SYDNEY TECHNICAL HIGH SCHOOL

Mathematics-Extension 1

2013

H.S.C ASSESSMENT TASK 1

Name ....

Teacher ....

General Instructions

- Working Time - 70 minutes.
- **Write using a blue or black pen.**
- Board Approved calculators may be used.
- All necessary working should be shown for every question.
- Begin each question on a fresh sheet of paper.

Total marks (53)

Section 1

5 marks

- Attempt Questions 1-5.
- Allow about 10 minutes for this section.

Section 1

48 marks

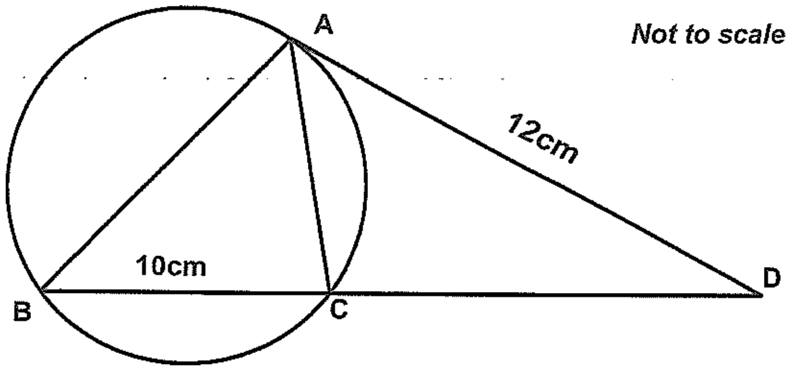
- Attempt Questions 6-11.
- Allow about 60 minutes for this section.

Question 1

A crystal measured 12.0 cm in length at the beginning of a chemistry experiment.
Each day it increased in length by 3%.
The length of the crystal after 14 days growth is closest to

A. 12.4 cm **B.** 16.7 cm **C.** 17.6 cm **D.** 18.2 cm

Question 2



ABC is a triangle inscribed in a circle. The tangent to the circle at A meets BC produced to D where $BC = 10cm$ and $AD = 12cm$. What is the length of CD?

A. 6 cm **B.** 7 cm **C.** 8 cm **D.** 9 cm

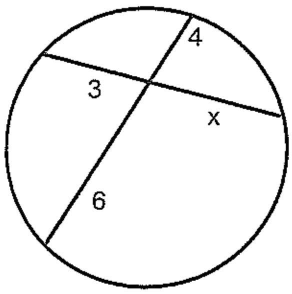
Question 3

The equation $2x^3 + x^2 - 13x + 6 = 0$ has roots $\alpha, \frac{1}{\alpha}$ and β. What is the value of β?

A. 3 **B.** 2 **C.** -3 **D.** -6

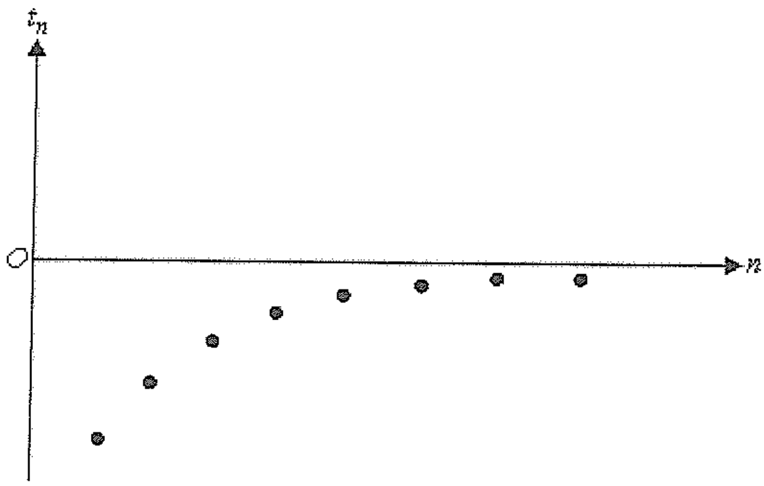
Question 4

Given the circle at the right with two intersecting chords.
Find the length represented as x.



A. 2 **B.** 6 **C.** 8 **D.** 10

Question 5
The graph below shows consecutive terms of a sequence.
The sequence could be



A. Geometric with common ratio r, where $r < 0$

B. Geometric with common ratio r, where $0 < r < 1$

C. Geometric with common ratio r, where $r > 1$

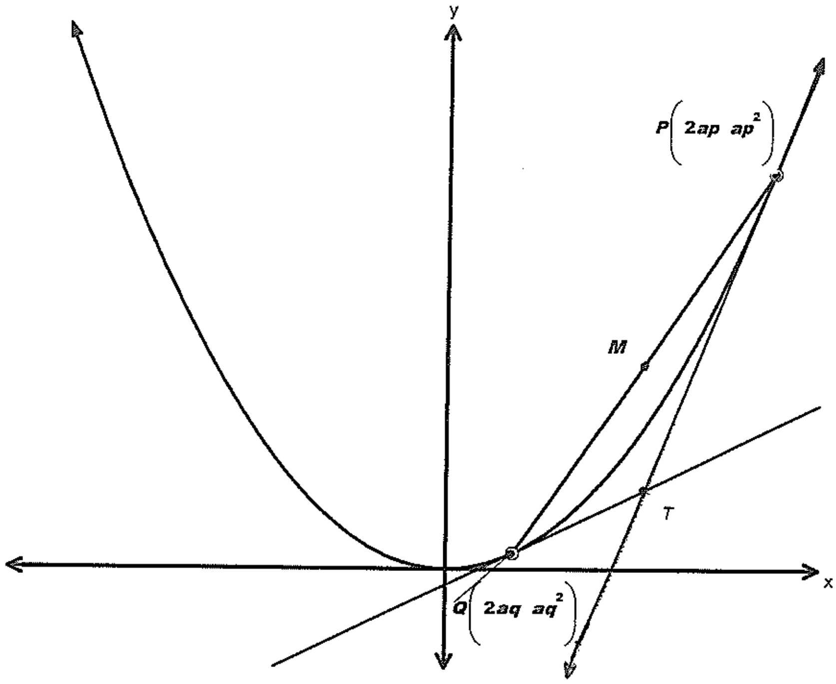
D. Arithmetic with common difference d, where $d < 0$

End of Multiple Choice questions

Question 6 (8 Marks) Use a Separate Sheet of paper

a) Find the equation of the normal to the parabola $x^2 = 8y$ at the point $(4p, 2p^2)$. **3**

b) The diagram below shows the tangents drawn at the point $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ on the parabola $x^2 = 4ay$. The tangents at P and Q intersect at T.



You may assume that the equation of the tangent at P is $y = px - ap^2$ and that the point T has coordinates $T[a(p+q), apq]$.

(i) Suppose that point T lies on the line $y = a$, show that $pq = 1$ **1**

(ii) Find the Cartesian equation of the locus of the midpoint, M of the chord PQ. **3**

(iii) State any restrictions on the x-coordinates of the locus of M. **1**

End of Question 6

Question 7 **(8 Marks)** Use a Separate Sheet of paper

a) i) Use the method of mathematical induction to prove for $n \geq 2$ **3**

$$\left(1-\frac{1}{2^2}\right)\times\left(1-\frac{1}{3^2}\right)\times\left(1-\frac{1}{4^2}\right)\times\ldots\times\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

ii) Hence evaluate $\frac{3}{4}\times\frac{8}{9}\times\frac{15}{16}\times\ldots\times\frac{9999}{10000}$ **1**

b) Harrison has started an exercise program to lose weight. When he started the program he weighed 105kg. In the first month he lost 5 kg, in the second he lost 4 kg and in the third month he lost 3.2 kg. If this weight loss trend continues

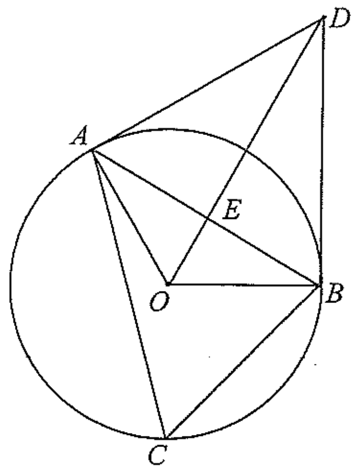
i) how much will Harrison lose in the fourth month? **1**

ii) what will be his ultimate weight? **3**

End of Question 7

Question 8 (8 Marks) Use a Separate Sheet of paper

a) The diagram shows points A, B and C on a circle centre O.
Tangents are drawn from A and B which meet at D. O is joined to D and the interval OD intersects AB at E.



Not to Scale

i) Prove that $\angle AOB = 2 \times \angle DAB$. **2**

ii) Prove that AOBD is a cyclic quadrilateral **1**

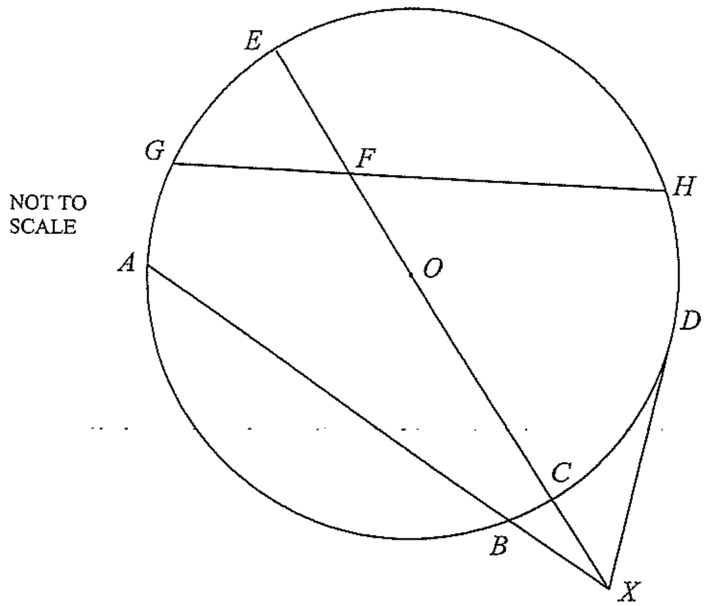
iii) Prove that E is the midpoint of AB. **2**

b) The polynomial $p(x) = x^3 + ax + b$ has $(x-5)$ as one of its factors and has a remainder of -60 when divided by $(x+5)$. Find the values of a and b. **3**

End of Question 8

Question 9 (8 Marks) Use a Separate Sheet of paper

a) The circle with center O has radius 6 cm. From an external point X, a tangent is drawn with a point of contact D. From X the secants XA and XE are also drawn.



i) If $DX = 8$ cm calculate the distance CX. **2**

ii) If F is the bisector of EO and $GF = 4.5$ cm, calculate the distance GH. **2**

b) In a geometric Series, the 3rd term is -8 and the 6th term is 216.
Find the 1st term and the common ratio. **2**

c) The chord of contact of the tangents to the parabola $x^2 = 4ay$ from an external point $A(x_1, y_1)$ passes through the point $B(0,2a)$.
Find the equation of the locus of the midpoint of AB. **2**

End of Question 9

Question 10 (8 Marks) Use a Separate Sheet of paper

a) Find the cartesian equation of the curve represented by the following parametric equations; **2**

$$x = 3t - 4$$
$$y = 2t^2 - t$$

b) Stephanie borrows \$50 000 at the beginning of 2013 from his local Building Society. The loan is to be repaid in equal monthly repayments of \$900, with interest charged at 7.2% p.a. at the end of each month, just before repayment.

Let A_n be the amount owing after the nth repayment.

i) Find an expression for A_1 and A_2 **2**

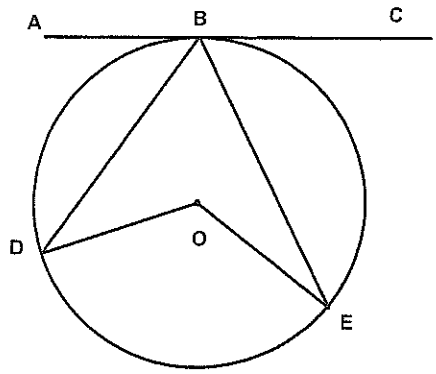
ii) **Show** that
$A_n = 50000(1.006)^n - 900(1 + 1.006 + 1.006^2 + \ldots + 1.006^{n-1})$. **1**

iii) After how many months will Stephanie have halved her loan? **3**

End of Question 10

Question 11 (8 Marks) Use a Separate Sheet of paper

a) In the diagram below, O is the centre of the circle, AC is a tangent at B and D and E are points on the circumference. If $\angle ABD = 80^\circ$ and $\angle DBE = 40^\circ$, find the size of $\angle BEO$, giving reasons. **3**



b) John wants to save for a holiday in 10 years' time. The interest rate is fixed at 9.08% p.a. compounding yearly for the first 5 years and will change to 10% p.a. compounding yearly for the next 5 years.
How much should John invest now so that he has $5000 in 10 years from now? Give your answer to the nearest dollar. **3**

c) The height of a tree was 10 metres and it increased by 2m during the next year. If in each succeeding year the growth is $\frac{2}{3}$ of that in the previous year, find the limiting height? **2**

End of Examination

Extension 1 Mathematics
2013
H.S.C Assessment Task 1

Multiple Choice

1 D
2 C
3 C
4 C
5 B

Question 6

a) $x^2 = 8y$

$y = \frac{x^2}{8}$

$y' = \frac{x}{4}$

When $x = 4p$ $m_1 = p$ $m_2 = -\frac{1}{p}$

Equation of the normal

$y - 2p^2 = -\frac{1}{p}(x - 4p)$

$yp - 2p^3 = -x + 4p$

$x + yp - 2p^3 - 4p = 0$

b) i) $apq = a$

$\therefore pq = 1$

ii) let $x = a(p+q)$

$y = \frac{a}{2}(p^2 + q^2)$

$x = a(p+q)$

$\frac{x}{a} = p + q$

$y = \frac{a}{2}(p^2 + q^2)$

$2y = a(p^2 + q^2)$

$\frac{2y}{a} = p^2 + q^2$

$\frac{2y}{a} = (p+q)^2 - 2pq$

$\frac{2y}{a} = \left(\frac{x}{a}\right)^2 - 2 \times 1$

$\frac{2y}{a} = \frac{x^2}{a^2} - 2a^2$

$2ay = x^2 - 2a^2$

$2ay + 2a^2 = x^2$

$2a(y + a) = x^2$

or

$y + a = \frac{x^2}{2a}$

$y = \frac{x^2}{2a} - a$

iii) M and T share the same x-co-ordinate, the extreme case will be when P and Q are NOT distinct points ie $p = q$

When this happens T (and M) will be $(2a, a)$. Since T must lie outside the parabola then $|x| > 2a$

or

$\frac{1}{2a}x^2 - a > a$

$\frac{1}{2a}x^2 > 2a$

$x^2 > 4a$

$x < -2a$. $x > 2a$

Question 7

a i) $\left(1 - \frac{1}{2^2}\right) \times \left(1 - \frac{1}{3^2}\right) \times \left(1 - \frac{1}{4^2}\right) \times \ldots \times \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$

Step 1 when $n = 2$

L.H.S $= 1 - \frac{1}{2^2} = \frac{3}{4}$

R.H.S $= \frac{2+1}{2 \times 2} = \frac{3}{4}$

$\therefore$ L.H.S $=$ R.H.S

Step 2

Assume true for $n = k$

$\left(1 - \frac{1}{2^2}\right) \times \left(1 - \frac{1}{3^2}\right) \times \left(1 - \frac{1}{4^2}\right) \times \ldots \times \left(1 - \frac{1}{k^2}\right) = \frac{k+1}{2k}$

Step 3.

Prove the formula is true for $n = k+1$ ie

$\left(1 - \frac{1}{2^2}\right) \times \left(1 - \frac{1}{3^2}\right) \times \left(1 - \frac{1}{4^2}\right) \times \ldots \times \left(1 - \frac{1}{k^2}\right) \times \left(1 - \frac{1}{(k+1)^2}\right) = \frac{k+2}{2(k+1)}$

L.H.S $= \frac{k+1}{2k} + \left(1 - \frac{1}{(k+1)^2}\right)$

$= \frac{k+1}{2k} \times \frac{(k+1)^2 - 1}{(k+1)^2}$

$= \frac{(k+1)^2 - 1}{2k(k+1)}$

$= \frac{k^2 + 2k}{2k(k+1)} = \frac{k+2}{2(k+1)}$

as required

We know that the formula is true for $n = 2$, so it must be true for $n = 3$. If it is true for $n = 3$, then it is true for $n = 4$ and so on

$\therefore$ it is true for all integers $n \geq$

a ii) When $n = 100$

$\frac{3}{4} \times \frac{8}{9} \times \frac{15}{16} \times \ldots \times \frac{9999}{10000}$

$\frac{9999}{10\,000} = 1 - \frac{1}{n^2}$

$n = 100$

$\therefore \frac{100 + 1}{2 \times 100} = \frac{101}{200}$

b i) $5, 4, 3.2$

$r = \frac{3.2}{4} = \frac{4}{5}$

$3.2 \times \frac{4}{5} = 2.56$ kg

ii) $a = 5$, $r = \frac{4}{5}$

$S_\infty = \frac{a}{1-r}$

$= \frac{5}{1-\frac{4}{5}}$

$= 25$ kg

Ultimate weight $= 105 - 25$

$= 80$ kg

Question 8

ai) $\angle DAB = \angle ACB$

(angle between tangent and chord is equal to the angle in the alternate segment)

$\angle AOB = 2 \times \angle ACB$

(angle at the centre is twice the angle at the circumference on the same arc)

$\angle AOB = 2 \times \angle DAB$ ($\angle DAB = \angle ACB$)

aii) $\angle DAO = \angle DBO = 90°$

(tangent is perpendicular to the radius)

$\angle DAO + \angle DBO = 180°$

(sum of two right angles)

$\therefore$ opposite angles of $AOBD$ are supplementary

$\therefore$ $AOBD$ is a cyclic quadrilateral

aiii) $AO = BO$ (equal radii)

$AD = BD$ (tangents from an external point are equal in length)

$\therefore$ $AOBD$ is a kite

OD bisects AB (symmetry of a kite)

$\therefore$ E is the midpoint of AB.

b) $p(x) = x^3 + ax + b$

$p(5) = 0$

$5^3 + 5a + b = 0$

$125 + 5a + b = 0$

$p(-5) = -60$

$(-5)^3 - 5a + b = -60$

$-125 - 5a + b = -60$

$125 + 5a + b = 0$

$-125 - 5a + b = -60$

$2b = -60$

$b = -30$

$5a + b + 125 = 0$

$5a - 30 + 125 = 0$

$5a = -95$

$b = -30$.

$a = -19$

Question 9

a i) $DX^2 = CX \cdot XE$

$8^2 = CX(CX + EC)$

$8^2 = CX(CX + 12)$

$64 = CX^2 + 12CX$

$0 = (CX)^2 + 12CX - 64$

$0 = (CX + 16)(CX - 4)$

$CX = -16$ $\quad CX = 4$ (only solution)

aii) $EF = FO = 3$ cm

$GF = 4.5$ cm

$GF \times FH = EF \times FC$

$4.5 \times FH = 3 \times 9$

$4.5 \times FH = 27$

$FH = 6$.

Distance of $GH = 6 + 4.5$

$= 10.5$ cm

b i) $T_3 = -8$

$T_6 = 216$

$ar^2 = -8$

$ar^5 = 216$.

$r^3 = -27$

$r = -3$

$a = -\frac{8}{9}$

c) Equation of the chord of contact $xx_1 = 2a(y + y_1)$

$B(0, 2a)$

$0 \times x = 2a(y + 2a)$

$0 = 2ay + 4a^2$

$-4a^2 = 2ay$

$y = -2a$

Locus of the midpoint AB

$A(x, -2a)$ $\quad B(0, 2a)$

$y = \frac{-2a + 2a}{2}$ $\quad x = \frac{0 + x_1}{2} = \frac{x}{2}$

$\therefore$ $y = 0$ is the equation of the locus of the midpoint AB.

Question 10

$x = 3t - 4$

$x + 4 = 3t$

$t = \frac{x+4}{3}$

$y = 2t^2 - t$

$y = 2\left(\frac{x+4}{3}\right)^2 - \left(\frac{x-4}{3}\right)$

$y = \frac{2(x^2 + 8x + 16)}{9} - \frac{x+4}{3}$

$y = \frac{2x^2 + 16x + 32 - 3x - 12}{9}$

$y = \frac{2x^2 - 13x + 20}{9}$

b)

$A_1 = 50\,000(1+0.006) - 900$
$= 50\,000(1.006) - 900.$

$A_2 = A_1(1.006) - 900$
$= [50\,000(1.006 - 900)]\,1.006 - 900$
$= 50\,000(1.006)^2 - 900(1.006) - 900$
$= 50\,000(1.006)^2 - 900(1 + 1.006)$

$A_3 = 50\,000(1.006)^3 - 900(1 + 1.006 + 1.006^2)$

$\therefore A_n = 50\,000(1.006)^n - 900(1 + 1.006 + 1.006^2 + \ldots + 1.006^{n-1})$

b ii) Let $n = 25\,000$

$25\,000 = 50000(1.006)^n - 900(1 + 1.006 + 1.006^2 + \ldots + 1.006^{n-1})$

$25\,000 = 50\,000(1.006)^n - 900\left(\frac{1.006^n - 1}{0.006}\right)$

$25\,000 = 50\,000(1.006)^n - 150\,000(1.006^n - 1)$

$1 = 2(1.006)^n - 6(1.006^n - 1)$
$1 = 2(1.006)^n - 6(1.006)^n + 6$
$-5 = -4(1.006)^n$
$\frac{5}{4} = (1.006)^n$

$\ln\left(\frac{5}{4}\right) = \ln(1.006)^n$

$n = \frac{\ln\left(\frac{5}{4}\right)}{\ln(1.006)}$

$n = 37.3$ months

* allow for guess and check method.

Question 11

a) $\angle BED = \angle ABD = 80°$
Construct Line DE
(angle between tangent and chord is equal to the angle in alternate segment)

$\angle DOE \times 2 \times \angle DBE = 80°$
(angle at the centre is twice the angle at the circumference of the same arc)

$DO = OE$ (Equal radii)

$\angle OED = \angle ODE = 50°$
(angles opposite equal sides are equal)

$\angle BEO = \angle BED - \angle OED$
$= 80° - 50°$
$= 30°$

11 b) $5000 = x(1.0908)^5 \times (1.1)^5$

$x = \frac{5000}{(1.0908)^5\,(1.1)^5}$

$x = \$2010.39$
$x = \$2010$ nearest dollar

11 c) $10 + 2 + \frac{4}{3} + \frac{8}{3} + \ldots$

$= 10 + S_\infty$

$= 10 + \frac{a}{1-r}$

$= 10 + \frac{2}{1 - 2/3}$

$= 10 + 6$

$= 16$