**QUESTION 1 ( 15 marks) Start on a new page**

Marks

(a) Consider the complex numbers $z_1 = \sqrt{2}\left(1 + i\sqrt{3}\right)$ *and* $z_2 = 2\sqrt{6}(1 + i)$

i) Express $z = \dfrac{z_1}{z_2}$ exactly in the form $x + iy$ where $x$ *and* $y$ are real numbers. 2

ii) Write $z_1, z_2$ *and* $z$ in modulus/argument form. 2

iii) Hence find the exact value of $\cos\dfrac{\pi}{12}$. 1

iv) On an Argand diagram draw the vectors $\vec{OA},\ \vec{OB}$ *and* $\vec{OC}$ to represent $z_1,\ z_2$ *and* $z_1 - z_2$ respectively 2

b) If $z = x + iy$ show that 2

$$z + \frac{|z|^2}{z} = 2\operatorname{Re}(z)$$

c) By applying De Moivre's theorem and also by expanding 4

$(\cos\theta + i\sin\theta)^4$, express $\cos 4\theta$ in terms of $\cos\theta$

(d) If z is a complex number such that 2

$z = k(\cos\theta + i\sin\theta)$, where k is real, show that

$$\arg(z + k) = \frac{\theta}{2}$$

**QUESTION 2 ( 15 marks)** **Start on a new page**

Marks

(a) i) By solving the equation $z^3 = 1$ find the three cube roots of one. 2

ii) Let ω be a cube root of one where ω is not real. Show that 1

$$1 + \omega + \omega^2 = 0$$

iii) Find the quadratic equation with integer coefficients that has roots $(4+\omega)$ and $(4+\omega^2)$ 3

iv) Given 2

$$x = a + b$$

$$y = a\omega + b\omega^2$$

and $z = a\omega^2 + b\omega$ prove that

$$x^2 + y^2 + z^2 = 6ab$$

(b) i) Show that $\dfrac{\left(1+i\sqrt{3}\right)^6}{\left(\sqrt{3}-i\right)^k} = 2^{6-k}\left(\cos\dfrac{k\pi}{6} + i\sin\dfrac{k\pi}{6}\right)$ 2

ii) For what value of k is $\dfrac{\left(1+i\sqrt{3}\right)^6}{\left(\sqrt{3}-i\right)^k}$ purely imaginary. 1

c) Let OABC be a square in the Argand diagram where O is the origin. The points A and C represent the complex numbers $z$ and $iz$ respectively.

i) Find the complex number represented by B. 1

ii) The square OABC is now rotated through $45^\circ$ in an anticlockwise direction about O to $OA'B'C'$. Find the complex numbers $A', B'$ and $C'$. 3

**QUESTION 3 ( 15 marks)** **Start on a new page**

Marks

(a) Find the square root of $5-12i$ and hence solve the equation 4

$$z^2+4z-1+12i=0$$

(b) Find the locus of z if 3

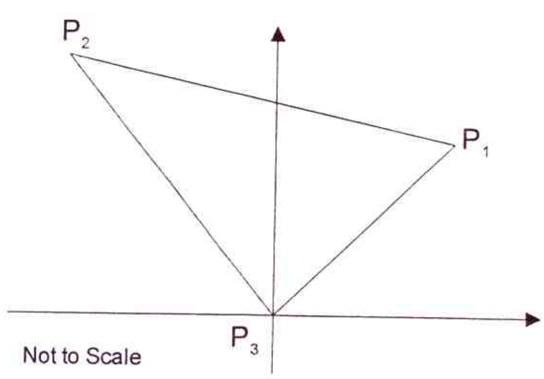
$$\frac{3z+i}{z-2} \text{ is purely imaginary.}$$

(c) Indicate on an Argand diagram the regions representing.

i) $\operatorname{Re}(z+iz)\geq 2$ 2

ii) $1\leq |z-1-i|\leq 3$ where $z=x+iy$ 2

(d)




The points $P_1, P_2$ and $P_3$ represent the complex numbers $z_1, z_2$ and $z_3$ respectively. (NOTE: $z_3=0$ )

i) If $P_1, P_2$ and $P_3$ are the vertices of an equilateral triangle, show that $\frac{z_2}{z_1}=\frac{1+i\sqrt{3}}{2}$ and deduce that $z_1^{\ 2}+z_2^{\ 2}=z_1z_2$ 2

ii) Deduce that if $z_1, z_2$ *and* $z_3$ are **ANY** three complex numbers at the vertices of an equilateral triangle then 2

$$z_1^{\ 2}+z_2^{\ 2}+z_3^{\ 2}=z_1z_2+z_2z_3+z_1z_3$$

**END OF EXAMINATION**

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a) i) $z = \dfrac{\sqrt{2}(1+i\sqrt{3})}{2\sqrt{6}(1+i)} \times \dfrac{(1-i)}{(1-i)}$

$= \dfrac{1}{2\sqrt{3}} \; \dfrac{(1+\sqrt{3})+(\sqrt{3}-1)i}{2}$

$= \dfrac{1+\sqrt{3}}{4\sqrt{3}} + \dfrac{(\sqrt{3}-1)i}{4\sqrt{3}}$

ii) $z_1 = 2\sqrt{2}\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$

$= 2\sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$z_2 = 2\sqrt{6}\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$= 4\sqrt{3}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$z = \dfrac{2\sqrt{2}}{4\sqrt{3}} \operatorname{cis}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

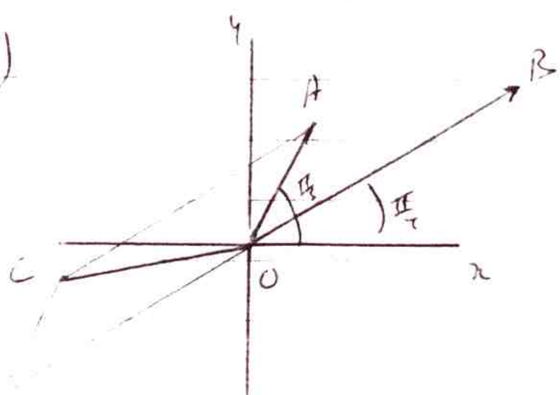
$= \dfrac{\sqrt{2}}{2\sqrt{3}}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

iii) $\dfrac{\sqrt{2}}{2\sqrt{3}} \cos\dfrac{\pi}{12} = \dfrac{1+\sqrt{3}}{4\sqrt{3}}$

$\cos\dfrac{\pi}{12} = \dfrac{1+\sqrt{3}}{4\sqrt{3}} \times \dfrac{2\sqrt{3}}{\sqrt{2}}$

$= \dfrac{1+\sqrt{3}}{2\sqrt{2}}.$

b) iv)




b) $z = x+iy$

$\therefore z + \dfrac{|z|^2}{z} = x+iy + \dfrac{x^2+y^2}{x+iy} \times \dfrac{x-iy}{x-iy}$

$= x+iy + (x^2+y^2) \cdot \dfrac{x-iy}{x^2+y^2}$

$= x+iy + x - iy$

$= 2x$

$= 2\,Re(z)$

c) $\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$

$\cos 4\theta = Re(\cos\theta + i\sin\theta)^4$

$= \cos^4\theta + 6i^2\cos^2\theta\sin^2\theta + i^4\sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

$= 8\cos^4\theta - 8\cos^2\theta + 1$

d) $z = k(\cos\theta + i\sin\theta)$

$z+k = k(\cos\theta + i\sin\theta) + k$

$= k(1+\cos\theta + i\sin\theta)$

$= k\left(1 + 2\cos^2\frac{\theta}{2} - 1 + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$

$= 2k\left(\cos^2\frac{\theta}{2} + i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$

$z+k = 2k\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$

$\arg(z+k) = \frac{\theta}{2}.$

3) a) i) $z^3=1$

let $z = r\,\mathrm{cis}\,\theta$

$z^3 = r^3 \mathrm{cis}\,3\theta = 1$

$|z^3| = 1 \therefore r = 1$

$\arg z^3 = 0$

$\therefore 3\theta = 0, 2\pi, 4\pi$

$\theta = 0, \frac{2\pi}{3}, -\frac{2\pi}{3}$

$z = 1, \mathrm{cis}\frac{2\pi}{3}, \mathrm{cis}\frac{-2\pi}{3}$

$= 1, \frac{-1+\sqrt{3}}{2}, \frac{-1-\sqrt{3}}{2}$.

ii) $z^3 = 1$

$z^3 - 1 = 0$

$(z-1)(z^2+z+1) = 0$

if $\omega$ is not real $\omega \neq 1$ a(w-1)≠0

$\therefore \omega^2+\omega+1 = 0$

OR $z^3 - 1 = 0$

$\Sigma\alpha = -\frac{b}{a} = 0 \therefore 1+\omega+\omega^2 = 0$

after shown $\omega^2$ is root.

iii) $\alpha+\beta = 4+\omega+4+\omega^2$

$= 8+\omega+\omega^2$

$= 7+(1+\omega+\omega^2)$

$= 7$

$\alpha\beta = (4+\omega)(4+\omega^2)$

$= 16+4\omega^2+4\omega+\omega^3$

$= 13+(4+4\omega+4\omega^2)$

$= 13$

$\therefore$ QUAD EQN IS $x^2 - 7x + 13 = 0$.

iv) $x^2+y^2+z^2 = (a+b)^2 + (a\omega+b\omega^2)^2 + (a\omega^2+b\omega)^2$

$= a^2+2ab+b^2+a^2\omega^2+2ab\omega^3+b^2\omega^4 + a^2\omega^4+ab\omega^3+b^2\omega^2$

$= a^2+2ab+b^2+a^2\omega^2+2ab + b^2\omega + a^2\omega + 2ab + b^2\omega^2$

$= a^2+\cancel{2ab}+b^2+ 2ab\ \cancel{2ab} + \omega(a^2+b^2) + \omega^2(a^2+b^2)$

$= a^2+b^2 + 6ab + (a^2+b^2)(\omega+\omega^2)$ but $\omega+\omega^2 = -1$

$\therefore = 6ab$

b) i) $\frac{(1+i\sqrt{3})^6}{(\sqrt{3}-i)^k} = \frac{2^6\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^6}{2^k\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^k}$

$= 2^{6-k}\frac{\mathrm{cis}\,6\times\frac{\pi}{3}}{\mathrm{cis}\,k\times-\frac{\pi}{6}}$

$= 2^{6-k}\mathrm{cis}\left(2\pi+\frac{k\pi}{6}\right)$

$= 2^{6-k}\left(\cos\left(2\pi+\frac{k\pi}{6}\right)+i\sin\left(2\pi+\frac{k\pi}{6}\right)\right)$

$= 2^{6-k}\left(\cos\frac{k\pi}{6}+i\sin\frac{k\pi}{6}\right)$

ii) purely imaginary if

$\frac{k\pi}{6} = \mp\frac{\pi}{2}, \mp\frac{5\pi}{2} \ldots$

$k = \mp 3\pi \mp 15\pi \ldots$

$= \mp 3(4n+1) \quad n = 0, 1, 2 \ldots$

OR SIMILAR.

c) i) $z_2 = z + iz = (1+i)z$

ii) $A' \Rightarrow z_1' = z\,\mathrm{cis}\frac{\pi}{4}$

$= \frac{z}{\sqrt{2}}(1+i)$

$B' \Rightarrow z_2' = (1+i)z \cdot \mathrm{cis}\frac{\pi}{4}$

$= z(1+i)\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)$

$= \frac{z}{\sqrt{2}}2i = \frac{2iz}{\sqrt{2}} = \sqrt{2}iz$

$C' \Rightarrow z_3' = iz_1'$

$= \frac{zi(1+i)}{\sqrt{2}}$

$= \frac{z(-1+i)}{\sqrt{2}}$

3) a) $z^2 = 5-12i$ let $z = x+iy$

∴ $x^2 - y^2 + 2ixy = 5-12i$

$x^2 - y^2 = 5$

$xy = -6$

let $y = \frac{-6}{x}$

∴ $x^2 - \frac{36}{x^2} = 5$

$x^4 - 5x^2 - 36 = 0$

$(x^2-9)(x^2+4) = 0$

$x = \mp 3$ $y = \pm 2$

$\sqrt{5-12i} = \mp(3-2i)$ *

$z^2 + 4z - 1 + 12i = 0$

$z = \frac{-4 \mp \sqrt{16 - 4(12i-1)}}{2}$

$= -2 \mp \frac{\sqrt{20-48i}}{2}$

$= -2 \mp \sqrt{5-12i}$

$= -2 \mp 2(3-2i)$

$= -8+4i$ or $4-4i$

b) let ~~z~~ $z = x+iy$

$\frac{3z+i}{z-2} = \frac{3(x+iy)+i}{x+iy-2}$

$= \frac{3x+(3y+1)i}{(x-2)+iy} \times \frac{(x-2)-iy}{(x-2)-iy}$

$R(z) = \frac{3x(x-2)+y(3y+1)}{(x-2)^2+y^2}$

$= 0$

$3x^2 - 6x + 3y^2 + y = 0$

$x^2 - 2x + 1 + y^2 + \frac{y}{3} + \frac{1}{36} = \frac{37}{36}$

$(x-1)^2 + (y+\frac{1}{6})^2 = \left(\sqrt{\frac{37}{36}}\right)^2$

∴ circle centre $\left(1, -\frac{1}{6}\right)$ $r = \frac{\sqrt{37}}{6}$

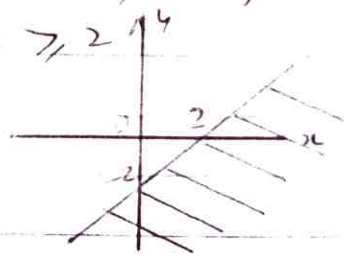
excluding $x=2$ $y=0$

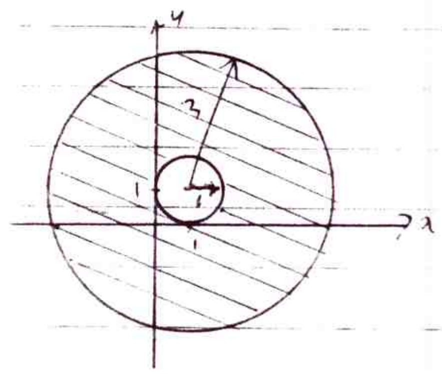
c) let $z = x+iy$ ∴ $z+iz$

∴ $z+iz = (x+iy) + xi - y$

$= (x-y) + (x+y)i$

$x - y \geq 2$




d) i) $z_2 = z_1 \operatorname{cis}\frac{\pi}{3}$

$\frac{z_2}{z_1} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$= \frac{1+i\sqrt{3}}{2}$ *

ii) $z_2 = z_1 \operatorname{cis}\frac{\pi}{3}$

$z_1^2 + z_2^2 = z_1^2 + z_1^2 \operatorname{cis}\frac{2\pi}{3}$

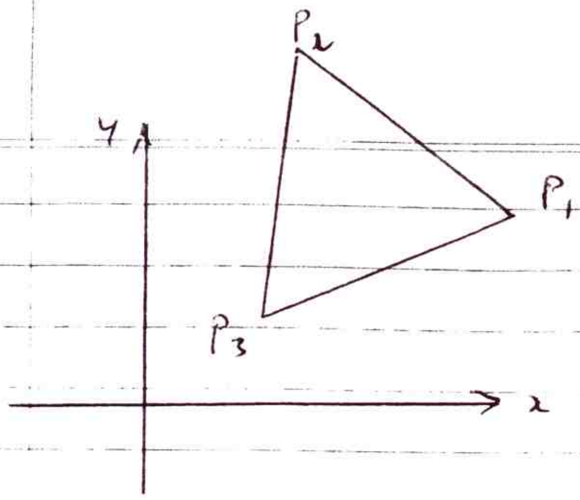
$= z_1^2 + z_1^2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= z_1\left(z_1 + z_1\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right)$

$= z_1\left(1 - \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)z_1$

$= z_1\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)z_1$

$= z_1 z_2$




(ii) from $z_1^2 + z_2^2 = z_1 z_2$

when $\triangle$ has $P_3$ at $O$

$$z_1 \rightarrow z_1 - z_3$$
$$z_2 \rightarrow z_2 - z_3$$

when $P_3$ not at $O$

$$\therefore (z_1 - z_3)^2 + (z_2 - z_3)^2 = (z_1 - z_3)(z_2 - z_3)$$

$$\therefore z_1^2 - 2z_1z_3 + z_3^2 + z_2^2 - 2z_2z_3 + \cancel{z_3^2} = z_1z_2 - z_1z_3 - z_2z_3 + \cancel{z_3^2}$$

$$\therefore z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_1z_3.$$