BAULKHAM HILLS HIGH SCHOOL

DECEMBER 2013

YEAR 12 TASK 1

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 50 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- Show all necessary working in questions 1 to 5
- Marks may be deducted for careless or badly arranged work

Total marks – 35
Exam consists of 3 pages.

Questions 1-5

- Attempt Questions 1-5

Topics Tested: Complex numbers

Question 1 **(8 marks) - Start a new page**

a) Let $z = 5 - i$ and $w = 2 + 3i$

Find in the form $a + ib$

(i) $\bar{z}w$ **2**

(ii) $\frac{w}{z}$ **2**

b) (i) Find the two square roots of $12 + 16i$ **2**

(ii) Hence solve $z^2 - 4z + 1 - 4i = 0$ **2**

Question 2 **(6 marks) - Start a new page**

a) (i) Express $-1 - \sqrt{3}i$ in mod-arg form **2**

(ii) Hence or otherwise evaluate $\left(-1 - \sqrt{3}i\right)^8$ **2**

Leave your answer in mod-arg form.

b) Given $1 - 2i$ is one root of the equation $x^2 + (1 + i)x + k = 0$, **2**

find the other root and the value of k

Question 3 **(8 marks) - Start a new page**

a) (i) If $|z| = 1$, show that $z^n - z^{-n} = 2i \sin n\theta$ **2**

(ii) Hence show that **2**

$$\sin^5 \theta = \frac{1}{16}(\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta)$$

[Note: $(a - b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$*]*

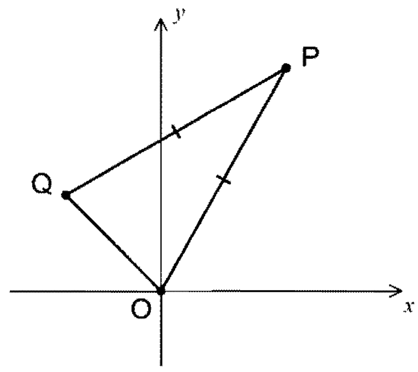
b) (i) Sketch the locus of $|z - 3i| = |z + 2 + 5i|$ **2**

(ii) Find the equation of this locus. **2**

Question 4 **(6 marks)** - **Start a new page**

a) Given $1+\sqrt{3}i = 2\text{cis}\frac{\pi}{3}$ and $-1-i = \sqrt{2}\text{cis}\frac{-3\pi}{4}$, find the exact value of $\cos\frac{-5\pi}{12}$ **2**

b) P is the point representing the complex number $p = 1+\sqrt{3}i$. Triangle OPQ is isosceles with $OP = PQ$ and $\angle OPQ = \frac{\pi}{6}$ as shown in the diagram below. **2**

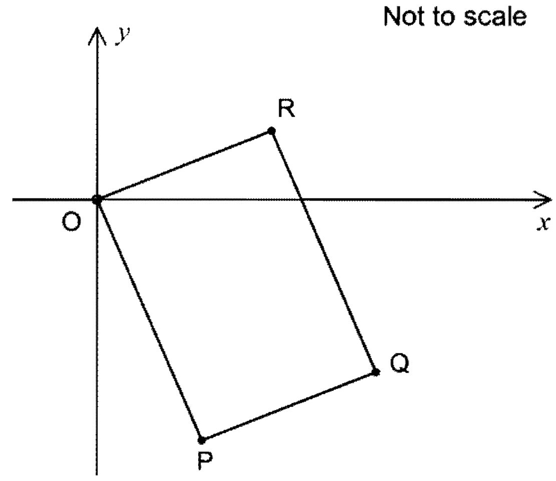


Find the complex number representing the vector $\overrightarrow{PQ}$ in the form $a+ib$.

c) On an Argand diagram, P represents the complex number $z = 1-\sqrt{3}i$. **2**

$OPQR$ is a rectangle where $|OP| = 2 \times |OR|$.

Not to scale



Find the complex numbers representing R and Q.

Question 5 **(7 marks)** - **Start a new page**

a) (i) Indicate on an Argand diagram the region defined by the pair of simultaneous inequalities $|z| \leq 6$ and $|z-5| \geq 5$ **2**

(ii) Hence find the range of values of $\arg z$. **2**

b) Find the Cartesian equation of the locus of z such that $\arg(z-2) = \arg(z^2)$. **3**

Describe the locus geometrically, noting any restrictions

- END OF PAPER -

Question 1 (8 marks) **BOS#:** Dec. Ass. X2 - 2013

a) $z = 5 - i \qquad w = 2 + 3i$

i) $\bar{z} \cdot w = \underbrace{(5+i)}_{①}(2+3i) = 10 + 3i^2 + 2i + 15i$

$= 7 + 17i$ ①

ii) $\dfrac{w}{z} = \dfrac{w \cdot \bar{z}}{z \cdot \bar{z}} = \dfrac{7+17i}{(5-i)(5+i)}$ ① $= \dfrac{7+17i}{26}$ ①

b) i) $\sqrt{12+16i} = z \qquad$ let $z = x + iy$

$\therefore 12 + 16i = z^2 = x^2 - y^2 + 2xyi$

(1) $12 = x^2 - y^2$

(2) $16 = 2xy \quad \therefore y = \dfrac{8}{x}$

$\therefore$ (1) $12 = x^2 - \dfrac{64}{x^2}$ ①

$\therefore 0 = x^4 - 12x^2 - 64 \quad \therefore x^2 = \dfrac{12 \pm \sqrt{400}}{2} = 16, -4$

$x^2 = 16 \quad \therefore x = \pm 4$

$x^2 = -4 \quad$ but x must be real $\therefore$ no real solns.

$\therefore x = 4 \ \therefore y = 2 \quad \therefore z = \sqrt{12+16i} = 4 + 2i$

$x = -4 \ \therefore y = -2 \quad \therefore z = \sqrt{12+16i} = -4 - 2i$

①

$\therefore \sqrt{12+16i} = \pm(4+2i)$

1b) ii) $z^2 - 4z + 1 - 4i = 0$

$\therefore z = \dfrac{4 \pm \sqrt{16 - 4(1-4i)}}{2} = \dfrac{4 \pm \sqrt{12+16i}}{2}$

$\therefore z = \dfrac{4 \pm (\pm 4 + 2i)}{2}$ ①

$\therefore z = \begin{cases} \dfrac{4 + (4+2i)}{2} = 4 + i \\ \dfrac{4 - (4+2i)}{2} = -i \end{cases}$ ①

You may ask for extra writing paper if you need more space to answer question 1

Question 2 (6 marks) **BOS#:** ______

a) i) $-1-\sqrt{3}\,i = 2\,\mathrm{cis}\left(-\frac{2\pi}{3}\right)$ (1) (1)

ii) $(-1-\sqrt{3}\,i)^8 = \left[2\,\mathrm{cis}\left(-\frac{2\pi}{3}\right)\right]^8 = 2^8\,\mathrm{cis}\left(8\times -\frac{2\pi}{3}\right)$

(1) - using DM's theorem $= 2^8\,\mathrm{cis}\left(-\frac{16\pi}{3}\right) = 2^8\,\mathrm{cis}\left(\frac{2\pi}{3}\right)$ (1) ISE

b) $x^2 + (1+i)x + k = 0$

let $\alpha = 1-2i$ the other root $\beta = a+ib$

$\therefore \alpha+\beta = -\frac{b}{a}$

$\therefore 1-2i + a+ib = -1-i$

$\therefore 1+a = -1$ and $-2i+ib = -i$

$a = -2$ $\quad$ $b = 1$

$\therefore \beta = -2+i$ (1)

now $\alpha\cdot\beta = \frac{c}{a}$

$\therefore (1-2i)(-2+i) = k$

$-2-2i^2+i+4i = k$

$-2+2+5i = k \quad \therefore k = 5i$ (1)

Question 3 - (8 marks) **BOS#:** ______

a) i)

$|z| = 1 \quad \therefore z = \mathrm{cis}\,\theta$

$\therefore z^n - z^{-n} = \mathrm{cis}(n\theta) - \mathrm{cis}(-n\theta)$ (1)

$= \cos(n\theta) + i\sin(n\theta) - [\cos(-n\theta) + i\sin(-n\theta)]$

$= \cos(n\theta) + i\sin(n\theta) - [\cos(n\theta) - i\sin(n\theta)]$

even (1) odd

$= 2i\sin(n\theta)$ $\therefore$ proven

ii) since $(z-z^{-1})^5 = (2i(\sin\theta))^5$

and $(z-z^{-1})^5 = z^5 - 5z^4 z^{-1} + 10z^3z^{-2} - 10z^2z^{-3} + 5z\,z^{-4} - z^{-5}$

equate both (1)

$\therefore (2i\sin\theta)^5 = (z^5-z^{-5}) - 5(z^3-z^{-3}) + 10(z-z^{-1})$

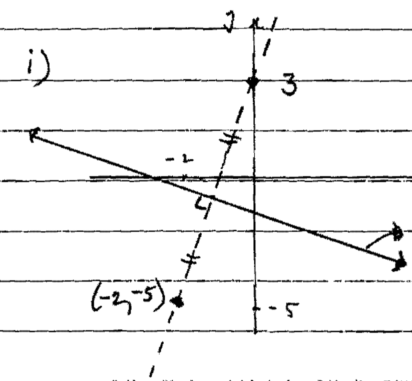
$i^5 = i$

$\therefore 32i\sin^5\theta = 2i\sin 5\theta - 5\times 2i\sin 3\theta + 10\times 2i\sin\theta$ (1)

$\therefore \sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{10}{32}\sin 3\theta + \frac{20}{32}\sin\theta$

$\therefore \sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$ $\therefore$ shown

b) i)



or (1) line

(2) perp & bisector must be shown

locus (2) is perpendicular bisector

3b) ii) $|z-3i| = |z+2+5i|$ let $z = x+iy$

$|x+iy-3i| = |x+iy+2+5i|$

$\sqrt{x^2+(y-3)^2} = \sqrt{(x+2)^2+(y+5)^2}$ (1)

$x^2+y^2-6y+9 = x^2+4x+4+y^2+10y+25$

$-6y+9 = 4x+10y+29$

$\therefore$ locus is $0 = 4x+16y+20$ (1)

or $0 = x+4y+5$

which is a straight line, being perpendicular bisector of line joining $(0,3)$ & $(-2,-5)$.

Question 4 (6 marks) **BOS#:**________________

a) $1+\sqrt{3}i = 2\operatorname{cis}\frac{\pi}{3}$
$-1-i = \sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$ } given

but $(1+\sqrt{3}i)(-1-i) = 2\operatorname{cis}\frac{\pi}{3} \times \sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$

$\therefore$ (1) multiplying both sides

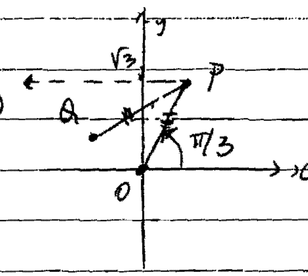
$\therefore \left[-1-\sqrt{3}i^2+i(-\sqrt{3}-1)\right] = 2\sqrt{2}\left[\operatorname{cis}\left(\frac{\pi}{3}+\left(-\frac{3\pi}{4}\right)\right)\right]$

$\left[(-1+\sqrt{3})+i(-\sqrt{3}-1)\right] = 2\sqrt{2}\left[\cos\left(-\frac{5\pi}{12}\right)+i\sin\left(-\frac{5\pi}{12}\right)\right]$

now equate real parts

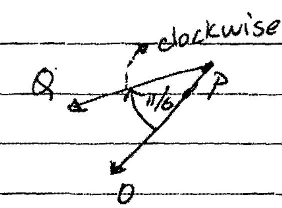
$\therefore -1+\sqrt{3} = 2\sqrt{2}\cos\left(-\frac{5\pi}{12}\right)$ (1)

$\therefore \cos\left(-\frac{5\pi}{12}\right) = \frac{-1+\sqrt{3}}{2\sqrt{2}}$

b) 

$p = 1+\sqrt{3}i = \overrightarrow{OP}$ $\arg(p) = \frac{\pi}{3}$

$\therefore \overrightarrow{PO} = -\overrightarrow{OP} = -1-\sqrt{3}i$



since $|OP| = |PQ|$

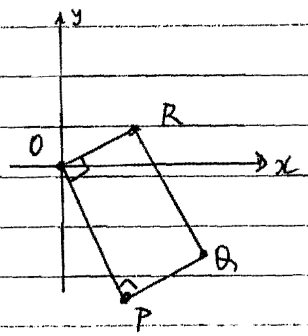
$\therefore \overrightarrow{PQ} = \overrightarrow{PO} \times \operatorname{cis}\left(-\frac{\pi}{6}\right)$ (1)

$\therefore \overrightarrow{PQ} = 2\operatorname{cis}\left(-\frac{2\pi}{3}\right) \times \operatorname{cis}\left(-\frac{\pi}{6}\right) = 2\operatorname{cis}\left(-\frac{5\pi}{6}\right)$

$= 2\cos(-5\pi/6) + 2i\sin(-5\pi/6)$

$= 2 \times \left(-\frac{\sqrt{3}}{2}\right) + 2\left(-\frac{1}{2}\right) = -\sqrt{3}-i$ (1)

4c)



$\vec{OP} = z = 1-\sqrt{3}\,i$

$\therefore \vec{OR} = \frac{1}{2}\,\text{cis}\,\frac{\pi}{2}\cdot\vec{OP} = \vec{OP}\cdot\frac{1}{2}i$

$= \frac{1}{2}i\,(1-\sqrt{3}\,i) = \frac{1}{2}i + \frac{\sqrt{3}}{2} = r$

$\therefore R\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is represented by $r = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ (1)

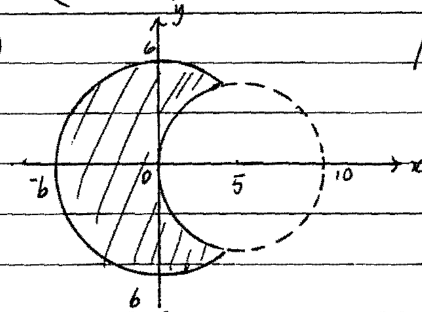
$\vec{OQ} = \vec{OP} + \vec{OR} = (1-\sqrt{3}i) + \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= \left(1+\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}-\sqrt{3}\right)i = q$ (1)

$\therefore$ Q is represented by $q = \left(1+\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}-\sqrt{3}\right)i$

Question 5 (7 marks) **BOS#:** ____________________

a) i)

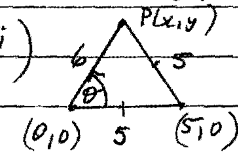


$|z|\le 6$ and $|z-5|\ge 5$

(1) - boundaries

(1) - shaded region

ii)



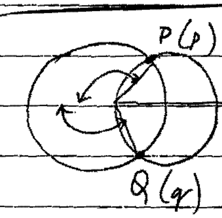
P(x,y) - pt. of intersection between circles

$\cos\theta = \frac{6^2+5^2-5^2}{2\times 6\times 5} = \frac{3}{5}$ (1) $\therefore\ \theta = \cos^{-1}\left(\frac{3}{5}\right) \doteq 53^\circ 7' 48.37''$ or other rounding off

$\therefore\ \cos^{-1}\left(\frac{3}{5}\right) \le \arg z \le \pi$, (1) $-\pi < \arg z \le -\cos^{-1}\left(\frac{3}{5}\right)$

OR $53^\circ 7' 48.37' \le \arg z \le 180^\circ$, $-180^\circ < \arg z \le -53^\circ 7' 48.37'$ → ignore if not stated cause of rounding

OR



pts of int. P(x,y) Q(x,y) $\begin{cases} x^2+y^2 = 6^2 \\ (x-5)^2 + y^2 = 25 \end{cases}$ $x = 3.6 = \frac{18}{5}$ $y = \pm 4.8$ $y = \pm\frac{24}{5}$

$\therefore\ P(3.6, 4.8)$ (1) $Q(3.6, -4.8)$

$\therefore\ p = 3.6 + 4.8i$ $q = 3.6 - 4.8i$

$\therefore\ \arg p = \tan^{-1}\frac{4.8}{3.6}$ $\arg q = -\tan^{-1}\frac{4}{3}$

$\arg p = \tan^{-1}\frac{4}{3}$

$\therefore\ \tan^{-1}\frac{4}{3} \le \arg z \le \pi$, $-\pi < \arg z \le -\tan^{-1}\frac{4}{3}$ (1)

OR

$53.1301...^\circ \le \arg z \le 180^\circ$ $-180^\circ < \arg z \le -53.1301...^\circ$

$0.927... \le \arg z \le \pi$, $-\pi < \arg z < -0.927295.$

5b) $\arg(z-2) = \arg(z^2)$

~~restriction: $z \neq 2$~~, $z \neq 0$

$\therefore$ algebraically: let $z = x+iy$ $\quad z-2 = x-2+iy$

$z^2 = x^2 - y^2 + 2xyi$

$\arg(z-2) = \tan^{-1}\dfrac{y}{x-2}$ and $\arg z^2 = \tan^{-1}\dfrac{2xy}{x^2-y^2}$

$\therefore \tan^{-1}\dfrac{y}{x-2} = \tan^{-1}\dfrac{2xy}{x^2-y^2}$

$\therefore \dfrac{y}{x-2} = \dfrac{2xy}{x^2-y^2}$ (1) $\quad \div y$ $\quad$ case (1) $y \neq 0$ $\quad$ case (2) $y = 0$

case (1): $\dfrac{1}{x-2} = \dfrac{2x}{x^2-y^2}$

$x^2 - y^2 = 2x^2 - 4x$ $\qquad$ since $z \neq 0$, $y \neq 0$, exclude $(0,0)$

$+4 + 0 = x^2 - 4x + y^2 + 4$

$4 = (x-2)^2 + y^2$ $\therefore$ circle centre $(2,0)$ $r=2$ (1)

case (2) $y = 0 \therefore x = 2$, but $z \neq 2 \therefore (2,0)$ not part of locus

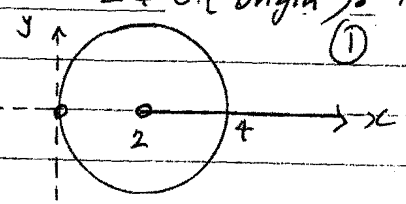
$y = 0 \therefore x > 2$ is o.k. $\quad$ ray

$\therefore$ locus is a circle centre $(2,0)$, $r = 2$

where $z \neq 0$ (excluding origin). The other part of locus is a ray (1)

~~purely real z~~ where $x > 2$

(since $z \neq 2$)



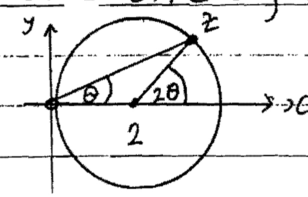
5b) geometrically

let $z = |z| \operatorname{cis}\theta \quad \therefore \arg z = \theta$

since $\arg(z^2) = 2 \times \arg z = 2 \times \theta$

$\therefore$ if $\arg(z-2) = \arg z^2 = 2 \times \arg z$ (1)

locus of z – circle, centre $(2,0)$, radius $= 2$



since angle at the centre is twice the angle at the circumf. on same arc

$\therefore$ locus $\therefore (x-2)^2 + y^2 = 4$ (1) excluding $(0,0)$

(1) $y = 0$ $\quad$ ray $x > 2$ $\quad$ (for z – purely real when $x > 2$)

restrictions $\therefore z \neq 0,\ z \neq 2$

OR