BAULKHAM HILLS HIGH SCHOOL

2015

HSC Assessment Task 1

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 50 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work
- Attempt all questions.
  Use the answer booklet provided.
- Answer each question on the appropriate page

Total marks –42
Exam consists of 4 pages.

This paper consists of TWO sections.

Section 1 – Page 2
Multiple Choice
Question 1-3 (3 marks)

Section 2 – Pages 3-5
Extended Response
Question 4-6 (39 marks)

Section I - 5 marks

Answer in the table provided in the answer booklet

1. What value of z satisfies $z^2 = 7 - 24i$?

(A) $3 - 4i$

(B) $-3 - 4i$

(C) $4 - 3i$

(D) $-4 - 3i$

2. Which of the following is true for all complex numbers of z?

(A) $\text{Im}(z) = \frac{z+\bar{z}}{2}$

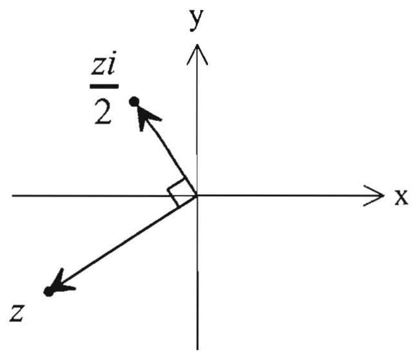
(B) $\text{Im}(z) = \frac{z-\bar{z}}{2}$

(C) $\text{Im}(z) = \frac{z+\bar{z}}{2i}$

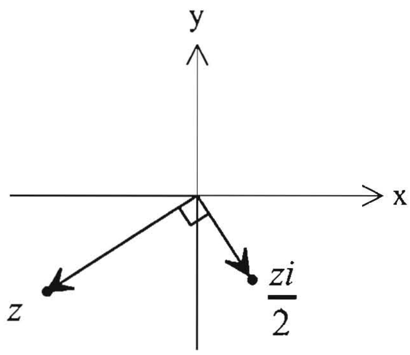
(D) $\text{Im}(z) = \frac{z-\bar{z}}{2i}$

3. Which Argand diagram could show the complex numbers z and $\frac{zi}{2}$?

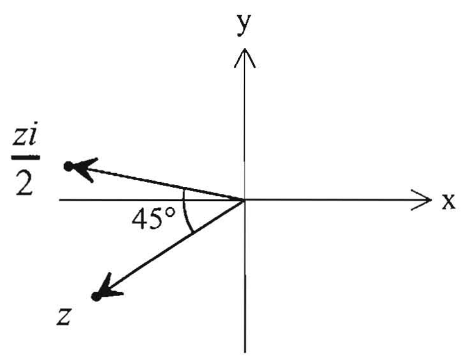
(A)



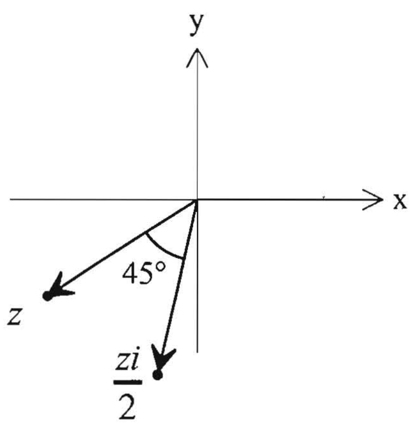
(B)



(C)



(D)



End of Section I

Section II – Extended Response

Attempt questions 6-10.
Answer each question on the appropriate page of the answer booklet, showing all necessary working.

Question 4 (12 marks) **Marks**

(a) (i) Express $1 + i$ in modulus argument form **2**

(ii) Hence evaluate $(1 + i)^{12}$ **2**

(b) Let $z = 3 + i$ and $w = 1 - i$. Find, in the form $x + iy$,

(i) $2z + iw$. **2**

(ii) $\overline{z}w$. **2**

(iii) $\frac{6}{w}$ **2**

(c) Find real numbers x and y such that $(1 + i)x + (2 - 3i)y = 10$ **2**

End of Question 4

Question 5 (13 marks)

a) On an Argand diagram, the points P, Q, R represent the complex roots $1, w$ and w^2. **3**
If $1, w$ and w^2 are the 3 roots of $z^3 - 1 = 0$.
Find the area of the triangle ΔPQR.

b) Let ω be one of the non-real roots of the equation $z^3 + 27 = 0$

(i) Factorise the cubic polynomial $z^3 + 27$ over the field of real numbers. **1**

(ii) Show that $\omega^2 - 3\omega + 9 = 0$ **1**

(iii) Hence find the value of $\left(\frac{\omega^2}{3} + 3\right)^6$ **3**

c) Sketch the following loci on separate argand diagrams

(i) $\text{Re}\left(\frac{z-4}{z}\right) = 0$ **3**

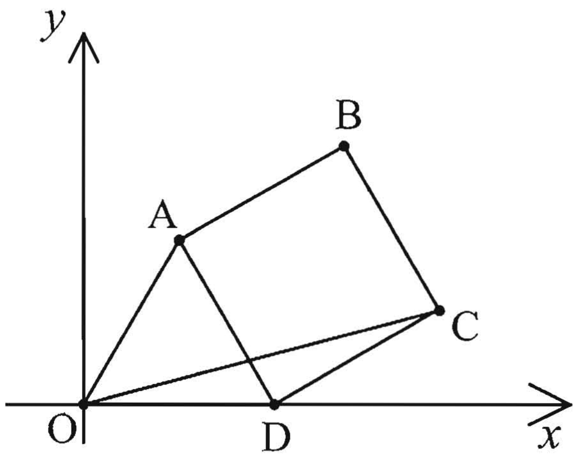
(ii) $\frac{3\pi}{4} \leq \arg(z - 1) \leq \pi$ and $|z - 1 - i| \leq 1$ **3**

End of Question 5

Question 6 **(14 marks)** **Marks**

a) It is given that $ABCD$ is a square and ABO is an equilateral triangle. **3**

If the vector $\overrightarrow{CB} = -1 + i\sqrt{3}$, find the complex number represented by the point C.



b) (i) Solve for $z^4 = 1$ for all z **1**

(ii) Hence or otherwise, solve $z^4 = (z-1)^4$ **3**

c) (i) Show that the roots $z^5 + 1 = 0$ on a unit circle in an Argand diagram. **2**

(ii) Factor $z^5 + 1$ into irreducible quadratic and linear factors with real coeffiecients **2**

(iii) Deduce that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$ and $\cos\frac{\pi}{5}\cos\frac{3\pi}{5} = -\frac{1}{4}$ **2**

(iv) Write a quadratic equations with integer coefficients which has roots $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$. **1**

End of Exam

BAULKHAM HILLS HIGH SCHOOL
EXTENSION 2 TRIAL HSC 2011 SOLUTIONS

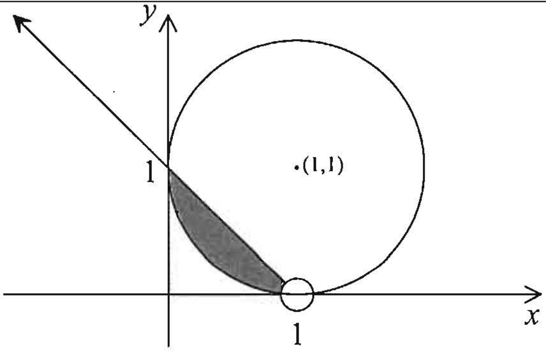
Solution	Marks	Comments
Section 1		
1 C - By expanding looking $2abi$ term and then $a^2 - b^2$	1	
2 D - $\frac{x+iy-(x-iy)}{2i}$	1	
3 B - $z \times \frac{i}{2} = z \times i \times \frac{1}{2}$, Multiplying by i rotates counter clockwise 90° Multiplying by $\frac{1}{2}$, halves the modulus	1	

	Solution	Marks	Comments
QUESTION 4			
4a (i)	$\lvert z\rvert = \sqrt{1^2+1^2} = \sqrt{2}$ $\arg(z) = \tan^{-1}\left(\frac{1}{1}\right) = 45°$ or $\frac{\pi}{4}$ $\therefore 1 + i = \sqrt{2}\text{ cis } 45°$	2	**2 marks** • equivalent correct expressions **1 mark** • correct modulus • correct argument
4a (ii)	$(1+i)^{12} = \left(\sqrt{2}\text{cis } 45°\right)^{12}$ $= 64\text{cis}180°$ $= -64$	2	**2 marks** • correct value **1 mark** • correct use of DeMoirves Theorem • simplified to a correct principle argument
4b (i)	$2z + iw = 2(3+i) + i(1-i)$ $= 6 + 2i + i - i^2$ $= 6 + 2i + i - i^2$ $= 7 + 3i$	2	**2 marks** • correct answer **1 mark** • correctly simplifies $i^2 = -1$ • correct expansion and simplifes
4b (ii)	$(3-i)(1-i) = 3 - 3i - i + i^2$ $= 2 - 4i$	1	**2 marks** • correct answer **1 mark** • correctly simplifies $i^2 = -1$ • correct expansion and simplifes
4c	$(1+i)x + (2-3i)y = 10$ $x + xi + 2y - 3i = 10$ Equating Real: $x + 2y = 10$……(1) Equating Imaginary: $x - 3y = 0$ $x = 3y$……(2) Sub (2) into (1): $3y + 2y = 10$ $y = 2$ Sub y → (2): $x = 6$ $\therefore x = 6, y = 2$	2	**2 marks** • correct answer **1 mark** • Equates Real • Equates imaginary

	Solution	Marks	Comments
QUESTION ~~7~~ 5			
7a	Method 1: Since the 3 roots are equally spaced then the angle between each root is 120°. $Area = 3\left(\frac{1}{2}\times 1\times 1\times \sin 120°\right)$ $Area = \frac{3\sqrt{3}}{4}$	2	**3 marks** • Correct solution **2 mark** • Significant progress towards solution with good reasoning **1 mark** • Progress towards solution with good reasoning
7b (i)	$z^3 + 27 = 0$ $(z+3)(z^2 - 3z + 9) = 0$	1	**1 mark** • Correct solution
7b (ii)	$since\ \omega\ is\ a\ root\ of\ z^3 + 27 = 0$ $then\ \omega^3 + 27 = 0$ $(\omega + 3)(\omega^2 - 3\omega + 9) = 0$ $Since\ \omega\ is\ non\text{-}real,\ then\ \omega \neq -3$ $\omega^2 - 3\omega + 9 = 0$ $\omega^2 + 9 = 3\omega$	1	**1 mark** • Correct solution
7b (iii)	$\omega^2 + 9 = 3\omega$ $\frac{\omega^2}{3} + 3 = \omega \quad (divided\ both\ sides\ by\ 3)$ $\left(\frac{\omega^2}{3} + 3\right)^6 = \omega^6 \quad (power\ of\ 6\ on\ both\ sides)$ $\left(\frac{\omega^2}{3} + 3\right)^6 = (\omega^3)^2$ $= (-27)^2 \quad (since\ \omega^3 = -27)$ $= 729$	3	**3 marks** • Correct solution **2 mark** • Manipulates • Equates imaginary **1 mark** • Correctly simplifies $\omega^3 = -27$ • Attempts to manipulate $\omega^2 - 3\omega + 9 = 0$
7c	$\|CB\| = \sqrt{1^2 + \sqrt{3}^2} = 2$ *AB = BC = CD = DA (sides of a square are equal)* *AD = DO = OA (sides of an equilateral triangle are equal)* $\therefore OD = 2$ $\overrightarrow{DC} = \overrightarrow{CB} \times -i$ $\overrightarrow{DC} = -i(-1 + i\sqrt{3})$ $\overrightarrow{DC} = \sqrt{3} + i$ $\overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{DC}$ $\overrightarrow{OC} = 2 + \sqrt{3} + i$	3	**3 marks** • Correct solution **2 mark** • Significant progress towards solution with good reasoning **1 mark** • Finds modulus of *CB* • Uses i to rotate to the correct direction

QUESTION ~~8~~ 6			
8a (i)	Let $z = x + iy$ $Re\left(\frac{x + iy - 4}{x + iy}\right) = 0$ $Re\left(\frac{(x - 4 + iy)(x - iy)}{(x + iy)(x - iy)}\right) = 0$ $Re\left(\frac{x(x-4) + y^2 + i(yx + y(x-4))}{x^2 + y^2}\right) = 0$ $\frac{x^2 - 4x + y^2}{x^2 + y^2} = 0$ $(x-2)^2 + y^2 = 4$ $\therefore it\ is\ a\ circle\ with\ centre\ (0,2) and\ radius\ 2$	2	**2 mark** • Correct solution and diagram **1 mark** • Progress towards solution with good reasoning • Uses $z = x + iy$ correctly • Extracts the real part of an expression

<table>
<tr><td>8a (ii)</td><td>



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3 mark
- Correct solution showing open circle and showing intersection at (1,0) and (0,1)

2 mark
- Understands both loci

1 mark
- Draws of the loci correctly.

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<tr><td>8b (i)</td><td>

$z = \pm 1, \pm i$

</td><td>1</td><td>

1 mark
- Correct solution

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<tr><td>8b (ii)</td><td>

$$z^4 = (z-1)^4$$
$$\frac{z^4}{(z-1)^4} = 1$$
$$\left(\frac{z}{z-1}\right)^4 = 1$$

$$\frac{z}{z-1} = \pm 1$$
$$z = \pm(z-1)$$

$z = +(z-1)$ | $z = -(z-1)$, $2z = +1$, $z = \frac{1}{2}$

$$\frac{z}{z-1} = \pm i$$

$z = +iz - i$, $z(1-i) = -i$, $z = \frac{-i}{1-i}$, $z = \frac{1-i}{2}$

$z = -iz + i$, $z(1+i) = i$, $z = \frac{i}{1+i}$, $z = \frac{1+i}{2}$

(or just realising that real coefficients will give roots in conjugate pairs)

$$\therefore z = \frac{1}{2}, \frac{-1 \pm i}{2}$$

(Note: there are only 3 roots as when you expand z^4 will cancel out)

</td><td>3</td><td>

3 mark
- Correct solution

2 mark
- Has 3 correct equations to solve for z.

1 mark
- Manipulates to an appropriate equation so solve.
- Solves for $\frac{z}{z-1}$ or $\frac{z-1}{z}$

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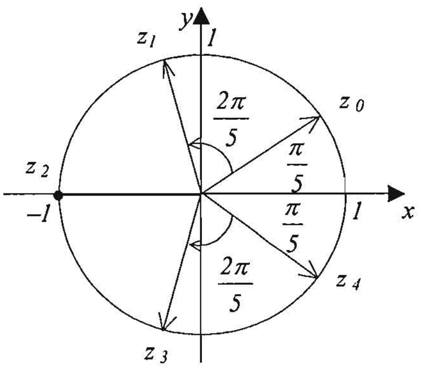
<tr><td>8c (i)</td><td>

$z^5 = -1$, $z^5 = cis(\pi + 2n\pi)$, where $n = 0, 1, 2, 3, 4$

$z = cis\left(\frac{\pi}{5} + \frac{2n\pi}{5}\right)$, $z_0 = cis\frac{\pi}{5}$

$z_1 = cis\frac{3\pi}{5}$, $z_2 = cis\pi = -1$, $z_3 = cis\frac{7\pi}{5} = cis\left(\frac{-3\pi}{5}\right)$,

$z_4 = cis\frac{9\pi}{5} = cis\left(-\frac{\pi}{5}\right)$



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2 marks
- correct solution

1 mark
- Finds roots only.
- Draws diagram only

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8c (ii)	$z^5+1=(z-z_0)(z-z_1)(z-z_2)(z-z_3)(z-z_4)$ $=[(z-z_0)(z-z_4)][(z-z_1)(z-z_3)](z+1)$ $z_0+z_4=cis\frac{\pi}{5}+cis\frac{-\pi}{5}=2\cos\frac{\pi}{5}$ $z_1z_4=1,$ $z_1+z_3=cis\frac{3\pi}{5}+cis\left(\frac{-3\pi}{5}\right)=2\cos\frac{3\pi}{5},$ $z_1z_3=1$ $z^5+1=(z+1)\left(z^2-2z\cos\frac{\pi}{5}+1\right)\left(z^2-2z\cos\frac{3\pi}{5}+1\right)$	2	**2 marks** • correct solution **1 mark** • Correctly simplifies z_1+z_2. Correctly simplifies z_1z_2.
8c (iii)	Substitute $z=i$: $(i+1)=(i+1)\left(-2i\cos\frac{\pi}{5}\right)\left(-2i\cos\frac{3\pi}{5}\right)$ $1=-4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$ $\cos\frac{\pi}{5}\cos\frac{3\pi}{5}=-\frac{1}{4}$ Substitute $z=1$: $2=2(2-2\cos\frac{\pi}{5})(2-2\cos\frac{3\pi}{5})$ $\frac{1}{4}=(1-\cos\frac{\pi}{5})(1-\cos\frac{3\pi}{5})$ $\frac{1}{4}=1-(\cos\frac{\pi}{5}+\cos\frac{3\pi}{5})+\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$ $\frac{1}{4}=1-(\cos\frac{\pi}{5}+\cos\frac{3\pi}{5})-\frac{1}{4}$ $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$	2	**2 marks** • correct solution **1 mark** • Proves one of the identities.
8c (iv)	$sum\ of\ roots=\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$ $product\ of\ roots=\cos\frac{\pi}{5}\cos\frac{3\pi}{5}=-\frac{1}{4}$ $x^2-\frac{b}{a}x+\frac{c}{a}=0$ $x^2-\frac{1}{2}x-\frac{1}{4}=0,$ $4x^2-2x-1=0$	1	**1 marks** • correct solution