Student Number: .... Teacher: ....

BAULKHAM HILLS HIGH SCHOOL

Assessment Task

December, 2010

Year 12

Mathematics Extension 2

Directions to candidates:

- Reading time: 5 minutes
- Working time: 50 minutes
- Attempt ALL questions
- Use black or blue pen only
- Start a new page for each question

Marks:

1	
2	
3	
4	
TOTAL /36	
Percentage	

Marks

Question 1 (9 marks) Start on a new page

a) If $z = 2 + 3i$ and $w = 1 - i$, express in the form $a + ib$

i) $\bar{z}$ — 1

ii) $\dfrac{z}{w}$ — 2

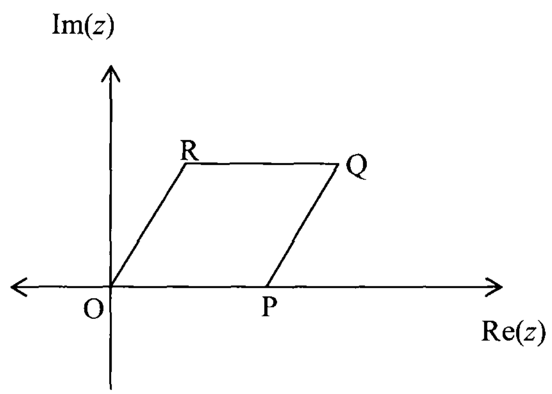
b) i) Find $\sqrt{6i - 8}$ expressing your answers in the form $a + ib$ — 2

ii) Hence solve $2z^2 - (3 + i)z + 2 = 0$ — 2

c) Find the least value of $|z|$ when $|z - 4 - 3i| = 3$ — 2

Question 2 (9 marks) Start on a new page

a) OPQR is a rhombus. O lies at the origin, P on the real axis and R corresponds to the complex number $1 + \sqrt{3}\, i$



i) Find the complex number corresponding to Q. — 2

ii) If the figure is rotated anticlockwise by 60° about O to form a new rhombus $OP'Q'R'$, show this on an Argand Diagram and find the complex number corresponding to the vertex at Q'. — 2

b) Draw neat sketches of the following

i) $\arg\left(\dfrac{z - 1 + i}{z + 2}\right) = -\dfrac{\pi}{3}$ — 3

ii) $0 < \arg(1 - i)z < \dfrac{\pi}{6}$ — 2

Marks

Question 3 (9 marks) Start on a new page

a) i) Solve the equation $z^5 - 1 = 0$ over the complex number field. **1**

ii) Hence factorise $z^5 - 1$ in terms of real linear and quadratic factors. **2**

iii) Hence, show that $\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{4}$ **3**

b) If $z = r(\cos\theta + i\sin\theta)$ show that $\frac{z}{z^2 + r^2}$ is purely real and find its value. **3**

Question 4 (9 marks) Start on a new page

a) If $1, w, w^2$ are the 3 cube roots of unity and given that $1+w+w^2=0$:

i) Show that $1 + w$ is a root of the equation $z^3 - 3z^2 + 3z - 2 = 0$ **2**

ii) Find the integer root of this equation AND also the third root in terms of w **2**

b) i) Show that the locus of $Re(z - 2) + 2 = |z - 2|$ is the parabola $y^2 = 4(x - 1)$. **2**

ii) Hence, show that when z satisfies the above relation, the principal argument of z lies between $\frac{-\pi}{4}$ and $\frac{\pi}{4}$ inclusive. **3**

End of Paper

DECEMBER EXT 2 2010 TASK

1a) i) $2-3i$ ✓

ii) $= \frac{2+3i}{1-i} \times \frac{1+i}{1+i}$ ✓

$= \frac{-1}{2} + \frac{5i}{2}$ ✓

bi) $\sqrt{6i-8} = x+iy$ (where x, y are real)

$6i-8 = x^2-y^2+2xyi$

Equating real & imaginary parts

$x^2-y^2=-8$ ①

$2xy = 6$ ② ✓

From ② $y = \frac{3}{x}$

sub in ① $x^2 - \frac{9}{x^2} = -8$

$x^4+8x^2-9=0$

$(x^2+9)(x^2-1)=0$

$x^2+9=0$ no real sol $\quad x^2-1=0$

$x = \pm 1$

$y = \pm 3$

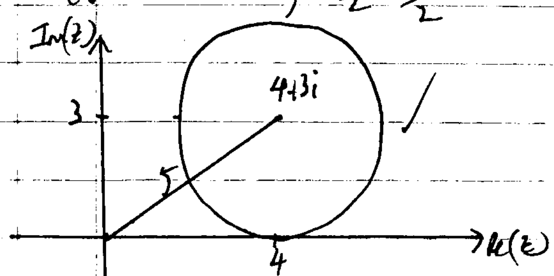
$\therefore \sqrt{6i-8} = \pm(1+3i)$ ✓

b ii) $z = \frac{3+i \pm \sqrt{8+6i-4\times 2\times 2}}{4}$

$= \frac{3+i \pm \sqrt{6i-8}}{4}$ ✓

$= \frac{3+i \pm (1+3i)}{4}$

$\therefore z = 1+i, \ \frac{1}{2} - \frac{i}{2}$ ✓



min $|z| = 5-3$

$= 2$ ✓

2a) i) $|\overrightarrow{OP}| = |\overrightarrow{OR}| = \sqrt{1+3} = 2$

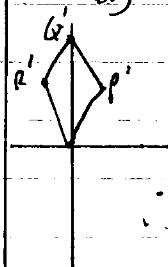
$\therefore P$ is $(2, 0)$ ✓

$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{OR} \quad (\overrightarrow{OP} = \overrightarrow{OR})$

$= 2+1+\sqrt{3}i$

$\therefore Q = 3+\sqrt{3}i$ ✓

ii) $\overrightarrow{OQ}' = \overrightarrow{OQ} \times \text{cis } 60°$



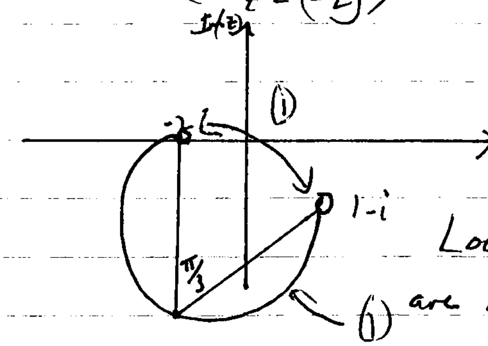
$= (3+\sqrt{3}i)(\frac{1}{2}+\frac{\sqrt{3}}{2}i)$ ✓

$= (\frac{3}{2}-\frac{3}{2}) + i(\frac{\sqrt{3}}{2}+\frac{3\sqrt{3}}{2})$

$\therefore Q$ is $(0, 2\sqrt{3}i)$ ✓

or $2\sqrt{3}\text{cis}\frac{\pi}{2}$

b) i) $\arg\left(\frac{z-(1-i)}{z-(-2)}\right) = -\frac{\pi}{3}$



(1) correct shape

(1) points and exclusions

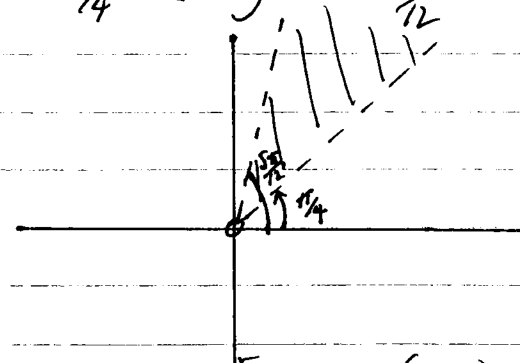
Locus is major arc of circle.

(ii) $0 < \arg(1-i)z < \frac{\pi}{6}$

$0 < \arg(1-i) + \arg(z) < \frac{\pi}{6}$ ✓

$0 < -\frac{\pi}{4} + \arg z < \frac{\pi}{6}$

$\frac{\pi}{4} < \arg z < \frac{5\pi}{12}$



✓

3 a) i) $z^5 = 1\,\text{cis}(2k\pi) \quad k=0,1,2,3,4.$

$\therefore z = \text{cis}\,\frac{2k\pi}{5} \quad k=0,1,2,3,4.$ ✓

a ii) $z_1 = 1$

$z_2 = \text{cis}\,\frac{2\pi}{5}$

$z_3 = \text{cis}\,\frac{4\pi}{5}$

$z_4 = \text{cis}\,\frac{6\pi}{5} = \overline{z_3}$

$z^5-1=(z-1)(z-z_2)(z-\bar{z}_2)(z-z_3)(z-\bar{z}_3)$

$=(z-1)(z^2-(z_2+\bar{z}_2)z+z_2\bar{z}_2)(z^2-(z_3+\bar{z}_3)z+z_3\bar{z}_3)$ ✓

$z^5-1=(z-1)(z^2-2\cos\frac{2\pi}{5}z+1)(z^2-2\cos\frac{4\pi}{5}z+1)$ ✓

iii) Since $z^5-1=(z-1)(z^4+z^3+z^2+z+1)$

$z^4+z^3+z^2+z+1=(z^2-2\cos\frac{2\pi}{5}z+1)(z^2-2\cos\frac{4\pi}{5}z+1)$

Equating coeff of z^2:

$1=1+4\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}+1$ ✓

$4\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}=-1$

$\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}=-\frac{1}{4}$

But $\cos\frac{4\pi}{5}=-\cos\frac{\pi}{5}$

$\therefore \cos\frac{2\pi}{5}\times\left(-\cos\frac{\pi}{5}\right)=-\frac{1}{4}$ ✓

$\cos\frac{\pi}{5}\cos\frac{2\pi}{5}=\frac{1}{4}$ as req'd.

3b) $\frac{z}{z^2+r^2}=\frac{r(\cos\theta+i\sin\theta)}{r^2(\cos2\theta+i\sin2\theta+1)}$ ✓

$=\frac{\cos\theta+i\sin\theta}{r(2\cos^2\theta+2i\sin\theta\cos\theta)}$ ✓

$=\frac{\cos\theta+i\sin\theta}{2r\cos\theta(\cos\theta+i\sin\theta)}$

$=\frac{1}{2r\cos\theta}$ or $\frac{1}{2r}\sec\theta$ ✓

4a) i) $1+\omega=-\omega^2$ since $1+\omega+\omega^2=0$

$P(-\omega^2)=(-\omega^2)^3-3(-\omega^2)^2+3(-\omega^2)-2$ ✓

$=-(\omega^3)^2-3\omega^4-3\omega^2-2$

$=-3(1+\omega^3.\omega+\omega^2)$

$=-3(1+\omega+\omega^2)$

$=-3\times0$

$=0$ ✓

$\therefore 1+\omega\ (-\omega^2)$ is a root.

ii) $P(2)=8-12+6-2$

$=0$

$\therefore 2$ is an integer root. ✓

Sum of roots one at a time (where α is 3rd root):

$2+(1+\omega)+\alpha=3$

$\alpha=-\omega$

$\therefore$ Third root is $-\omega$. ✓

4b) i) let $z=x+iy$

$Re(z-2)+2=|z-2|$

$x-2+2=|x-2+iy|$ ✓

$x=\sqrt{(x-2)^2+y^2}$

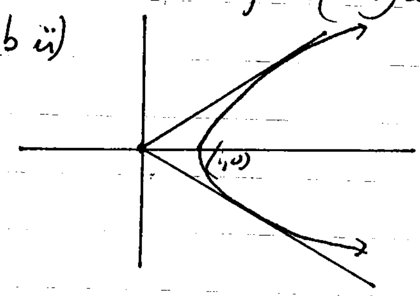
$x^2=x^2-4x+4+y^2$

$y^2=4x-4$

$y^2=4(x-1)$ ✓

$\therefore$ Parabola $y^2=4(x-1)$ is locus.

b ii)



let $y=mx$. Max and min arguments will be when $y=mx$ is a tangent to parabola

$y^2=4(x-1)$

$(mx)^2=4x-4$ ✓

$m^2x^2=4x-4$

$m^2x^2-4x+4=0$

tangent will occur when $\Delta=0$ ✓

$(-4)^2-4m^2.4=0$

$16-16m^2=0$

$m^2=1$

$m=\pm1$

$\therefore y=x$ and $y=-x$ are tangents

$\therefore$ arguments lie between

$-\frac{\pi}{4}<\arg z<\frac{\pi}{4}$ ✓