SYDNEY TECHNICAL HIGH SCHOOL

MANNERS MAKYTH MAN

MATHEMATICS DEPARTMENT

YEAR 11 EXTENSION 1

H.S.C. ASSESSMENT TASK 1, DECEMBER 2011

Name:______________________________ Teacher:______________________

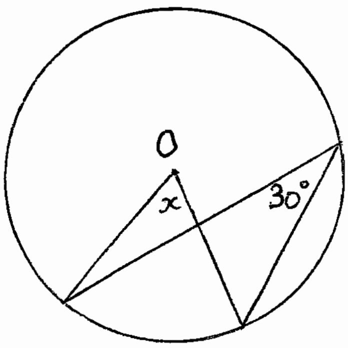
- Time allowed: 70 minutes.
- Start each question on a new page.
- Diagrams are not to scale.
- Show necessary working.
- Full marks may not be awarded for poorly arranged work or illegible writing.

Question 1	Question 2	Question 3	Question 4	Question 5	TOTAL
/10	/11	/9	/10	/10	/50

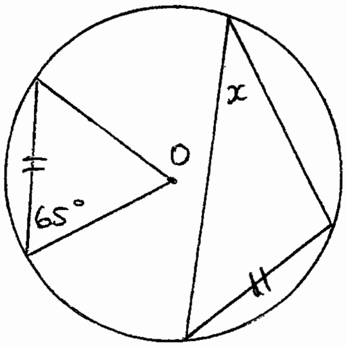
Question 1

a) Find the value of each pronumeral. Reasons are <u>not</u> required. O is the centre of each circle and diagrams are not to scale. 5

i)



ii)



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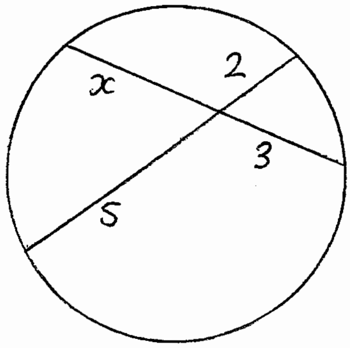

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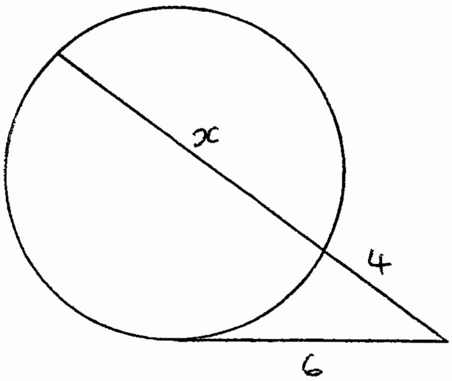

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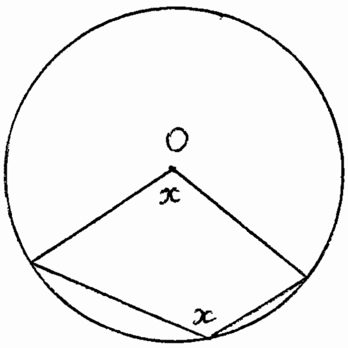


b) A sequence is given by $T_n = \frac{n-1}{n}$.

i) Which term of the sequence is 0.99? 1

ii) Simplify $T_{n+1} : T_n$ 1

c) Evaluate $\sum_{n=20}^{100}(2n-4)$ 2

d) Find the equation of the chord of contact from (-1,-2) to the parabola $x^2 = 4y$ 1

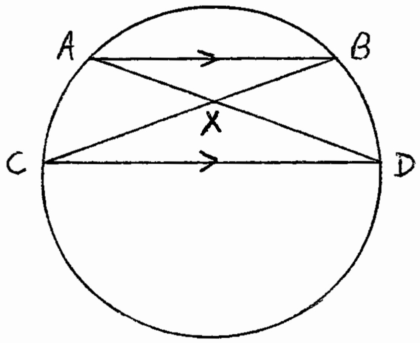
Question 2 (start a new page)

a) For a certain series, the sum to n terms is $S_n = n^2 - 4n$. Find:

i) the seventh term. 1

ii) the nth term in simplest form. 2

b)



AB and CD are parallel chords. AD and CB intersect at X.

Prove that ΔCXD is isosceles. 2

c) The parabola $x = 4t,\ y = 2t^2$ has points P and Q with parameters "p" and "q".

i) Find the equation of the chord PQ. 2

ii) If PQ is a focal chord, show that $pq = -1$ 1

iii) Find the coordinates of M, the midpoint of PQ. 1

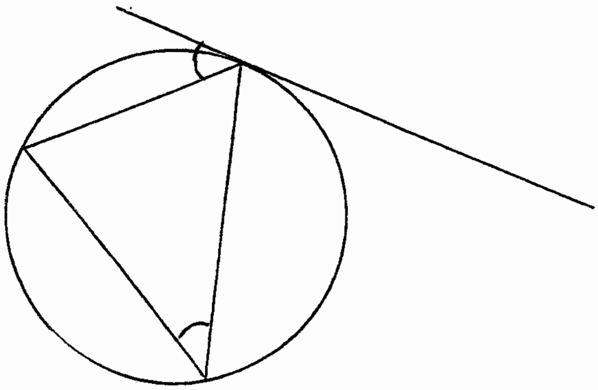
iv) Show that the locus of M is the parabola $x^2 = 4y - 8$. 2

Question 3 (start a new page)

a) Rewrite $3 + 5 + 7 + \ldots\ldots + 99$ using sigma notation, starting with $n = 1$. 1

b) Find the sum to 30 terms of the sequence $T_n = 2 + 2^n - 2n$. 3

c)



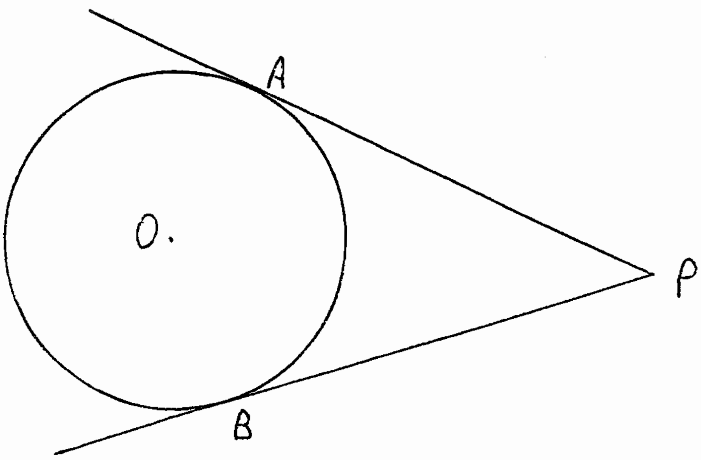
Write the fully worded property that applies to the marked angles above. 1

d) Prove by Mathematical Induction that the sum of the first n terms of a geometric series

$a + ar + ar^2 + \ldots\ldots + ar^{n-1}$, is $S_n = \frac{a(r^n-1)}{r-1}$ for positive integers n ($r \neq 1$). 4

Question 4 (start a new page)

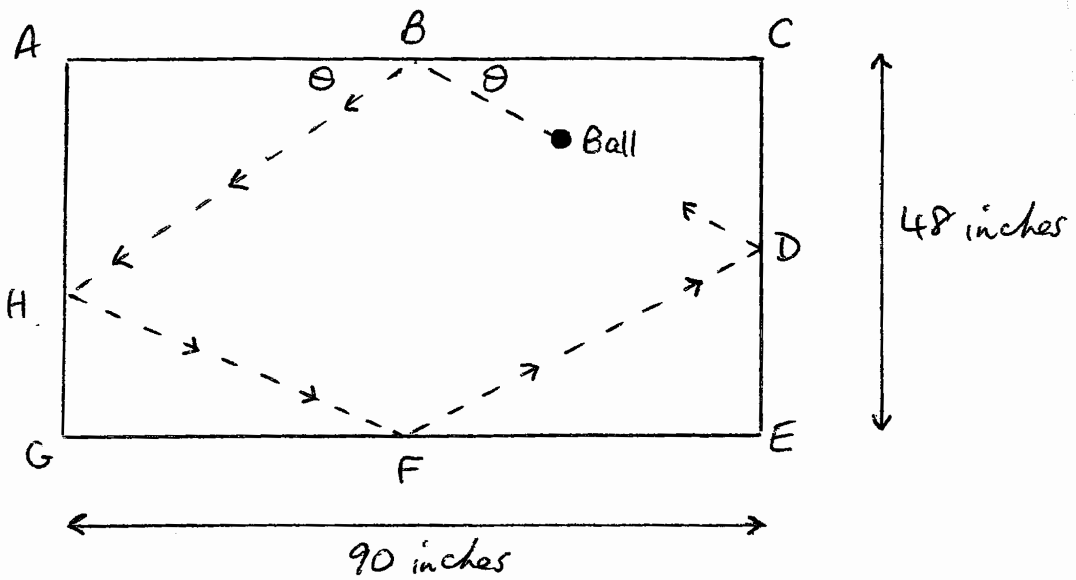
a)



Two tangents are drawn to a circle, centre O, from an external point P to touch the circle at A and B.

Prove that the tangents are equal in length. 2

b)



A rectangular pool table is 90 inches long and 48 inches wide (it's American!).

When hit, the ball shown makes angles of θ on the first rebound at B and continues rebounding perfectly (equal angles) off each side, returning to its starting position.

i) Copy the diagram neatly and mark all angles in terms of θ. 1

ii) Why is $BHFD$ a parallelogram? 1

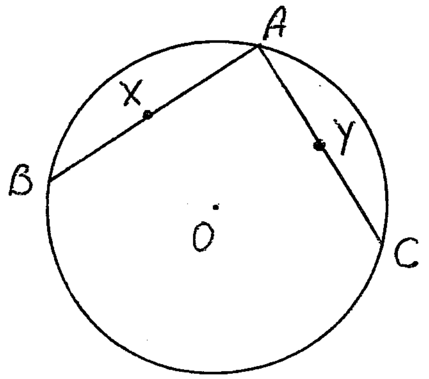
iii) Which congruence test is used to prove that $\Delta BCD \equiv \Delta FGH$? (do not prove congruence) 1

iv) Let $BC = m,\ CD = n.$ Show that $m:n = 15:8$ 3

v) Find the perimeter of parallelogram $BHFD$. 2

Question 5 (start a new page)

a)



AB and AC are chords of a circle, centre O. X and Y are midpoints of AB and AC.

i) Prove that A, X, O, Y form a cyclic quadrilateral. 3

ii) Describe where the centre of the circle $AXOY$ is. 1

b) A man borrows \$5000 from the bank at a reducible interest rate of 12% p.a. He repays \$400 per month.

Let A_n represent the amount still owing on the loan after n months.

i) Write an expression for A_1 and show that $A_2 = 5000 \times 1.01^2 - 400(1.01 + 1)$. 2

ii) Show that $A_n = 5000 \times 1.01^n - 40000(1.01^n - 1)$. 2

iii) Hence show that the number of months, n, could be found using $1.01^n = {}^8/_7$ 2

END OF TEST

SOLUTIONS

(1) a) i) $60°$ ii) $25°$ iii) $3x = 10$ iv) $\frac{x}{2} + x = 180°$ v) $4(x+4) = 36$

$x = 3\frac{1}{3}$ $3x = 360$ $x = 5$

$x = 120$

b) i) $\frac{n-1}{n} = 0.99$

$n - 1 = 0.99n$

$0.01n = 1$

$n = 100$

ii) $\frac{n}{n+1} : \frac{n-1}{n} = \frac{n}{n+1} \times \frac{n}{n-1}$

$= \frac{n^2}{n^2-1}$

c) $S_{81} = \frac{81}{2}(36 + 196)$

$= 9396$

d) $-x = 2(y+2)$

$\therefore x + 2y + 4 = 0$

(2) a) i) $T_7 = S_7 - S_6$

$= (49 - 28) - (36 - 24)$

$= 21 - 12$

$= 9$

ii) $T_n = S_n - S_{n-1}$

$= n^2 - 4n - [(n-1)^2 - 4(n-1)]$

$= \cancel{n^2} - 4n - (\cancel{n^2} - 2n + 1 - 4n + 4)$

$= 2n - 5.$

b) $\angle A = \angle D$ (alternate angles, parallel lines)

$\angle A = \angle C$ (equal angles on same arc BD)

$\therefore \angle C = \angle D$

$\therefore \triangle CXD$ is isosceles (equal base angles)

c) i) $P(4p, 2p^2)$ $Q(4q, 2q^2)$

chord PQ: $\dfrac{y-2p^2}{x-4p} = \dfrac{2q^2-2p^2}{4q-4p}$

$$= \frac{2\cancel{(q-p)}(q+p)}{4\cancel{(q-p)}}$$

$$= \frac{p+q}{2}$$

$\therefore 2y - 4p^2 = (p+q)(x-4p)$

ii) Subst. $(0,2)$

$4 - 4p^2 = (p+q)(-4p)$

$4 - \cancel{4p^2} = -\cancel{4p^2} - 4pq$

$-4pq = 4$

$pq = -1$ as reqd.

iii) $M\left(\dfrac{4p+4q}{2}, \dfrac{2p^2+2q^2}{2}\right)$

$= M(2p+2q, p^2+q^2)$

iv) $x = 2p+2q$

$= 2(p+q)$

$\therefore p+q = \frac{x}{2}$, $y = p^2+q^2$

$= (p+q)^2 - 2pq$

$= \left(\frac{x}{2}\right)^2 + 2$

$= \frac{x^2}{4} + 2$

$\therefore 4y = x^2 + 8$ or $x^2 = 4y - 8$ as reqd.

(3) a) $\sum_{n=1}^{49} (2n+1)$

b) sum G.P. (2^n) + sum A.P. $(-2n+2)$

$= \frac{2(2^{30}-1)}{2-1} + \frac{30}{2}(0-58)$

$= 2147483646 - 870$

$= 2,147,482,776$

c) The angle between a tangent and a chord, at the point of contact, is equal to the angle in the alternate segment standing on the same arc/chord

d) Prove true for $n=1$

$$LHS = a, \quad RHS = \frac{a(r^1-1)}{r-1}$$

$$= a$$

$$= LHS$$

Assume true for $n=k$, ie. assume $S_k = \dfrac{a(r^k-1)}{r-1}$

Prove true for $n=k+1$, ie. prove that $S_{k+1} = \dfrac{a(r^{k+1}-1)}{r-1}$

Use $S_{k+1} = S_k + T_{k+1}$

$$= \frac{a(r^k-1)}{r-1} + ar^k$$

$$= \frac{a(r^k-1) + ar^k(r-1)}{r-1}$$

$$= \frac{\cancel{ar^k} - a + ar^{k+1} - \cancel{ar^k}}{r-1}$$

$$= \frac{a(-1 + r^{k+1})}{r-1}$$

$$= \frac{a(r^{k+1}-1)}{r-1} \quad \text{as reqd.}$$

$\therefore$ if the result is true for $n=k$, then it has been proven true for $n=k+1$

The result is true for $n=1$, and from above it must be true for $n = 1+1 = 2$, then $n = 2+1 = 3$ and so on for all pos. integral n.

Question 4.

a) $OA = OB$ (equal radii)

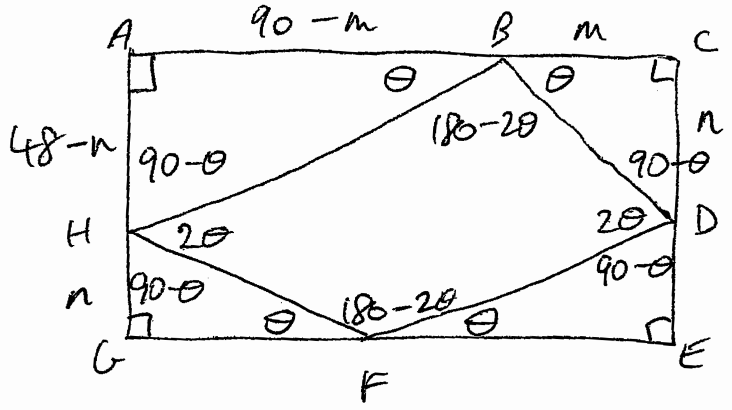
OP is common

$\angle OAP = \angle OBP = 90^\circ$ (radius $\perp$ tangent)

$\therefore \Delta OAP \equiv \Delta OBP$ (RHS)

$\therefore AP = BP$ (corresponding sides in congruent triangles)

b)



c) i) above

ii) opposite angles are equal

iii) AAS

iv) Since congruent, then $HG = n$ (corresponding sides)

$\therefore AH = 48 - n$

Now $\Delta ABH \parallel\!| \Delta BCD$ (equiangular from i)

$\therefore \dfrac{m}{n} = \dfrac{90 - m}{48 - n}$ (equal ratio corresp. sides)

$\therefore 48m - \cancel{mn} = 90 - \cancel{mn}$

$\therefore \dfrac{m}{n} = \dfrac{90}{48}$

$\therefore m : n = 15 : 8$

v) Let $BC = 15$, $CD = 8$

$\therefore BD = 17$ (Pythagoras)

Also, $AB = 75$, $AH = 40$

$\therefore BH = 85$ (Pythag.)

$\therefore$ perimeter $BHFD$

$= 2 \times 17 + 2 \times 85$

$= 204$ inches.

⑤ a) i) $\angle OXA = 90°$ (centre to midpoint of chord $\perp$ chord)

Similarly $\angle OYA = 90°$

$\therefore AXOY$ is a cyclic quadrilateral (opposite angles supplementary)

ii) centre is midpoint of OA.

b) i) $A_1 = 5000 \times 1.01 - 400$

$A_2 = A_1 \times 1.01 - 400$

$= 5000 \times 1.01^2 - 400 \times 1.01 - 400$

$= \underline{5000 \times 1.01^2 - 400(1.01 + 1)}$

ii) $A_n = 5000 \times 1.01^n - 400\underbrace{(1.01^{n-1} + 1.01^{n-2} + \ldots + 1)}$

$$S_n = \frac{1(1.01^n - 1)}{1.01 - 1}$$

$\therefore A_n = 5000 \times 1.01^n - \dfrac{400(1.01^n - 1)}{0.01}$

$= 5000 \times 1.01^n - 40000(1.01^n - 1)$ as reqd.

iii) $A_n = 0$ at end of loan

$\therefore 40000(1.01^n - 1) = 5000 \times 1.01^n$

$40000 \times 1.01^n - 40000 = 5000 \times 1.01^n$

$\therefore 35000 \times 1.01^n = 40000$

$\therefore 1.01^n = \dfrac{40000}{35000}$

$= \dfrac{8}{7}$ as reqd.