BAULKHAM HILLS HIGH SCHOOL

MATHEMATICS EXTENSION 2 ASSESSMENT

December 2012

Time allowed: 50 minutes
plus 5 minutes reading time

BOS NUMBER :____________________

TEACHER'S NAME:____________________

QUESTION	MARK
1	/10
2	/10
3	/7
4	/11
TOTAL	/38
PERCENTAGE	

Topics Tested: Complex Numbers

BAULKHAM HILLS HS PERSEVERE	# Extension 2 Mathematics *December 2012* *Time: 50 minutes + 5 minutes reading time*

DIRECTIONS

- Full working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Use black or blue pen only (*not pencils*) to write your solutions.
- No liquid paper is to be used. If a correction is to be made, one line is to be ruled through the incorrect answer.
- Approved Maths aids and calculators may be used
- Marks are indicated next to each question

QUESTION 1 Start on the appropriate page in your booklet (10 marks)

a If $u = 1 - i$ and $v = -2 + 4i$ evaluate the following

i. $u^2 + v$ **1**

ii. $Im(u + \bar{v})$ **2**

iii. $\left|\frac{u+v+1}{u+v+i}\right|$ **2**

b Let z the complex number $z = \sqrt{3} + i$

Express in mod-arg form (with principal argument)

i. z **2**

ii. iz **1**

iii. z^9 **2**

QUESTION 2 Start on the appropriate page in your booklet (10 marks)

a Find both square roots of $-3 + 4i$ **2**

b Sketch the locus of z if $\arg(z - 1) = \arg(z - i)$ **2**

c i. Sketch $|z - 2i| \leq 1$ **2**

ii. Find the largest value of $\arg(z)$ when $|z - 2i| = 1$ **1**

d Sketch the locus of z if $Re\left(\frac{-i}{z}\right) = 2\ Im\left(-\frac{i}{z}\right)$ **3**

CONTINUED ON NEXT PAGE

QUESTION 3 Start on the appropriate page in your booklet (7 marks)

a i. Given $|z| = 1$, show that $z^n + z^{-n} = 2\cos n\theta$. **2**

ii. Solve $z^6 + 1 = 0$ and **3**

hence factorise $z^6 + 1$ into real quadratic factors

iii. Hence deduce that $\cos 3\theta = \cos\theta\left(2\cos\theta + \sqrt{3}\right)\left(2\cos\theta - \sqrt{3}\right)$ **2**

QUESTION 4 Start on the appropriate page in your booklet (11 marks)

a Given that $z + \frac{1}{z} = k$ where k is real and $z = x + iy$

i. Show that either $y = 0$ or $x^2 + y^2 = 1$ **2**

ii. Show that if $y = 0$ then $|k| \geq 2$ **2**

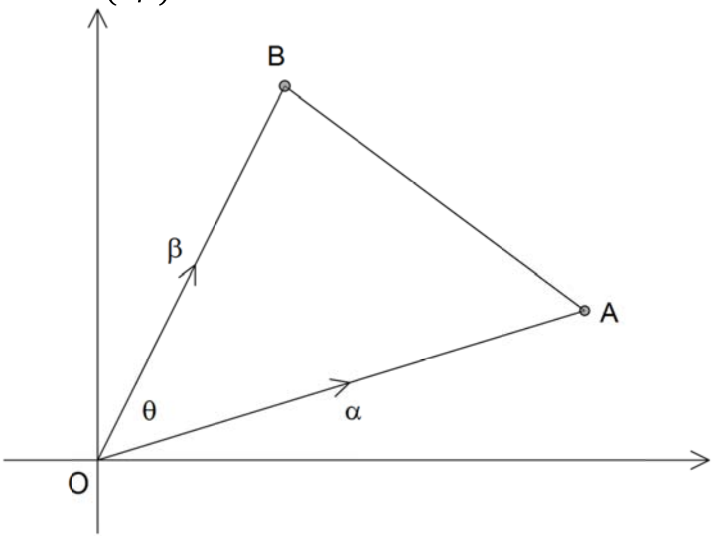
iii. Show that if $x^2 + y^2 = 1$ then $|k| \leq 2$ **2**

b i. Use the results $z + \bar{z} = 2Re(z)$ and $|z|^2 = z\bar{z}$ to show that **3**

$$|\alpha|^2 + |\beta|^2 - |\alpha - \beta|^2 = 2\,Re\left(\alpha\bar{\beta}\right)$$

ii. The diagram shows the angle θ between the complex numbers α and β **2**

Prove that $|\alpha\beta|\cos\theta = Re\left(\alpha\bar{\beta}\right)$



~ END OF EXAM ~

EXTENSION 2 DECEMBER TASK 2012

1 a) i) $(1-i)^2 - 2 + 4i$

$= 1 - 2i + i^2 - 2 + 4i$

$= -2 + 2i$ ✓

ii) $\text{Im}\left(1 - i + (-2 - 4i)\right)$ ✓ ① for conjugate

$= \text{Im}(-1 - 5i)$

$= -5$ ✓

iii) $\left|\dfrac{1 - i - 2 + 4i + 1}{1 - i - 2 + 4i + i}\right|$

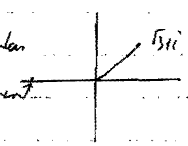
$= \left|\dfrac{3i}{-1 + 4i}\right|$

$= \dfrac{|3i|}{|-1 + 4i|}$ or equivalent: $\left|\dfrac{3i\,(-1-4i)}{(-1+4i)(-1-4i)}\right|$ ✓

$= \dfrac{3}{\sqrt{1 + 4^2}}$

$= \dfrac{3}{\sqrt{17}}$ or $\dfrac{3\sqrt{17}}{17}$ ✓

1 b) i) $z = |z|(\cos 30° + i\sin 30°)$ ① modulus ① argument



$= \sqrt{3+1}\,(\cos 30° + i\sin 30°)$

$z = 2(\cos 30° + i\sin 30°)$ ✓✓

ii) $iz = 2(\cos 120° + i\sin 120°)$ ✓

iii) $z^9 = \left(2(\cos 30° + i\sin 30°)\right)^9$

$z = 512(\cos 270° + i\sin 270°)$ ✓

$z = 512(\cos(-90°) + i\sin(-90°))$ ✓

2 a) $x + iy = \sqrt{-3 + 4i}$

$x^2 - y^2 + 2xyi = -3 + 4i$

By inspection $\sqrt{-3 + 4i} = \pm(1 + 2i)$ ✓✓

(or $x^2 - y^2 = -3$ $\quad 2xy = 4$ (OR ✓

$y = \frac{2}{x}$

$x^2 - \frac{4}{x^2} = -3$

$x^4 - 4 = -3x^2$

$x^4 + 3x^2 - 4 = 0$

$(x^2 + 4)(x^2 - 1) = 0$

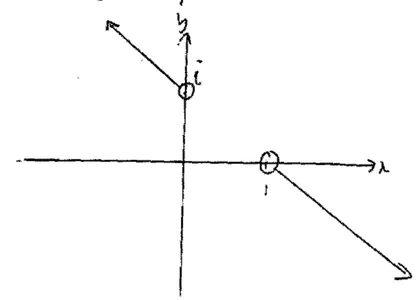
no real soln $\quad x^2 - 1 = 0$

$x = \pm 1$

$\therefore y = \pm 2$

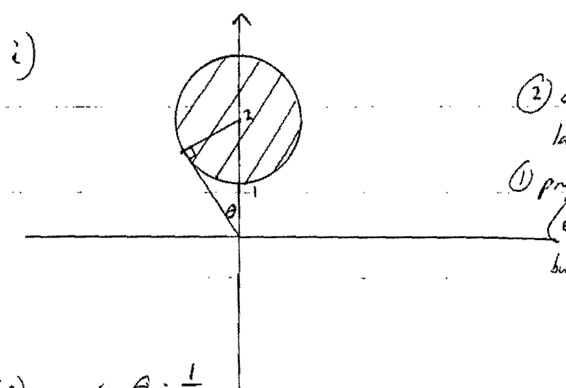
$\therefore \sqrt{-3 + 4i} = \pm(1 + 2i)$) ✓)

b)



① endpoints ✓
① line excluding interval from i to 1. ✓

c) i)



② circle, centre $2i$, with i & $3i$ labelled, & shaded inside ✓
① progress toward (eg centre $2i$, shaded inside but no clear radius ✓

ii) $\sin\theta = \frac{1}{2}$

$\theta = 30°$ ✓

$\therefore$ max argument $= 120°$ or $\left(\frac{2\pi}{3}\right)$ ✓

2d) $\operatorname{Re}\left(\frac{-i}{z}\right) = 2\operatorname{Im}\left(\frac{-i}{z}\right)$ NB $z \neq 0$

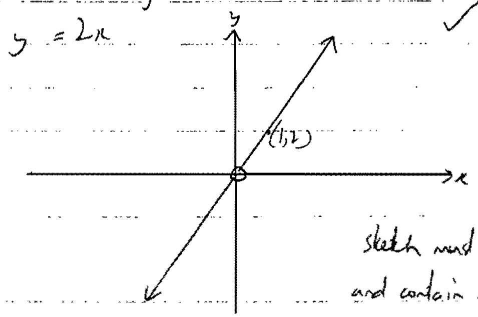
let $z = x+iy$

$$\operatorname{Re}\left(\frac{-i(x-iy)}{z\bar{z}}\right) = 2\operatorname{Im}\left(\frac{-i(x-iy)}{z\bar{z}}\right)$$

$$\operatorname{Re}\left(\frac{-y-ix}{x^2+y^2}\right) = 2\operatorname{Im}\left(\frac{-y-ix}{x^2+y^2}\right) \quad ✓$$

$$\frac{-y}{x^2+y^2} = \frac{2(-x)}{x^2+y^2} \quad ✓$$

$$y = 2x$$



✓

sketch must exclude $z=0$ and contain a point/scale.

a) i) let $z = 1(\cos\theta + i\sin\theta)$ $|z|=1$

$$z^n + z^{-n} = (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$$

$$= \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta)$$

$$= \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$$

$$z^n + z^{-n} = 2\cos n\theta \quad \text{as req'd.}$$

(ii) $z^6 = -1$

$$z_k = \operatorname{cis}\left(\frac{\pi + 2k\pi}{6}\right) \quad k = 0,1,2,\ldots 5 \quad \text{(or equival.)} \quad ✓$$

$z_0 = \operatorname{cis}\frac{\pi}{6}$

$z_1 = \operatorname{cis}\frac{3\pi}{6}$

$z_2 = \operatorname{cis}\frac{5\pi}{6}$

$z_3 = \operatorname{cis}\frac{7\pi}{6} = \bar{z}_2$

$z_4 = \operatorname{cis}\frac{9\pi}{6} = \bar{z}_1$

$z_5 = \operatorname{cis}\frac{11\pi}{6} = \bar{z}_0$

$$k^2 - 4\times1\times1 \geq 0$$
$$k^2 - 4 \geq 0$$
$$k^2 \geq 4$$
$$|k| \geq 2 \quad ✓$$

iii) $\left|z + \frac{1}{z}\right| = |k|$

$|z| + \left|\frac{1}{z}\right| \geq |k|$ as $|z| + \left|\frac{1}{z}\right| \geq \left|z + \frac{1}{z}\right|$ ✓

$|z| + \frac{1}{|z|} \geq |k|$

$1 + \frac{1}{1} \geq |k|$ as $|z| = 1$ if $x^2+y^2=1$ ✓

$|k| \leq 2$

(method 2. From above

$(x+iy)(x^2+y^2) + x - iy = k(x^2+y^2)$

$x+iy + x - iy = k$ (as $x^2+y^2=1$)

$2x = k$ ✓

Since $x^2+y^2=1$ $-1 \leq x \leq 1$

$\therefore -2 \leq 2x \leq 2$

$-2 \leq k \leq 2$

ie $|k| \leq 2$) ✓

b i) $|\alpha|^2 + |\beta|^2 - |\alpha-\beta|^2$

$= \alpha\bar{\alpha} + \beta\bar{\beta} - (\alpha-\beta)\overline{(\alpha-\beta)}$ ✓

$= \alpha\bar{\alpha} + \beta\bar{\beta} - (\alpha-\beta)(\bar{\alpha}-\bar{\beta})$ ✓

$= \cancel{\alpha\bar{\alpha}} + \cancel{\beta\bar{\beta}} - (\cancel{\alpha\bar{\alpha}} + \cancel{\beta\bar{\beta}} - \alpha\bar{\beta} - \bar{\alpha}\beta)$

$= \alpha\bar{\beta} + \bar{\alpha}\beta$

$= \alpha\bar{\beta} + \overline{\alpha\bar{\beta}}$ ✓

$= 2\operatorname{Re}(\alpha\bar{\beta})$

$$z^6+1=(z-z_1)(z-\bar{z}_1)(z-z_0)(z-\bar{z}_0)(z-z_2)(z-\bar{z}_2)$$

$$=\left(z^2-(z_1+\bar{z}_1)z+z_1\bar{z}_1\right)\left(z^2-(z_0+\bar{z}_0)z+z_0\bar{z}_0\right)\left(z^2-(z_2+\bar{z}_2)z+z_2\bar{z}_2\right)$$

$$=\left(z^2-2\cos\tfrac{\pi}{2}z+1\right)\left(z^2-2\cos\tfrac{\pi}{6}z+1\right)\left(z^2-2\cos\tfrac{5\pi}{6}z+1\right)$$

iii)

$$\frac{z^6+1}{z^3}=\frac{(z^2+1)}{z}\left(\frac{z^2-\sqrt{3}z+1}{z}\right)\left(\frac{z^2+\sqrt{3}z+1}{z}\right)$$

$$z^3+\frac{1}{z^3}=\left(z+\frac{1}{z}\right)\left(z+\frac{1}{z}-\sqrt{3}\right)\left(z+\frac{1}{z}+\sqrt{3}\right)$$

$$2\cos 3\theta=2\cos\theta\left(2\cos\theta-\sqrt{3}\right)\left(2\cos\theta+\sqrt{3}\right)$$

$$\cos 3\theta=\cos\theta\left(2\cos\theta-\sqrt{3}\right)\left(2\cos\theta+\sqrt{3}\right)$$

OR

$$\left(z+\frac{1}{z}\right)^3=z^3+3z+\frac{3}{z}+\frac{1}{z^3}$$

$$(2\cos\theta)^3=z^3+\frac{1}{z^3}+3\left(z+\frac{1}{z}\right)$$

$$8\cos^3\theta=2\cos 3\theta+6\cos\theta$$

$$\cos 3\theta=4\cos^3\theta-3\cos\theta$$

$$\cos 3\theta=\cos\theta\left(4\cos^2\theta-3\right)$$

$$\cos 3\theta=\cos\theta\left(2\cos\theta+\sqrt{3}\right)\left(2\cos\theta-\sqrt{3}\right)$$

Q4 a) i) $z+\frac{1}{z}=k$

$$x+iy+\frac{1}{x+iy}=k$$

$$x+iy+\frac{x-iy}{x^2+y^2}=k$$

$$(x+iy)(x^2+y^2)+x-iy=k(x^2+y^2)+0i$$

Equating imaginary parts $y(x^2+y^2)-y=0$

$$y(x^2+y^2-1)=0$$

$\therefore y=0$ or $x^2+y^2-1=0$

ie $x^2+y^2=1$

ii) If $y=0$ $z+\frac{1}{z}=k$ becomes

$$x+\frac{1}{x}=k$$

since x and k are real $\Delta\geq 0$

$$x^2+1=kx$$

$$x^2-kx+1=0$$

b ii) In $\triangle OBA$

$$\cos\theta=\frac{|\alpha|^2+|\beta|^2-|\alpha-\beta|^2}{2|\alpha||\beta|}$$

$$|\alpha||\beta|\cos\theta=\frac{2\,\mathrm{Re}(\alpha\bar{\beta})}{2}$$

$$|\alpha\beta|\cos\theta=\mathrm{Re}(\alpha\bar{\beta})$$