawn at P. Prove that AB is parallel to CD.

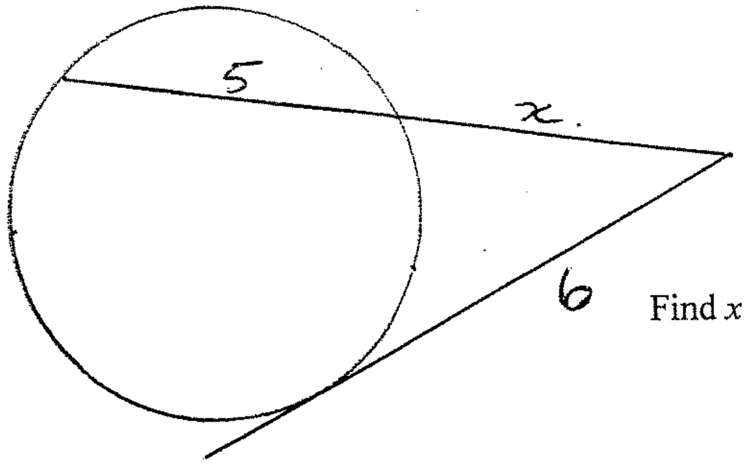
(4 marks)

Question 2 (Start a new page)

a) The nth term of a sequence is given by

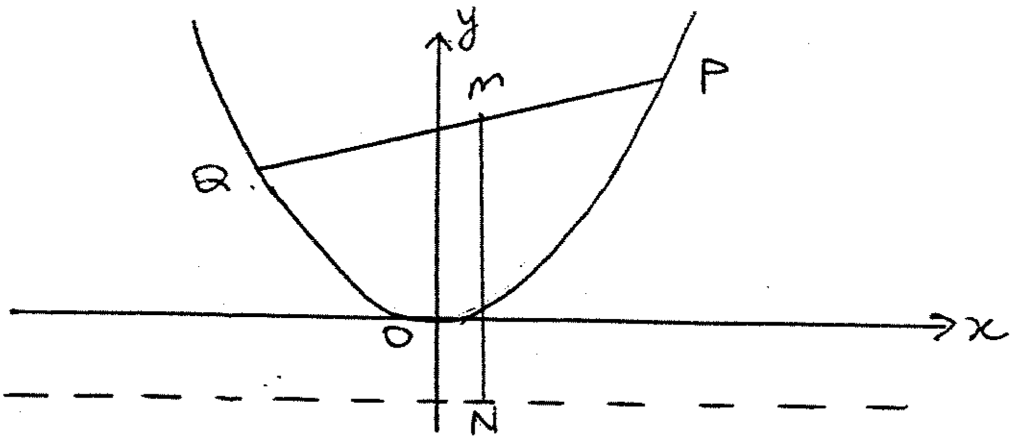
$$T_n = a\left(\frac{1}{2}\right)^n + bn$$

If the first 3 terms are 11, 10, 11 find a and b, and hence the fourth term. (3 marks) b)



Find x (2 marks)

Question 3 (Start a new page)



Let $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ be points on the parabola $x^2 = 4ay$ as shown in the diagram .

a) Show that the equation of PQ is

$$y = \frac{p+q}{2}x - apq$$

(2 marks)

b) Show that if the chord PQ passes through the focus (o, a),

then $pq = -1$ (1 mark)

c) M is the midpoint of the focal chord PQ and N lies on the directrix vertically below M. T is the midpoint of MN.

Write down

i) the co-ordinates of M (1 mark)

ii) the co-ordinates of N (1 mark)

iii) show that T has co-ordinates

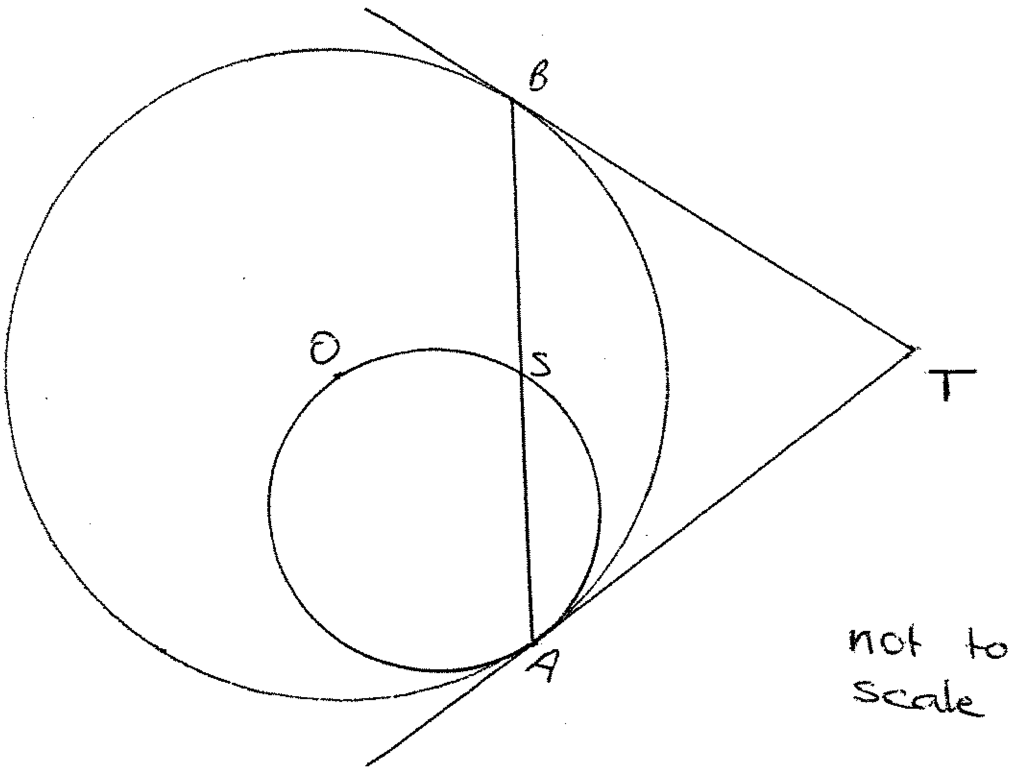
$$[a(p+q), \frac{a}{4}(p^2+q^2-2)]$$ (1 mark)

iv) show that the locus of T is $x^2 = 4ay$ (2 marks)

Question 4 (start a new page)

a) The sum of three consecutive terms of an arithmetic series is 21, and the sum of their squares is 155. Find the three terms by letting a be the middle term. (5 marks)

b)



Two circles touch internally at a point A and the smaller of the two circles passes through O, the centre of the larger circle. AB is any chord of the larger circle, cutting the smaller circle at S. The tangents to the larger circle at A and B meet at a point, T.

Prove i) AB is bisected at S (3 marks)

ii) O, S and T are collinear (2 marks)

Question 5 (start a new page)

a) The normal at any point $P(2at, at^2)$ on the parabola $x^2 = 4ay$ cuts the y axis at Q and is produced to a point R such that $PQ = QR$

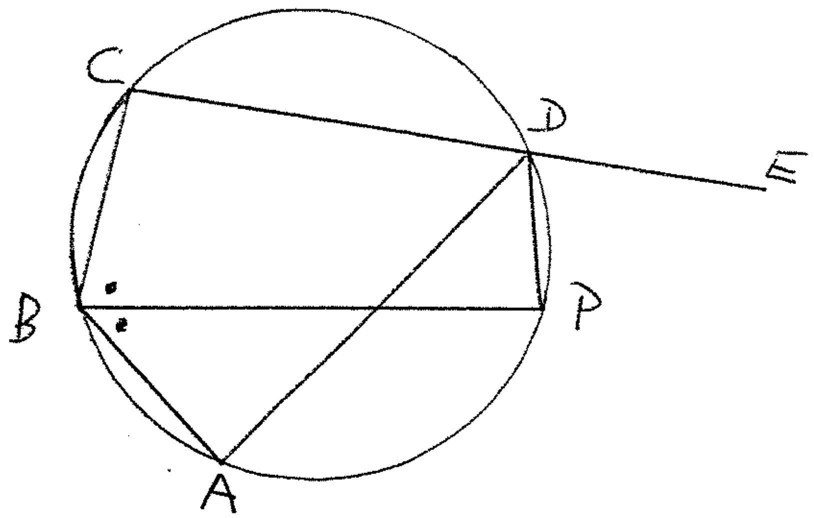
i) show that the equation of the normal is $x + ty - at^3 - 2at = 0$ (1 mark)

ii) find the co-ordinates of Q (1 mark)

iii) write down the coordinates of R (2 marks)

iv) by eliminating t show that the locus of R is $x^2 = 4a(y - 4a)$ (2 marks)

b)



In the diagram $ABCD$ is a cyclic quadrilateral. CD is produced to E.
P is a point on the circle such that $\angle ABP = \angle PBC$

i) copy the diagram

ii) give a reason why $\angle ABP = \angle ADP$ (1 mark)

iii) show that PD bisects $\angle ADE$ (2 marks)

iv) if, in addition, $\angle BAP = 90°$ and $\angle APD = 90°$ state where the centre of the circle is located. (1 mark)

Question 6 (start a new page)

A man borrows \$30 000 at 12 % p a compound interest. If the principal plus interest are to be paid by 20 equal annual instalments,

i) Write an expression for A_1 the amount owing after 1 year. Let the annual instalment be M. (1 mark)

ii) Show that the amount owing at the end of 2 years is given by

$$A_2 = 30\,000\,(1.12)^2 - M(1.12 + 1)$$ (1 mark)

iii) Find the annual instalment (3 marks)

b) Prove by mathematical induction that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$ (4 marks)

FILE

Teacher's Name: Student's Name/Nº:

Mathematics - Extension 1

December 2006.

Question 1

a) $\frac{a}{1-r} = \frac{3}{2} \implies 2a = 3-3r$ ①

$\frac{a}{1-r/2} = \frac{12}{17} \implies 17a = 12-6r$ ②

① × 2 $\quad 4a = 6-6r$ ③

② − ③ $\quad 13a = 6$

$\therefore a = \frac{6}{13}$

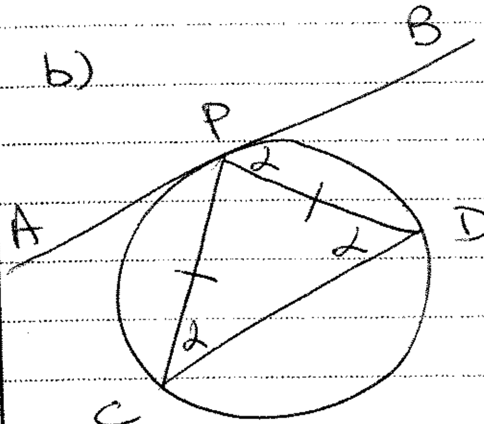
Put $a = \frac{6}{13}$ into ①

$\frac{12}{13} = 3-3r$

$3r = \frac{27}{13}$

$r = \frac{9}{13}.$

b)



$\angle BPD = \angle PCD$ (angle in the alternate segment)

$\angle PDC = \angle PCD$ (base angles of an isosceles triangle)

$\therefore \angle BPD = \angle PDC$

Since a pair of alternate angles are equal $AB \parallel CD$.

Question 2

a) $T_n = a\left(\frac{1}{2}\right)^n + bn$

T_1 : $a\left(\frac{1}{2}\right) + b = 11$

ie $a + 2b = 22$ ①

T_2 : $a\left(\frac{1}{2}\right)^2 + 2b = 10$

ie $a + 8b = 40$ ②

① − ② $\quad -6b = -18$

$b = 3$

Teacher's Name: _______________ Student's Name/N°: _______________

Put $b=3$ into ①

$a + 6 = 22$

$\therefore a = 16$

$T_4 = 16(\frac{1}{2})^4 + 3(4)$

$= 1 + 12$

$= 13$

b) $x(5+x) = 6^2$

$5x + x^2 = 36$

$x^2 + 5x - 36 = 0$

$(x+9)(x-4) = 0 \Rightarrow x = -9$ or $x = 4$

But x must be positive

$\therefore x = 4$

Question 3

a) Using the two point form

$$\frac{y - ap^2}{x - 2ap} = \frac{ap^2 - aq^2}{2ap - 2aq}$$

$$= \frac{\cancel{a}\cancel{(p-q)}(p+q)}{2\cancel{a}\cancel{(p-q)}} \qquad p \neq q,\ a \neq 0.$$

$\therefore 2y - 2ap^2 = (x - 2ap)(p+q)$

$2y - \cancel{2ap^2} = px + qx - \cancel{2ap^2} - 2apq$

$$y = \frac{p+q}{2}x - apq$$

b) Since passes through focus, $(0,a)$ satisfies the equation

ie $a = \frac{p+q}{2}(0) - apq$

$pq = -1$

c) By midpoint formula

(i) $M \equiv \left(\frac{2ap + 2aq}{\cancel{2}}, \frac{ap^2 + aq^2}{2}\right)$

$\equiv \left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$

Teacher's Name: ______________ Student's Name/Nº: ______________

(ii) $N \equiv \left(a(p+q), -a \right)$

(iii) By midpoint formula

$$T \equiv \left(a(p+q), \frac{\frac{a(p^2+q^2)}{2} - a}{2} \right)$$

$$\equiv \left(a(p+q), \frac{a(p^2+q^2) - 2a}{4} \right)$$

$$\equiv \left(a(p+q), \frac{a(p^2+q^2-2)}{4} \right)$$

(iv) $x = a(p+q)$

$\Rightarrow p+q = \frac{x}{a}$

$$y = \frac{a\left((p+q)^2 - 2pq - 2\right)}{4}$$

$$= \frac{a\left(\left(\frac{x}{a}\right)^2 - 2(-1) - 2\right)}{4}$$

$$= \frac{\cancel{a}}{4}\left(\frac{x^2}{a^{\cancel{2}}}\right) \qquad a \neq 0$$

$\Rightarrow x^2 = 4ay$

Question 4

a) Let the terms be

$a-d, \; a, \; a+d$

Then

$(a-d) + a + (a+d) = 3a$

and $3a = 21$ (given)

$\therefore a = 7$

$(a-d)^2 + a^2 + (a+d)^2 = 155$

$a^2 \cancel{-2ad} + d^2 + a^2 + a^2 \cancel{+2ad} + d^2 = 155$

$3a^2 + 2d^2 = 155$

$3(49) + 2d^2 = 155$

$2d^2 = 8$

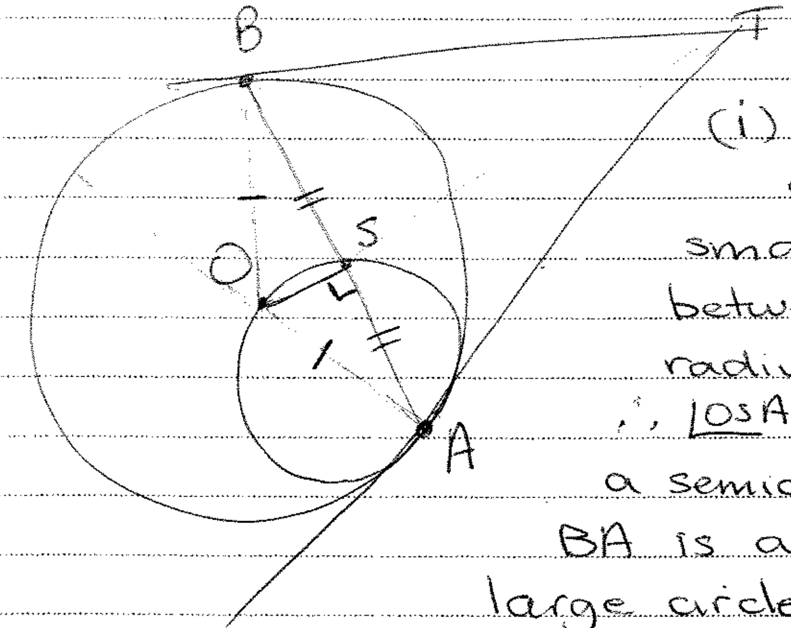
$d^2 = 4$

Teacher's Name: Student's Name/N°:

$\therefore\ d = \pm 2$

$\therefore$ Terms are 9, 7, 5 or 5, 7, 9.

b)



(i) OA is a diameter of the small circle (angle between tangent and radius is 90°)

$\therefore \angle OSA = 90°$ (angle in a semicircle)

BA is a chord of the large circle and OS is perpendicular to it $\Rightarrow$ OS bisects AB

(ii) Δ BTA is isosceles (tangents from an external point equal)

$\angle TSA = 90°$ (TS is an altitude of an isosceles triangle)

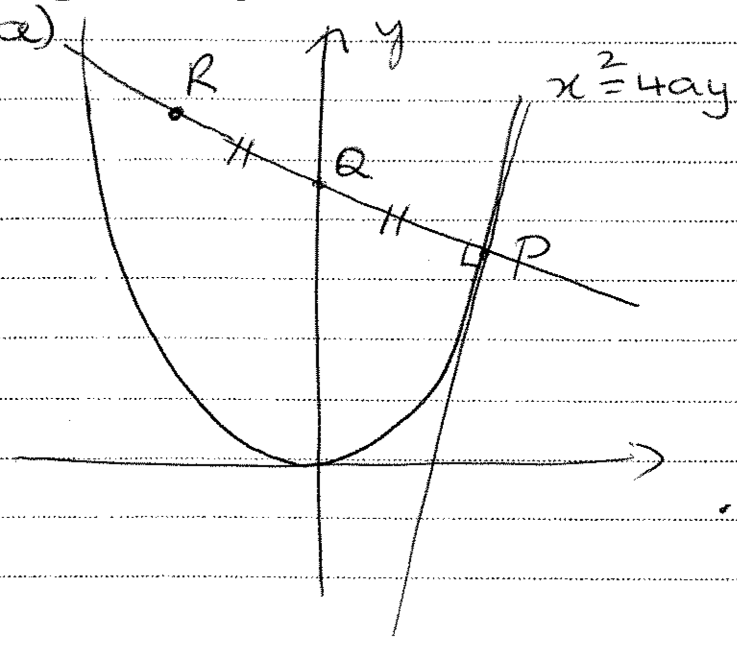
$\therefore \angle OST = \angle OSA + \angle TSA$

$= 180°$

$\therefore$ O, S, T are collinear.

Question 5

a)



(i) $x = 2at$

$\frac{dx}{dt} = 2a$

$y = at^2$

$\frac{dy}{dt} = 2at$

$\therefore \frac{dy}{dx} = \frac{2at}{2a} = t$

Teacher's Name: ______ Student's Name/N°: ______

$\therefore$ gradient normal $= -\frac{1}{t}$

Equation normal

$$y - at^2 = -\frac{1}{t}(x - 2at)$$
$$ty - at^3 = -x + 2at$$
$$x + ty - at^3 - 2at = 0$$

(ii) Put in $x = 0$.

$$ty = at^3 + 2at$$
$$y = at^2 + 2a \qquad t \neq 0.$$
$$\therefore Q \equiv (0,\ a(t^2+2)).$$

(iii) Let $R \equiv (x_1, y_1)$

Then using midpoint formula.

$$0 = \frac{x_1 + 2at}{2} \Rightarrow x = -2at.$$

$$a(t^2+2) = \frac{y_1 + at^2}{2}$$

$$2a(t^2+2) = y_1 + at^2$$
$$\Rightarrow y_1 = at^2 + 4a$$
$$= a(t^2+4) \qquad R \equiv (-2at,\ a(t^2+4))$$

(iv) From $x = -2at$

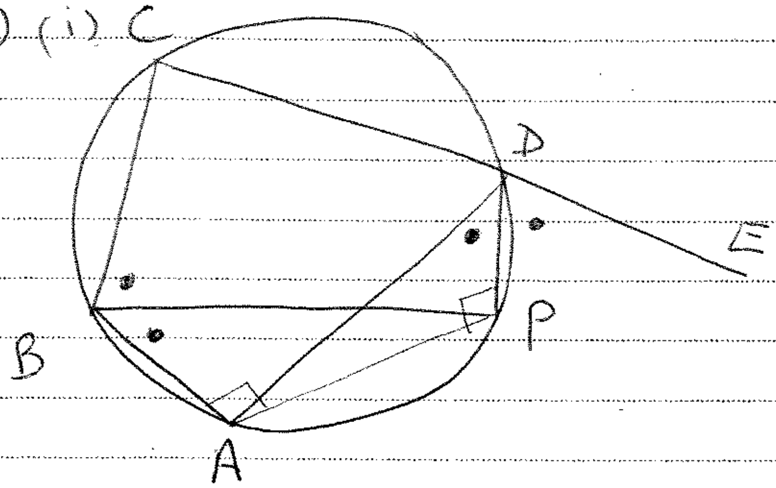
$$t = \frac{x}{-2a}$$

$$y = a\left(\frac{x^2}{4a^2} + 4\right)$$
$$y = \frac{x^2}{4a} + 4a$$

$$4ay = x^2 + 16a^2$$
$$x^2 = 4a(y - 4a)$$

Teacher's Name: ____________ Student's Name/N°: ____________

b) (i)



(ii) $\angle ABP = \angle ADP$ (angles standing on the same arc).

(iii) $\angle PDE = \angle CBD$ (exterior angle of a cyclic quadrilateral)

$\therefore$ $\angle PDE = \angle ADP$ (~~since~~ from (ii))

ie PD bisects $\angle ADE$.

(iv) Circle centre at intersection of BP and AD (as have two angles in a semicircle).

Question 6

(i) $A_1 = 30000(1 + .12) - m$

(ii) $A_2 = [30000(1.12) - m](1.12) - m$

$= 30000(1.12)^2 - m(1.12 + 1)$.

(iii) $A_{20} = 30000(1.12)^{20} - m(1.12^{19} + 1.12^{18} + \dots + 1)$.

$= 30000(1.12)^{20} - m\left(\dfrac{1.12^{20} - 1}{1.12 - 1}\right)$

(using GP formula with $a = 1$, $r = 1.12$, $n = 20$)

$A_{20} = 0$ since loan finished.

$\therefore 30000(1.12)^{20} = \dfrac{m(1.12^{20} - 1)}{1.12 - 1}$

Teacher's Name: Student's Name/N°:

$$M = \frac{30000(1\cdot12)^{20} \times \cdot12}{1\cdot12^{20} - 1}$$

$\doteqdot \$4016\cdot36.$

b) $\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1)$

ie $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6} n(n+1)(2n+1).$

Let $n=1$

$LHS = 1^2 = 1$

$RHS = \frac{1}{6}(1)(1+1)(2\times1+1)$

$= \frac{1}{6}(2)(3)$

$= 1 \quad = LHS \qquad \therefore$ True for $n=1$.

let $n=k$ and assume result true

ie $\sum_{r=1}^{k} r^2 = \frac{1}{6} k(k+1)(2k+1)$

Let $n=k$ and try to show result still holds

ie $\sum_{r=1}^{k+1} r^2 = \frac{1}{6}(k+1)(k+2)(2k+3)$,

$LHS = \sum_{r=1}^{k+1} r^2$

$= \sum_{r=1}^{k} r^2 + (k+1)^2 \qquad [S_{k+1} = S_k + T_{k+1}]$

$= \frac{1}{6}(k)(k+1)(2k+1) + (k+1)^2$

using assumption.

$\therefore LHS = (k+1)\left[\frac{1}{6} k(2k+1) + (k+1)\right]$

$= \frac{1}{6}(k+1)\left[2k^2 + k + 6(k+1)\right]$

$= \frac{1}{6}(k+1)\left[2k^2 + 7k + 6\right]$

$= \frac{1}{6}(k+1)(k+2)(2k+3)$

$\therefore$ If true for $n=k$ also true for $n=k+1$

Since true for $n=1$ also true for $n=2$ and so

by the induction hypothesis true for all