STUDENT NUMBER____________________
TEACHER NAME__________________

BAULKHAM HILLS HIGH SCHOOL

HSC ASSESSMENT TASK 1
2011

MATHEMATICS
EXTENSION 2

Time allowed - 50 minutes plus 5 minutes reading time

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- Begin each Question on a new page. Write your name on each page.
- All necessary working should be shown. Marks may be deducted for careless or poorly arranged work.
- Approved calculators and templates (Math Aids) may be used. Do NOT use liquid paper or correction tape. At the end of the exam, attach this cover sheet to the front of your solutions,

Question	Mark
1	/12
2	/11
3	/11
4	/10
Total	/45

QUESTION 1 (12 marks)

a) If $z = 3 - 4i$ and $w = 2 + i$, find in the form $x + iy$:

(i) $z + iw$ **1**

(ii) $z\overline{w}$ **1**

b) (i) Express $1 - i$ in mod-arg form **1**

(ii) Hence or otherwise, evaluate $(1 - i)^{12}$ **2**

c) Sketch the locus:

(i) $|z + 2 + 3i| = 2$ **2**

(ii) $0 \leq \arg(z - i) \leq \frac{\pi}{4}$ **2**

d) If $z = x + iy\ (x, y\ real)$ and $z^2 = -5 - 12i$, find z **3**

QUESTION 2 (11 marks) ***Start a new page now***

a) (i) Sketch the locus of $arg\left(\frac{z+1}{z-1}\right) = \frac{\pi}{4}$ **2**

(ii) Find the equation of this locus **2**

(iii) Find the maximum value of $|z|$ **1**

b) ω is a complex cube root of 1 and $\omega \neq 1$.

(i) Show that ω^2 is also a root **1**

(ii) Show that $1 + \omega + \omega^2 = 0$ **1**

(iii) Find the value of $\frac{(1-\omega+\omega^2)^3}{2-\omega-\omega^2}$ **2**

(iv) Simplify $\frac{2+3\omega+4\omega^2}{4+2\omega+3\omega^2}$ **2**

QUESTION 3 (11 marks) ***Start a new page now***

a) (i) Find the roots of z^7-1=0, and plot them on an Argand diagram **2**

(ii) Factorise $z^7 - 1$ into real linear and quadratic factors **2**

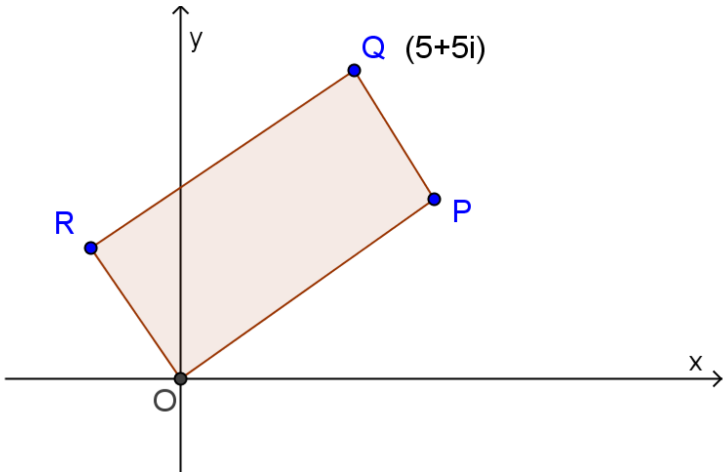
(iii) Hence evaluate $cos\frac{2\pi}{7} - cos\frac{3\pi}{7} - cos\frac{\pi}{7}$ **3**

Question 3 continues over page

QUESTION 3 Continued

b) In the diagram below (not to scale), OPQR is a rectangle and OP=2.OR

Point Q represents the complex number 5+5i



(i) Find the complex numbers represented by points P and R **3**

(ii) Find the complex number represented by the vector $\overrightarrow{PR}$ **1**

QUESTION 4 **(10 marks)** ***Start a new page now***

a) The complex numbers z and w are such that $w = \frac{z+2}{z}$

Find the Cartesian equation of the locus of w if $|z| = 1$ **3**

b) c is a real number and $c \neq 0$.

It is given that $(1 + ic)^5$ is real.

(i) Expand and simplify $(1 + ic)^5$ **1**

(ii) Show that $c^4 - 10c^2 + 5 = 0$ **2**

(iii) Hence show that

$c = \sqrt{5 - 2\sqrt{5}},\ -\sqrt{5 - 2\sqrt{5}},\ \sqrt{5 + 2\sqrt{5}}\ or\ -\sqrt{5 + 2\sqrt{5}}$ **2**

(iv) Let $1 + ic = rcis\theta$. Show that the smallest positive value of θ is $\frac{\pi}{5}$. **1**

(v) Hence evaluate $tan\frac{\pi}{5}$. **1**

End of examination

X2 Solns Task 1 (Dec 2011)

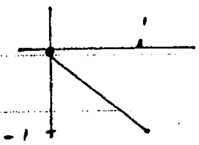
Question 1.

a) i) $3-4i + 2i - 1 \quad = \underline{2-2i}$ 1

ii) $(3-4i)(2-i) \quad = 6 - 3i - 8i - 4$

$= \underline{2 - 11i}$ 1

b) i)

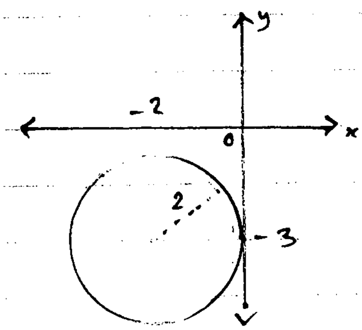


$\underline{\sqrt{2}\ \text{cis}\left(-\frac{\pi}{4}\right)}$ 1

ii) $\sqrt{2}^{12} \cdot \text{cis}\left(12 \times -\frac{\pi}{4}\right) \quad = 64\ \text{cis}(-3\pi)$ 1

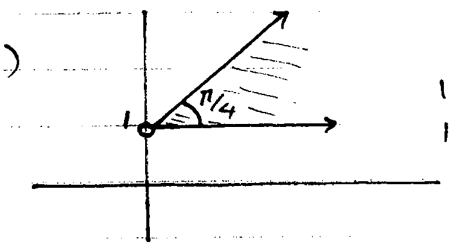
$= \underline{-64}$ 1

c) (i)



1 centre
1 radius

(ii)



1 region
1 endpt excl.

d)
$$(x+iy)^2 = -5-12i$$
$$x^2 - y^2 + 2ixy = -5-12i$$
$$\left.\begin{aligned} x^2 - y^2 &= -5 \\ xy &= -6 \end{aligned}\right\} \quad 1$$
$$x^2 - \left(-\frac{6}{x}\right)^2 = -5$$
$$x^2 - \frac{36}{x^2} = -5$$
$$x^4 + 5x^2 - 36 = 0 \quad 1$$
$$(x^2+9)(x^2-4) = 0$$
$$\cancel{x^2 = -9} \qquad x^2 = 4$$
$$(x \text{ real}) \qquad x = 2, -2$$
If $x = 2,\ y = -3$

If $x = -2,\ y = 3$

$\therefore \quad \underline{z = 2-3i,\ -2+3i} \quad \text{or} \quad \pm(2-3i)$ 1

$x = 2,\ y = -3$

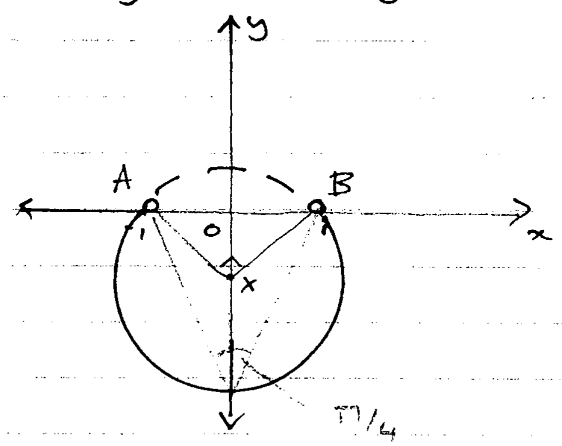
$x = -2,\ y = 3$

$\therefore z = 2-3i,\ -2+3i$

1 each if by inspection

Question 2.

a) (i) $\arg(z+1) - \arg(z-1) = \pi/4$



1 major arc
1 below x-axis.
must exclude endpts.

ii) $AX = BX = r$ $\quad 2r^2 = 4$ (Pythag).

$r^2 = 2 \quad r = \sqrt{2}$.

$OX = OB = y \quad 2y^2 = 2$

$y^2 = 1 \quad y = 1$

$\therefore$ Circle has centre $(0, -1)$ and $r = \sqrt{2}$

$\therefore$ Eqn of locus :

$$x^2 + (y+1)^2 = 2 \quad (y < 0)$$

OR $|z - i| = \sqrt{2}$ 1 LHS, 1 RHS

iii) $1 + \sqrt{2}$.   1

b) (i) $\omega^3 = 1$ since ω is a root of 1

$(\omega^2)^3 = (\omega^3)^2 = 1^2 = 1 \quad \therefore \omega^2$ is also a root

(ii) $1 + \omega + \omega^2$ = sum of roots $[\, z^3 + 0z^2 + 0z - 1 = 0\,]$

$= -\frac{b}{a}$ 1

$= -0/1 \quad = 0$

(iii) $1 + \omega + \omega^2 = 0 \quad \therefore 1 + \omega^2 = -\omega \;,\; -\omega - \omega^2 = 1$

$$\therefore \frac{(1 - \omega + \omega^2)^3}{2 - \omega - \omega^2} = \frac{(-2\omega)^3}{2 + 1}$$

$$= -\frac{8\omega^3}{3} \quad (\omega^3 = 1)$$

$$= -8/3.$$

1 num.
1 denom.

Using $1+\omega+\omega^2=0$ $\to 1$

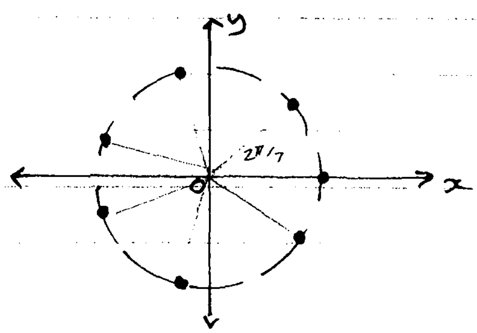
iv) $\dfrac{2+3\omega+4\omega^2}{4+2\omega+3\omega^2} \times \dfrac{\omega}{\omega}$ 1

$= \dfrac{2\omega+3\omega^2+4}{(4+2\omega+3\omega^2)\,\omega}$

$= \underline{\dfrac{1}{\omega} \quad \text{or} \quad \omega^2} \quad \underline{\text{or } -1-\omega}$ 1

Question 3.

a) i) $z = 1, \ \operatorname{cis}\left(\pm \frac{2\pi}{7}\right), \ \operatorname{cis}\left(\pm \frac{4\pi}{7}\right), \ \operatorname{cis}\left(\pm \frac{6\pi}{7}\right)$



1 roots
1 diagram

ii) $z^7-1 = (z-1)(z-\operatorname{cis}\frac{2\pi}{7})(z-\operatorname{cis}-\frac{2\pi}{7})(z-\operatorname{cis}\frac{4\pi}{7})$
$(z-\operatorname{cis}\frac{-4\pi}{7})(z-\operatorname{cis}\frac{6\pi}{7})(z-\operatorname{cis}\frac{-6\pi}{7})$

$= (z-1)(z^2-2\cos\frac{2\pi}{7}.z+1)(z^2-2\cos\frac{4\pi}{7}.z+1,$
$(z^2-2\cos\frac{6\pi}{7}+1)$ 2

iii) $z^7-1 = (z-1)(z^6+z^5+z^4+z^3+z^2+z+1)$

$\therefore z^6+z^5+z^4+z^3+z^2+z+1$
$= (z^2-2z.\cos\frac{2\pi}{7}+1)(z^2-2z.\cos\frac{4\pi}{7}+1)(z^2-2z.\cos\frac{6\pi}{7}+1$

Equating coefficients of z:

$1 = -2\cos\frac{2\pi}{7} - 2\cos\frac{4\pi}{7} - 2\cos\frac{6\pi}{7}$ 1

$-\frac{1}{2} = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$

But $\cos\frac{4\pi}{7} = -\cos\frac{3\pi}{7}$, $\cos\frac{6\pi}{7} = -\cos\frac{\pi}{7}$ 1

$\therefore \cos\frac{2\pi}{7} - \cos\frac{3\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2}$

b) i) Let $R = x + iy$, $P = \cancel{\tfrac{1}{2}} - 2i(x+iy)$

$= 2y - 2ix$

$(x+2y) + i(y-2x) = 5 + 5i$

$\left.\begin{array}{l} x + 2y = 5 \\ -2x + y = 5 \end{array}\right\}$ $\quad \begin{array}{l} 2x + 4y = 10 \\ -2x + y = 5 \end{array}$ 1

$5y = 15 \quad \therefore y = 3 , x = -1.$

$\therefore R = -1 + 3i$ 1

$P = 6 + 2i$ 1

ii) Vector from P to R $(\vec{OR} - \vec{OP})$

$= -1 + 3i - 6 - 2i$

$= -7 + i$ 1

Question 4.

a) $zw = z + 2$

$z(w-1) = 2$

$z = \dfrac{2}{w-1}$ 1

$|z| = \left|\dfrac{2}{w-1}\right| = 1$

$\dfrac{2}{|w-1|} = 1$

$|w-1| = 2$ 1

$\therefore (x-1)^2 + y^2 = 4.$ 1

b) (i) $(1+ic)^5 = 1 + 5(ic) + 10(ic)^2 + 10(ic)^3 + 5(ic)^4 + (ic)^5$

$= 1 + 5ic - 10c^2 - 10ic^3 + 5c^4 + ic^5$ 1

(ii) If $(1+ic)^5$ is real, then imag. part $= 0$ 1

$5c - 10c^3 + c^5 = 0$

$(\div c)$ $5 - 10c^2 + c^4 = 0$ ie $c^4 - 10c^2 + 5 = 0$ 1

(iii) $c^2 = \dfrac{10 \pm \sqrt{100-20}}{2} = \dfrac{10 \pm \sqrt{80}}{2}$ $(\sqrt{80} = 4\sqrt{5})$

$= 5 \pm 2\sqrt{5}$ 1

$\therefore$ $c = \pm\sqrt{5 \pm 2\sqrt{5}}$ (the four given solns). 1

(iv) $(1+ic)^5 = r^5 \operatorname{cis} 5\theta$ is purely real.

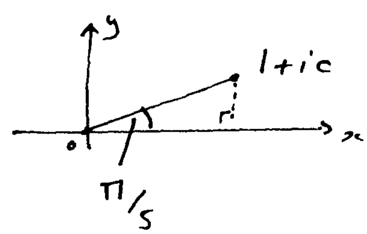
$5\theta = n\pi$ (n = integer)

$5\theta = \pi$ (smallest positive n)

$\theta = \pi/5.$ 1

(v) $\tan\dfrac{\pi}{5} = \dfrac{c}{1} = c$

$= \sqrt{5 - 2\sqrt{5}}.$



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