STUDENT'S NAME: ______________________________

BAULKHAM HILLS HIGH SCHOOL

DECEMBER ASSESSMENT

2007

MATHEMATICS

EXTENSION 2

GENERAL INSTRUCTIONS:

- Time allowed - 70 minutes.
- Write using blue or black pen.
- Start each question on a new page.
- Write your name at the top of each page.
- Calculators may be used
- **ALL** necessary working should be shown in every Question.

QUESTION 1 (16 marks) Start on a new page

Marks

(a) If $z_1 = 5 - 2i$ and $z_2 = i - 4$ express the following in the form a + ib where a and b are real numbers.

i) $z_1 + z_2$ 1

ii) $z_1 z_2$ 2

iii) $\frac{z_1}{z_2}$ 2

b) Sketch the region in the complex plane where the inequalities:

$|z - 1 - 2i| < 5$ and $0 < \arg(z - 1 - 2i) < \frac{\pi}{3}$ both hold 3

(c) If z = a + ib where a and b are real numbers, find

i) $\text{Im}(4iz - 3)$ 2

ii) $\overline{3iz}$ in the form x + iy where x and y are real numbers 3

(d) Find a and b such that

$(1 + i\sqrt{3})^8 = a + ib$ 3

QUESTION 2 (17 marks) **Start on a new page**

Marks

(a) Find $\sqrt{6i-8}$ and hence solve the quadratic equation: 4

$2z^2-(3+i)z+2=0$ expressing your answer in the form

x + iy

(b) If ω is a complex root of the equation:

$z^5-1=0$

i) Show that ω^2, ω^3 *and* ω^4 are the other complex roots. 2

ii) Prove that $1+\omega+\omega^2+\omega^3+\omega^4=0$ 1

iii) Form the quadratic equation with roots 3

$\left(\omega^2+\omega^3\right)$ *and* $\left(\omega+\omega^4\right)$

(c) A represents the complex number 2 + 3i in the complex plane

Find a possible complex number represented by B such that triangle OAB is a right angled isosceles triangle with right angle at

i) O 1

ii) A 1

iii) B 2

(d) Given $z=\frac{1}{2}(\cos\theta+i\sin\theta)$ where θ is real, show that the imaginary part of $\frac{1}{1-z}$ is 3

$$\frac{2\sin\theta}{5-4\cos\theta}$$

QUESTION 3 (15 marks) Start on a new page

Marks

(a) By letting $z = \cos\theta + i\sin\theta$ and using De Moivre's Theorem, find an expression for both $\sin 3\theta$ and $\cos 3\theta$ and hence show that 4

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

(b) i) If $z = \cos\theta + i\sin\theta$ show that: 3

$z + z^{-1} = 2\cos\theta$ and

$z^n + z^{-n} = 2\cos n\theta$ and find corresponding expressions for

$z - z^{-1}$ and $z^n - z^{-n}$

ii) By expanding $\left(z + \frac{1}{z}\right)^4$ find an expression for $\cos^4\theta$ in the form 2

$A\cos 4\theta + B\cos 2\theta + C$

c) Find the equation of the locus of z if 3

$$Arg\left(\frac{z+1}{z-3}\right) = \frac{\pi}{2}$$ stating any restrictions.

d) By considering the triangle inequality 3

$|z_1 - z_2| \geq |z_1| - |z_2|$ find the greatest value of $|z|$ which is satisfied by the equation:

$$\left|z - \frac{7}{z}\right| = 4$$

.

END OF EXAMINATION

1 a) i) $1-i$

ii) $(5-2i)(i-4)$

$= 5i-20-2i^2+8i$

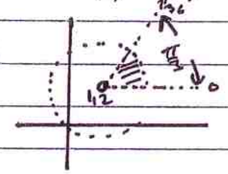
$= 13i-18$

iii) $\dfrac{5-2i}{-(4+i)} \times \dfrac{4-i}{4-i}$

$= \dfrac{-(20-5i-8i+2i^2)}{17}$

$= \dfrac{-18}{17} + \dfrac{13i}{17}$

b)



c) i) $4iz-3$

$= 4i(a+ib)-3$

$= 4ia-4b-3$

$\therefore Im(4iz-3) = 4a$

ii) $3iz = 3i(a+ib)$

$= 3ai-3b$

$\overline{3iz} = -3b-3ai$

d) $(1+i\sqrt{3}) = 2\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)$

$= 2\,\text{cis}\,\frac{\pi}{3}$

$(1+i\sqrt{3})^8 = 2^8\,\text{cis}\,\frac{8\pi}{3}$

$= 2^8\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)$

$= -128+128\sqrt{3}i$

2 a) let $z = x+iy = \sqrt{6i-8}$

$x^2-y^2+2xyi = 6i-8$

$x^2-y^2 = -8$

$xy = 3$

let $y = \frac{3}{x}$

$\therefore x^2 - \frac{9}{x^2} = 8$

$x^4+8x^2-9=0$

$(x^2+9)(x^2-1)=0$

$\therefore x = \mp 1 \quad y = \mp 3$

$\therefore \sqrt{6i-8} = \mp(1+3i)$

$2z^2-(3+i)z+2=0$

$z = \dfrac{3+i \mp \sqrt{9+6i-1-16}}{4}$

$= \dfrac{3+i \mp \sqrt{6i-8}}{4}$

$= \dfrac{3+i \mp (1+3i)}{4}$

$= \dfrac{4+4i}{4}$ or $\dfrac{2-2i}{4}$

$= (1+i)$ or $\frac{1}{2}(1-i)$

b) $z^5-1=0$

if ω is a soln

$\omega^5-1=0$

$\omega^5=1$

now $(\omega^2)^5 = (\omega^5)^2 = 1$

$(\omega^3)^5 = (\omega^5)^3 = 1$

$(\omega^4)^5 = (\omega^5)^4 = 1$

$\therefore \omega^2, \omega^3, \omega^4$ are also soln.

ii) $z^5-1=0$

~~$(z-1)(\omega^4+\omega^3+\omega^2+\omega+1)=0$~~

$(z-1)(1+z+z^2+z^3+z^4)=0$

if ω is a soln $\omega \neq 1$ as complex

$\therefore 1+\omega+\omega^2+\omega^3+\omega^4=0$

Can also do it by $\Sigma\alpha$

$\Sigma\alpha = 1 \quad \therefore 1+\omega+\omega^2+\omega^3+\omega^4=0$

iii) $\alpha+\beta = \omega^2+\omega^3+\omega+\omega^4 = -1$

$\alpha\beta = (\omega^2+\omega^3)(\omega+\omega^4)$

$= \omega^3+\omega^6+\omega^4+\omega^7$

$= \omega^3+\omega+\omega^4+\omega^2 = -1$

$\therefore$ quad is $z^2+z-1=0$.

c) i) $i(2+3i)$ or $-i(2+3i)$

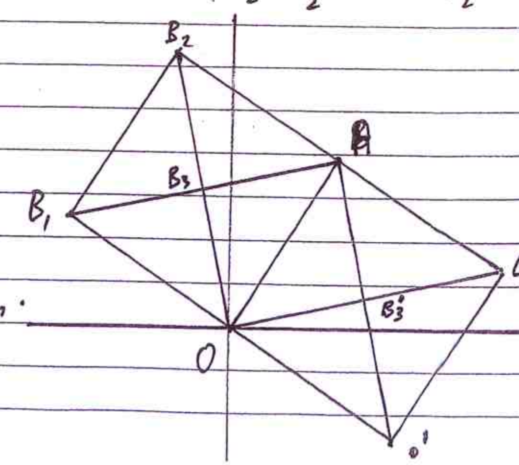
$= -3+2i \qquad 3-2i$

ii) $(-3+2i)+(2+3i)$ or $(3-2i)+(2+3i)$

$-1+5i \qquad 5+i$

iii) $\frac{1}{2}(-1+5i)$ or $\frac{1}{2}(5+i)$

$= -\frac{1}{2}+\frac{5i}{2} \qquad \frac{5}{2}+\frac{i}{2}$



d) $z = \frac{1}{2}(\cos\theta + i\sin\theta)$

$$\frac{1}{1-z} = \frac{1}{1-\frac{1}{2}(\cos\theta + i\sin\theta)}$$

$$= \frac{2}{2-\cos\theta - i\sin\theta}$$

$$= \frac{2}{(2-\cos\theta) - i\sin\theta} \times \frac{(2-\cos\theta)+i\sin\theta}{(2-\cos\theta)+i\sin\theta}$$

$$= \frac{(4-2\cos\theta)+2i\sin\theta}{4-4\cos\theta+\cos^2\theta+\sin^2\theta}$$

$$= \frac{(4-2\cos\theta)+2i\sin\theta}{5-4\cos\theta}$$

$$\therefore \operatorname{Im}\left(\frac{1}{1-z}\right) = \frac{2\sin\theta}{5-4\cos\theta}$$

③ a) $z = \cos\theta + i\sin\theta$

$(\cos\theta + i\sin\theta)^3 = \cos3\theta + i\sin3\theta$

$\therefore \cos3\theta + i\sin3\theta = \cos^3\theta + 3\cos^2\theta\sin\theta\, i + 3\cos\theta\sin^2\theta\, i^2 + \sin^3\theta\, i^3$

$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + (3\cos^2\theta\sin\theta - \sin^3\theta)i$ 1

$\therefore \sin3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$

$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$ 1

$\cos3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$. 1

$$\tan3\theta = \frac{\sin3\theta}{\cos3\theta}$$

$$= \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$$

÷ top and bottom by $\cos^3\theta$

$$\therefore \tan3\theta = \frac{\frac{3\cos^2\theta\sin\theta}{\cos^3\theta} - \frac{\sin^3\theta}{\cos^3\theta}}{\frac{\cos^3\theta}{\cos^3\theta} - \frac{3\cos\theta\sin^2\theta}{\cos^3\theta}}$$ 1

$$= \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}.$$

b) i) $z = \cos\theta + i\sin\theta$.

$z^{-1} = \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta - i\sin\theta$. 1

$\therefore z + z^{-1} = 2\cos\theta$ $\quad z - z^{-1} = 2i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos n\theta - i\sin n\theta$ 1

$\therefore z^n + z^{-n} = 2\cos n\theta$ $\quad z^n - z^{-n} = 2i\sin n\theta$. 1

ii) $\left(z + \frac{1}{z}\right)^4 = z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$

$= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$ 1

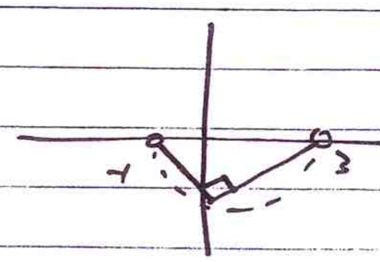
(3) cont

$(2\cos\theta)^4 = 2\cos 4\theta + 4(2\cos 2\theta) + 6$

$16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ 1

c) $\arg(z+1) - \arg(z-3) = \frac{\pi}{2}$



semi circle $y<0$ with diameter between $(-1,0)$ $(3,0)$ 1

centre is $(1,0)$ radius $= 2$

$\therefore$ ~~$(x-1)^2+(y-1)^2=4$~~ 1 1

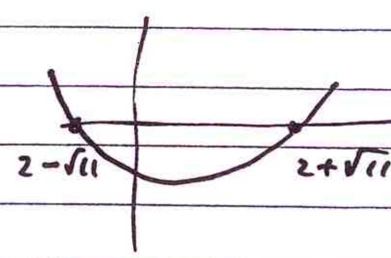
$(x-1)^2 + y^2 = 4$ $y<0$

d) $\left|z - \frac{7}{z}\right| \geq |z| - \frac{7}{|z|}$

$\therefore |z| - \frac{7}{|z|} \leq 4$ 1

$|z|^2 - 7 \leq 4|z|$

$|z|^2 - 4|z| - 7 \leq 0$ 1



if $|z|^2 - 4|z| - 7 = 0$

$|z| = \frac{4 \mp \sqrt{16+28}}{2}$

$= 2 \mp \sqrt{11}$

$\therefore 2-\sqrt{11} \leq |z| \leq 2+\sqrt{11}$

$\therefore \max |z| = 2+\sqrt{11}$ 1