Name : ______________________

Teacher/ Class : ______________________

SYDNEY TECHNICAL HIGH SCHOOL

HSC ASSESSMENT TASK 1

DECEMBER 2007

MATHEMATICS - EXTENSION 1

Time Allowed : **70 minutes**

Instructions:

- Write your name and class at the top of each page.
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.
- Start **each** question on a new **page.**
- Diagrams unless otherwise stated are not to scale.

Q1	Q2	Q3	Q4	Q5	TOTAL
/10	/10	/10	/10	/10	/50

Question 1

a) Given the parabola described by $x = 6t, y = -3t^2$

(i) Write the cartesian equation of this parabola 1

(ii) Find the coordinates of the focus and the equation of the directix 2

(iii) Find the length of the latus rectum 1

b) The sum to n terms of a certain series is given by $Sn = \frac{n}{n+1}$

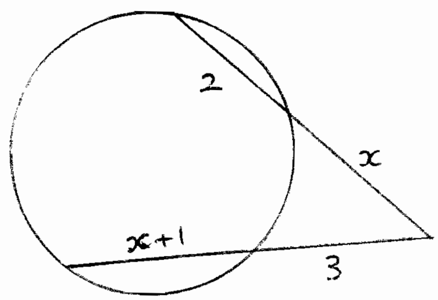
(i) Find the first and second terms of the sequence 2

(ii) Find the sum of the 9^{th}, 10^{th} and 11^{th} terms 2

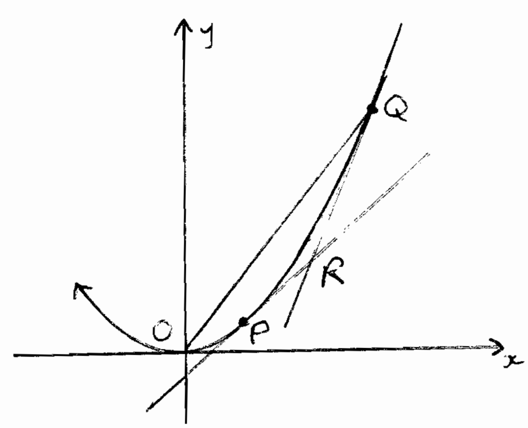
(iii) Find a simplified expression for the nth term, Tn 2

Question 2

a) Find the value of x 2



b) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. OQ is a chord passing through the vertex. Tangents at P and Q intersect at R.



(i) Derive the equation of the tangent at P. 2

(ii) Find the coordinates of R 2

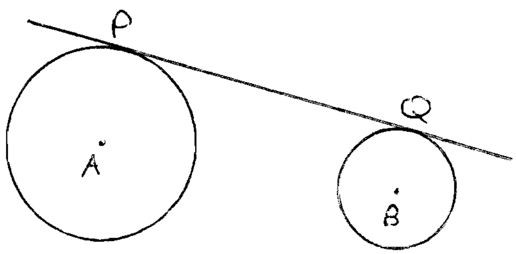
(iii) Find the coordinates of the point where the tangent at P intersects the directrix 1

(iv) Chord OQ is parallel to the tangent at P. Find a condition linking p and q 1

(v) Show that, as P moves on its parabola, R moves on another parabola $x^2 = \frac{9}{2}ay$ 2
(You may use the result from part iv)

Question 3

(a)



Two circles with centres A and B have radii 14 cm and 7 cm respectively. The distance between the centres is 25 cm. 1

Find the length of the common tangent PQ.

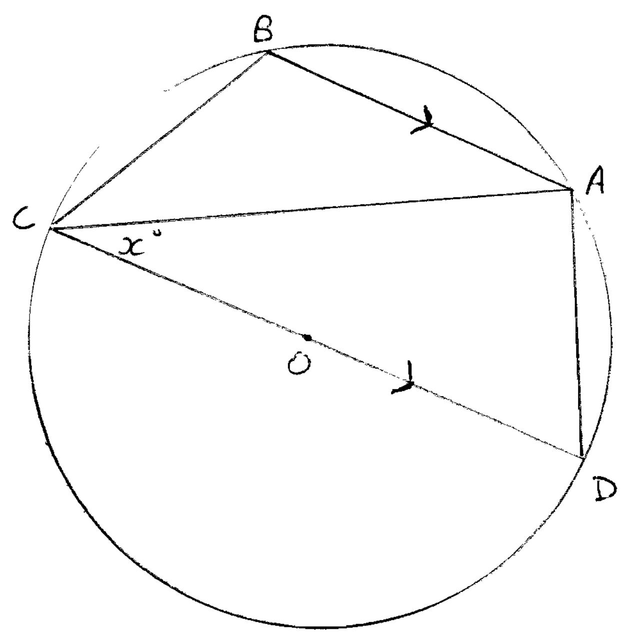
(b) Find the number that must be added to each of 1, 3, 4 in order to form a geometric sequence. 2

(c) Find the sum of all the integers from 501 to 1600 (inclusive), but excluding all multiples of 8. 4

(d) Points A, B, C, D are points on the circle, centre O. 3

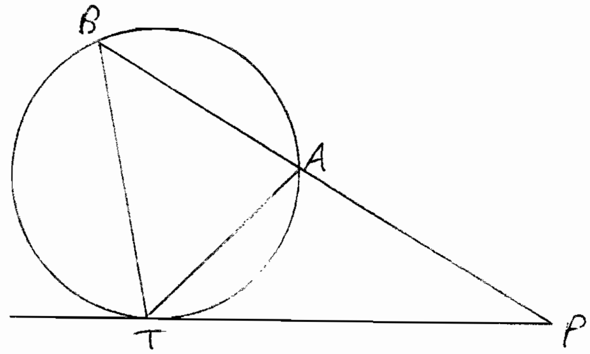
CD//BA and $\angle ACD = x^\circ$

Find an expression for $\angle BCD$ in terms of x, giving reasons.



Question 4

(a) TP is a tanget to the circle and BP is a secant. BP cuts the circle at A.



i) Prove that triangles TAP and BTP are similar. 3

ii) Hence show that $PT^2 = PA.PB$ 1

(b) The curve $y = ax^2 + bx$ has a turning point at (1,-2). Find the values of a and b. 3

(c) Sketch a curve $y = f(x)$ satisfying the following conditions: 3

$f'(1) = f'(2) = f'(3) = 0$

$f'(x) < 0$ for $x < 1$, for $1 < x < 2$, for $x > 3$

$f'(x) > 0$ for $2 < x < 3$.

Question 5

(a) Use mathematical induction to prove that $8^{n} - 5^{n}$ is divisible by 3 for all positive integers n. 4

(b) A man borrows \$40, 000 for a new car. The loan is to be repaid over 5 years by equal monthly instalments of \$R per month. Interest of 12% p.a is calculated on the balance owing.

Let A_n be the amount owing after n months.

(i) Write an expression for A_1 and show that $A_2 = 40\,000(1.01)^2 - 1.01R - R$ 1

(ii) Find the amount of each monthly repayment to the nearest cent. 3

(iii) Suppose instead that the man repays \$1200 every month. How many months will it take to repay the loan? 2

Solutions

1 a) i) $t = \frac{x}{6} \Rightarrow y = -3\left(\frac{x}{6}\right)^2$

$$= -3 \times \frac{x^2}{36}$$

$$= \frac{-x^2}{12}$$

$$\therefore x^2 = -12y$$

ii) $F(0,-3)$

directrix $y = 3$

iii) $LR = 12$ units

b) i) $T_1 = S_1 = \frac{1}{2}$

$$T_2 = S_2 - S_1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

ii) $T_9 + T_{10} + T_{11} = S_{11} - S_8$

$$= \frac{11}{12} - \frac{8}{9} = \frac{1}{36}$$

iii) $T_n = S_n - S_{n-1}$

$$= \frac{n}{n+1} - \frac{n-1}{n}$$

$$= \frac{n^2 - (n-1)(n+1)}{n(n+1)}$$

$$= \frac{1}{n(n+1)}$$

2 a) $x(x+2) = 3(x+4)$

$$x^2 + 2x - 3x - 12 = 0$$

$$x^2 - x - 12 = 0$$

$$(x-4)(x+3) = 0$$

$$\therefore x = 4 \text{ only}$$

b) i) $y = \frac{x^2}{4a} \Rightarrow \frac{dy}{dx} = \frac{x}{2a}$

when $x = 2ap$, $\frac{dy}{dx} = \frac{2ap}{2a} = p$

$\therefore$ eqn tangent at P:

$$y - ap^2 = p(x - 2ap) = px - 2ap^2$$

$$\therefore y = px - ap^2$$

iii) $y = -a \Rightarrow -a = px - ap^2$

$$\therefore px = ap^2 - a$$

$$\therefore x = \frac{ap^2 - a}{p}$$

$\therefore$ point is $\left(\frac{ap^2 - a}{p}, -a\right)$

ii) $y = px - ap^2$, $y = qx - aq^2$

At R: $px - ap^2 = qx - aq^2$

$$px - qx = ap^2 - aq^2$$

$$\therefore x = \frac{a(p+q)\cancel{(p-q)}}{\cancel{p-q}}$$

Subst. $\Rightarrow y = p \times a(p+q) - ap^2$

$$= \cancel{ap^2} + apq - \cancel{ap^2}$$

$$= apq$$

iv) $M_{OQ} = p$

$\therefore \dfrac{aq^2 - 0}{2aq - 0} = p$

$\therefore \dfrac{q}{2} = p$ or $q = 2p$

v) For R: $x = a(p+q)$
$y = apq$

Subst. $p = \dfrac{q}{2}$:

$\therefore x = a\left(\dfrac{q}{2} + q\right)$

$= \dfrac{3aq}{2}$ ①

$\therefore y = a\dfrac{q}{2}q$

$= \dfrac{aq^2}{2}$ ②

From ①: $q = \dfrac{2x}{3a}$

Sub in ②: $y = \dfrac{a}{2}\left(\dfrac{2x}{3a}\right)^2$

$= \dfrac{a}{2} \times \dfrac{4x^2}{9a^2}$

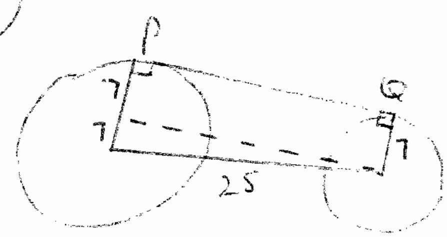
$= \dfrac{2x^2}{9a}$

$\therefore 9ay = 2x^2$

$\therefore x^2 = \dfrac{9}{2}ay$

③

a)



$PQ^2 = 25^2 - 7^2$

b)

$1+x,\ 3+x,\ 4+x$ forms a G.P.

if $\dfrac{3+x}{1+x} = \dfrac{4+x}{3+x}$

$\therefore \cancel{x^2} + 6x + 9 = \cancel{x^2} + 5x + 4$

$\therefore x = -5$. (ie. $-4, -2, -1$)

c) Sum $= (501 + 502 + \dots + 1600)$
$- (504 + 512 + \dots + 1600)$

$= \dfrac{1100}{2}(501 + 1600) - \dfrac{138}{2}(504 + 1600)$

$= 1155550 - 145176$

$= 1010374$

d) $\angle CAB = x^\circ$ (alternate angles, $BA \parallel CD$)
$\angle CAD = 90^\circ$ (angle in semicircle)

$\therefore \angle BAD = 90^\circ + x^\circ$

$\therefore \angle BCD = 180 - (90 + x)^\circ$ (supplementary opposite angles, cyclic quad.)

$= (90 - x)^\circ$

④ a)

i) $\angle P$ is common
$\angle ATP = \angle TBP$ (angle in alternate segment rule)

$\therefore \triangle TAP \parallel\!\parallel \triangle BTP$ (equiangular)

ii) $\dfrac{PT}{PB} = \dfrac{PA}{PT}$ (equal ratio of sides in similar triangles)

$PT^2 = PA \cdot PB$

b) $\frac{dy}{dx} = 2ax + b$

Sub $\frac{dy}{dx} = 0$, $x = 1$:

$\Rightarrow 0 = 2a + b$ — ①

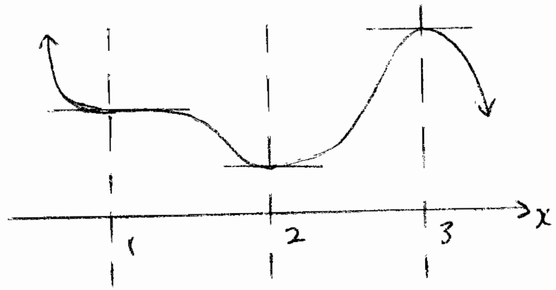
Sub $x = 1, y = -2$:

$\Rightarrow -2 = a + b$ — ②

① − ②: $\boxed{a = 2}$

Sub in ①: $\boxed{b = -4}$

c)



⑤ a) Step 1 Prove true for $n=1$

$8^1 - 5^1 = 3$ (divisible by 3)

Step 2 Assume true for $n=k$

ie assume $8^k - 5^k = 3P$ (P is integral)

Step 3 Prove true for $n=k+1$

ie prove $8^{k+1} - 5^{k+1} = 3Q$ (Q is integral)

Now, $8^{k+1} - 5^{k+1} = 8 \times 8^k - 5 \times 5^k$

$= 8(8^k - 5^k) + 3 \times 5^k$

$= 8 \times 3P + 3 \times 5^k$

Step 4 The result is true for $n=1$. From step 3, it must also be true for $n = 1+1 = 2$, then $n = 2+1 = 3$ and so on for all positive, integral n.

b) i) $A_1 = 40000 \times 1{\cdot}01 - R$

$A_2 = A_1 \times 1{\cdot}01 - R$

$= 40000 \times 1{\cdot}01^2 - 1{\cdot}01R - R$

ii) $A_{60} = 40000 \times 1{\cdot}01^{60} - 1{\cdot}01^{59}R - \ldots - R$

But $A_{60} = 0$,

$\therefore 40000 \times 1{\cdot}01^{60} - (1{\cdot}01^{59}R + \ldots + R) = 0$

$\therefore 40000 \times 1{\cdot}01^{60} = \frac{R(1{\cdot}01^{60} - 1)}{0{\cdot}01}$

$\therefore R = 40000 \times 1{\cdot}01^{60} \frac{\times 0{\cdot}01}{1{\cdot}01^{60} - 1}$

$= \$889{\cdot}78$ per month.

iii)

$1200 = 40000 \times 1{\cdot}01^n \frac{\times 0{\cdot}01}{1{\cdot}01^n - 1}$

$= \frac{400 \times 1{\cdot}01^n}{1{\cdot}01^n - 1}$

$\therefore 1200 \times 1{\cdot}01^n - 1200 = 400 \times 1{\cdot}01^n$

$800 \times 1{\cdot}01^n = 1200$

$1{\cdot}01^n = 1{\cdot}5$

$(n > 40)$

$\therefore$ 41 months needed to repay the loan