BAULKHAM HILLS HIGH SCHOOL

2014
HSC Assessment Task 1

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work
- Attempt all questions. Use the answer booklet provided.
- Answer each question on the appropriate page

Total marks –46
Exam consists of 5 pages.

This paper consists of TWO sections.

Section 1 – Pages 2-3
Multiple Choice
Question 1-5 (5 marks)

Section 2 – Pages 3-5
Extended Response
Question 6-10 (41 marks)

Section I - 5 marks

Answer in the table provided in the answer booklet

1. The argument of iz where $z = 1 + i$ is

(A) $-\frac{\pi}{4}$

(B) $\frac{3\pi}{4}$

(C) $\frac{5\pi}{4}$

(D) $-\frac{3\pi}{4}$

2. If w is a non-real cube root of unity, the value of $\frac{1}{1+w^2} + \frac{1}{1+w}$ is equal to

(A) -1

(B) 0

(C) 1

(D) none of the above

3. $\arg(z) + \arg(\bar{z})$ equals

(A) 0

(B) $n\pi$

(C) $-n\pi$

(D) $\frac{\pi}{4}$

4. Consider the two statements below, for any complex number z with $|z| = 1$:

I. $z + \frac{1}{z} = 2Re(z)$

II. $z - \frac{1}{z} = 2Im(z)$

Which of these statements is always true?

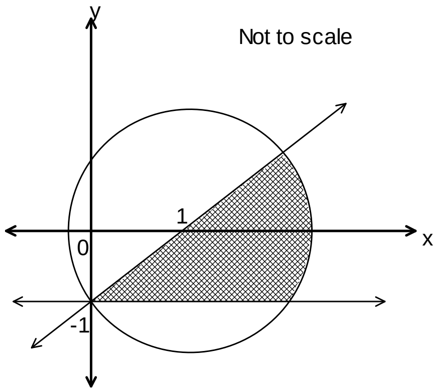
(A) Only statement I

(B) Only statement II

(C) Both statements

(D) Neither of the statements

5. The centre of the circle below is (1,0). Which inequalities define the shaded area?



(A) $|z-1| \leq \sqrt{2}$ and $0 \leq \arg(z-i) \leq \frac{\pi}{4}$

(B) $|z-1| \leq \sqrt{2}$ and $0 \leq \arg(z+i) \leq \frac{\pi}{4}$

(C) $|z-1| \leq 1$ and $0 \leq \arg(z-i) \leq \frac{\pi}{4}$

(D) $|z-1| \leq 1$ and $0 \leq \arg(z+i) \leq \frac{\pi}{4}$

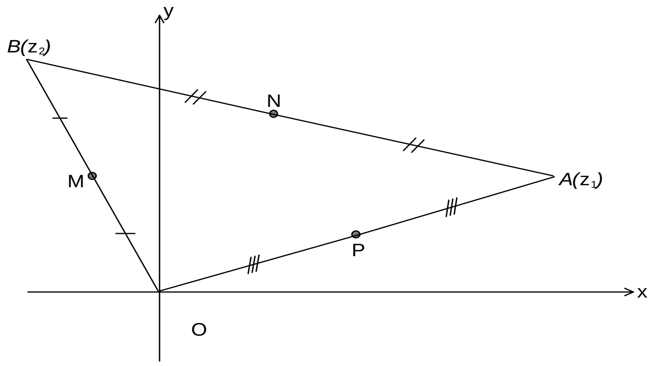
End of Section I

Section II – Extended Response

Attempt questions 6-10.
Answer each question on the appropriate page of the answer booklet, showing all necessary working.

Question 6 (9 marks)		**Marks**
(a)	Let $z = 1 + i$ and $w = 4 - 2i$, find	
	(i) zw	**1**
	(ii) $z + iw$	**1**
	(iii) $\left\|\frac{w}{z}\right\|$	**1**
(b)	(i) Find all the complex numbers $z = a + ib$ such that $(a+ib)^2 = 5 - 12i$, where a, b are real numbers.	**3**
	(ii) Hence solve $z^2 - (1-4i)z - (5-i) = 0$	**3**
	End of Question 6	

Question 7 (9 marks)		Marks
a)	If $z = \sqrt{3} + i$ and $w = 1 - i$,	
	(i) Write $\frac{z}{w}$ in the form $a+ib$ where a and b are real numbers.	2
	(ii) Write $\frac{z}{w}$ in mod-arg form	2
	(iii) Hence find the exact value of $\sin\frac{5\pi}{12}$	1
b)	z satisfies $\lvert z - 2i \rvert = 1$ and the point P represents z on an Argand diagram.	
	(i) Sketch the locus of P	1
	(ii) Find the maximum and minimum values of $\arg z$, where $-\pi < \arg z < \pi$	2
	(iii) Find the value of z when $\arg z$ takes its maximum value, expressing your answer in modulus-argument form	1
	End of Question 7	

Question 8 (7 marks)		Marks
a)	w is a complex cube root of unity (ie w is a root of $z^3 = 1$)	
	(i) Prove that w^2 is also a cube root of unity	1
	(ii) Evaluate $(1 + w)^3$, showing all working	2
b)		
	ΔABO lies on an Argand diagram. Points A and B represent the complex numbers z_1 and z_2 respectively. M, N and P are the midpoints of OB, AB and OA respectively.	
	(i) Which complex number is represented by N?	1
	(ii) Express $\overrightarrow{AM}$ and $\overrightarrow{BP}$ in terms of z_1 and z_2	2
	(iii) Hence simplify $\overrightarrow{ON} + \overrightarrow{AM} + \overrightarrow{BP}$	1
	End of Question 8	

Question 9 (8 marks)		**Marks**
(i)	Solve $z^5 + 1 = 0$ by using De Moivre's theorem (Leave your answer in modulus-argument form)	**2**
(ii)	Explain why the solutions of $z^4 - z^3 + z^2 - z + 1 = 0$ are the non-real solutions of $z^5 + 1 = 0$	**1**
(iii)	Show that if $z^4 - z^3 + z^2 - z + 1 = 0$ where $z = \text{cis}\theta$, then $4\cos^2\theta - 2\cos\theta - 1 = 0$	**3**
(iv)	Hence find the exact value of $\cos\frac{3\pi}{5}$	**2**
	End of Question 9	

Question 10 (8 marks)		
a)	The region R in the Argand diagram is defined by $\|z - 1\| \leq \|z - i\|$ and $\|z - 2 - 2i\| \leq 1$	
	(i) Sketch the region R	**3**
	(ii) If z lies on the **boundary** of region R, and $\arg(z - 1) = \frac{\pi}{4}$, find z in the form $a + ib$	**3**
b)	If $z = r\text{cis}\theta$, show that $\frac{z^2 - r^2}{z}$ is purely imaginary and state its value in terms of r and θ	**2**

End of Task

X2 HSC Assess. Task 1 Dec. 2014 SOLUTIONS.

SECTION I.

1. $\arg z = \frac{\pi}{4}$

$\arg iz = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$ (B)

2. $\omega^3 = 1$, $1 + \omega + \omega^2 = 0$

$\frac{1}{1+\omega^2} + \frac{1}{1+\omega} = \left(\frac{1}{-\omega} + \frac{1}{-\omega^2}\right) \times \omega^3$

$= -\omega^2 - \omega$

$= 1$ (C)

3. o (A)

4. I is true

II is false (should be $2i \cdot \text{Im}\, z$)

$\therefore$ (A)

5. Circle: $r = \sqrt{2}$, centre $(1,0)$

$|z-1| \le \sqrt{2}$

Arg from $(0,-1)$: between 0 and $\pi/4$

$0 \le \arg(z+i) \le \frac{\pi}{4}$

(B)

II.

Q6.

a) i) $zw = (1+i)(4-2i)$

$= 4 - 2i + 4i + 2$

$= 6 + 2i$ (1)

ii) $z + iw = 1 + i + i(4-2i)$

$= 1 + i + 4i + 2$

$= 3 + 5i$ (1)

iii) $\left|\frac{w}{z}\right| = \frac{|w|}{|z|} = \frac{\sqrt{20}}{\sqrt{2}} = \sqrt{10}$ (1)

b) i) $a^2 - b^2 + 2iab = 5 - 12i$ (1)

$a^2 - b^2 = 5$, $ab = -6$ (1) $\therefore a = 3, b = -2$ or $a = -3, b = 2$

$\therefore z = 3-2i, \ -3+2i$ (1)

ie. $z = \pm(3-2i)$

ii) $z = \frac{1-4i \pm \sqrt{(-(1-4i))^2 - 4.1.-(5-i)}}{2}$

$= \frac{1-4i \pm \sqrt{-15-8i+20-4i}}{2}$

$= \frac{1-4i \pm \sqrt{5-12i}}{2}$ (1)

$= \frac{1-4i \pm (3-2i)}{2}$

$= \frac{4-6i}{2}, \frac{-2-2i}{2}$

$= 2-3i, \ -1-i$ (2) ← 1 each

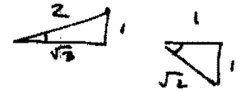
Q7.

a) i) $\frac{z}{w} = \frac{\sqrt{3}+i}{1-i} \cdot \frac{1+i}{1+i}$ (1) realise den.

$= \frac{\sqrt{3} + \sqrt{3}i + i - 1}{2}$

$= \frac{\sqrt{3}-1}{2} + \frac{\sqrt{3}+1}{2} \cdot i$ (1)

ii) $\frac{z}{w} = \frac{2 \text{ cis } \pi/6}{\sqrt{2} \text{ cis } -\pi/4}$ (1)

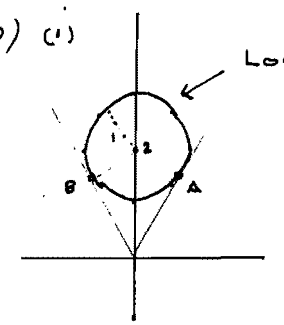


$= \sqrt{2} \text{ cis } \frac{5\pi}{12}$ (1)

iii) $\sqrt{2} \sin \frac{5\pi}{12} = \text{Im}\left(\frac{z}{w}\right) = \frac{\sqrt{3}+1}{2}$.

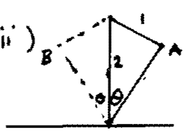
$\therefore \sin \frac{5\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$. (1)

b) (i)



Locus of P (circle). (1)

(ii)



$\sin \theta = 1/2$

$\therefore \theta = \pi/6$ —— (1)

Max arg $z = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$ (at B)

Min arg $z = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$. (at A) } (1)

(iii) $OB = \sqrt{3}$ (by Pythagoras)

At B, $z = \sqrt{3} \text{ cis } \frac{2\pi}{3}$ (1)

Q8.

a)(i) $w^3 = 1$ since w is a root of $z^3 = 1$

$(w^2)^3 = (w^3)^2 = 1^2 = 1$ (1)

$\therefore w^2$ is also a root.

(ii) $(1+w)^3 = (-w^2)^3$ (1) for $(-w^2)$

$= -w^6$

$= -(w^3)^2$

$= -1$ (1)

OR $(1+w)^3 = 1 + 3w + 3w^2 + w^3$

$= 1 + 3(w + w^2) + 1$ (1) binomial + $w^3 = 1$

$= 2 + 3(-1)$

$= -1$. (1)

b) i) $N = \frac{z_1 + z_2}{2}$ (1)

ii) $\vec{AM} = \frac{z_2}{2} - z_1$ (1)

$\vec{BP} = \frac{z_1}{2} - z_2$ (1)

iii) $\vec{ON} + \vec{AM} + \vec{BP}$

$= \frac{z_1 + z_2}{2} + \frac{z_2}{2} - z_1 + \frac{z_1}{2} - z_2$

$= \frac{z_1 + z_2 + z_2 - 2z_1 + z_1 - 2z_2}{2}$

$= 0$. (1)

Q9.

i) Let $z = r \operatorname{cis}\theta$ and $z^5 = -1$

$r^5 \operatorname{cis} 5\theta = \operatorname{cis}\pi$ (by De Moivre's theorem)

$r^5 = 1$ and $5\theta = \pi + 2k\pi$ (k = integer)

$r = 1$ $\quad \theta = \frac{(2k+1)\pi}{5}$ — (1)

$\therefore z = \operatorname{cis}\frac{\pi}{5}, \operatorname{cis}\frac{3\pi}{5}, \operatorname{cis}\pi, \operatorname{cis}\frac{7\pi}{5}, \operatorname{cis}\frac{9\pi}{5}$

or (1)

$z = \operatorname{cis}\left(\pm\frac{\pi}{5}\right), \operatorname{cis}\left(\pm\frac{3\pi}{5}\right), -1.$

ii) $z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1) = 0$

when $z + 1 = 0$ or $z^4 - z^3 + z^2 - z + 1 = 0$

$z = -1$ (the real soln of $z^5 + 1 = 0$)

The soln's will be the non-real solns of $z^5 + 1 = 0$. (1)

iii) If $z^4 - z^3 + z^2 - z + 1 = 0$ (divide by z^2)

then $z^2 - z + 1 - \frac{1}{z} + \frac{1}{z^2} = 0$

$\left(z^2 + \frac{1}{z^2}\right) - \left(z + \frac{1}{z}\right) + 1 = 0$ (1)

Since $|z| = 1 \therefore z + \frac{1}{z} = 2\cos\theta$, $z^2 + \frac{1}{z^2} = 2\cos 2\theta$ (1)

$\therefore 2\cos 2\theta - 2\cos\theta + 1 = 0$

$2(2\cos^2\theta - 1) - 2\cos\theta + 1 = 0$ (1)

$4\cos^2\theta - 2\cos\theta - 1 = 0$

iv) $z = \operatorname{cis}\frac{3\pi}{5}$ is one of the solutions of $z^4 - z^3 + z^2 - z + 1 = 0$ (from ii)

$\therefore \theta = \frac{3\pi}{5}$ is one of the solns of $4\cos^2\theta - 2\cos\theta - 1 = 0$

$\therefore \cos\theta = \frac{2 \pm \sqrt{4 - 4.4.-1}}{8}$

$= \frac{2 \pm \sqrt{20}}{8}$ (1)

$= \frac{2(1 \pm \sqrt{5})}{8}$

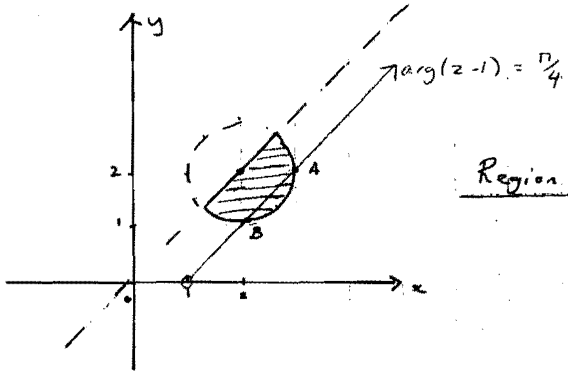
$= \frac{1 \pm \sqrt{5}}{4}$

Now $\frac{3\pi}{5}$ is obtuse $\therefore \cos\frac{3\pi}{5} < 0$

$\therefore \cos\frac{3\pi}{5} = \frac{1 - \sqrt{5}}{4}$. (1)

Q10.

a) (i)



Region R shaded.

(1) – circle
(1) – line
(1) – shading

(ii) $\arg(z-1) = \frac{\pi}{4} \Rightarrow y = x-1$ in Cartesian form ①

$|z-2-2i| = 1 \Rightarrow (x-2)^2 + (y-2)^2 = 1$ " " " ②

z is given by the two points of intersection A and B.

$(x-2)^2 + (x-3)^2 = 1$

$x^2 - 4x + 4 + x^2 - 6x + 9 = 1$

$2x^2 - 10x + 12 = 0$ (1)

$2(x^2 - 5x + 6) = 0$

$2(x-3)(x-2) = 0$

$x = 3$, $y = 2$ $\qquad$ $x = 2$, $y = 1$

$\therefore z = 3+2i$ (1), $2+i$ (1)

b) $\frac{z^2 - r^2}{z} = \frac{z^2 - z\bar{z}}{z}$ (1) ... use $z\bar{z}$

$= z - \bar{z}$

$= 2i \cdot \text{Im}(z)$

$= 2i \cdot r\sin\theta$. (1)

OR

$z - \frac{z\bar{z}}{z} = z - \bar{z}$ (1) use $z\bar{z}$

$= 2i \cdot \text{Im}(z)$

$= 2i \cdot r\sin\theta$ (1)

OR

$\frac{z^2 - r^2}{z} = \frac{r^2 \operatorname{cis} 2\theta - r^2}{r \operatorname{cis}\theta}$

$= \frac{r^2(\operatorname{cis} 2\theta - 1)}{r \operatorname{cis}\theta}$

$= r\left(\frac{\operatorname{cis} 2\theta}{\operatorname{cis}\theta} - \frac{1}{\operatorname{cis}\theta}\right)$

$= r(\operatorname{cis}\theta - \operatorname{cis}(-\theta))$ ... (1)

$= r(\cos\theta + i\sin\theta - (\cos\theta - i\sin\theta))$

$= r(\cos\theta + i\sin\theta - \cos\theta + i\sin\theta)$

$= r \cdot 2i \cdot \sin\theta$

$= 2i \cdot r \cdot \sin\theta$ } (1)