STION 1 (13 marks)

(a) Express $z_1 = 2+2i$ and $z_2 = -\sqrt{3}+i$ in mod arg form and use to evaluate $\frac{z_1^3}{z_2^2}$ [3]

(leave answer in mod arg form)

(b) If $z = x+iy$ show $z + \frac{|z|^2}{z} = 2\,Re\,z$ [3]

(c) Sketch the region in the A
rgand diagram defined by

i) $\frac{\pi}{3} \le Arg\,z \le \frac{2\pi}{3}$ [2]

ii) $|z-i| = |z|$ [2]

(d) Solve for z (giving answer in the form a+bi.) $z(1+i) = 2+6i$ [3]

QUESTION 2 (13 marks)

(a) If $z = 2-i$. Find the real numbers p and q such that $pz + \frac{q}{z} = 1$ [3]

(b) i) Find the roots of $z^5 + 1 = 0$ and show their position on a unit circle in an Argand diagram. [3]

ii) Since the sum of these roots add to zero show $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$ [2]

(c) The locus of a point P(x,y), which moves in the complex plane is represented by the equation $|z-3i| = 2$

i) Draw this locus, give a geometrical description and hence write down its Cartesian equation [2]

ii) Show that the minimum value of arg z is $\cos^{-1}\left(\frac{2}{3}\right)$ and find the modulus of z when P is in the position of minimum argument. [3]

QUESTION 3 (15 marks)

(a) i) Using the fact that if $z = \cos\vartheta + i\sin\vartheta$, then $z^n + z^{-n} = 2\cos n\vartheta$, show $\cos^3\vartheta = \frac{1}{4}(\cos 3\vartheta + 3\cos\vartheta)$ [2]

ii) Use the above to evaluate $\int_0^{\frac{\pi}{3}} \cos^3\vartheta \, d\vartheta$

(b)) If z=a+bi where $a^2 + b^2 \neq 0$

i) Show that if Im(z)>0 then $\text{Im}\left(\frac{1}{z}\right) < 0$ [2]

ii) Prove that $\left|\frac{1}{z}\right| = \frac{1}{|z|}$ [2]

(c) z_1 and z_2 are complex numbers such that $|z_1| = 1$ and $|z_2| = 4$ and $\arg\frac{z_2}{z_1} = \frac{2\pi}{3}$

i) Show $\left|\sqrt{z_1} - \sqrt{z_2}\right| = \sqrt{3}$ [3]

ii) If arg $\sqrt{z_1}$ = α and (0 < α < $\frac{\pi}{2}$) find arg $(\sqrt{z_1} - \sqrt{z_2})$ in terms of α

[2]

iii) Explain, with the aid of a diagram, why there are two solutions to part (ii) if $0 \leq \alpha \leq 2\pi$ [1]

Solutions Ext. 2 term 4 2002

Question 1.

a) $z_1 = 2\sqrt{2}\,\text{cis}\,\pi/4, \quad z_2 = 2\,\text{cis}\,5\pi/6$ ✓ [3]

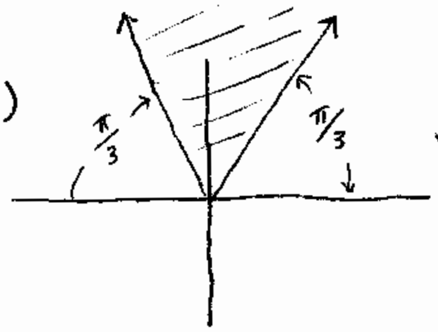
$$\frac{z_1^3}{z_2^2} = \frac{(2\sqrt{2})^3\,\text{cis}\,3\pi/4}{2^2\,\text{cis}\,10\pi/6} = 2\sqrt{2}\,\text{cis}\left(\frac{9\pi}{12} - \frac{20\pi}{12}\right) = 4\sqrt{2}\,\text{cis}\,\frac{-11\pi}{12}$$ ✓✓

b) $$z + \frac{|z|^2}{z} = x + yi + \frac{x^2+y^2}{x+iy} = \frac{(x+iy)^2 + x^2 + y^2}{x+iy}$$ ✓ [3]

$$= \frac{x^2 + 2ixy - y^2 + x^2 + y^2}{x+iy} = \frac{2x^2 + 2ixy}{x+iy} = \frac{2x(x+iy)}{x+iy} = 2x$$ ✓

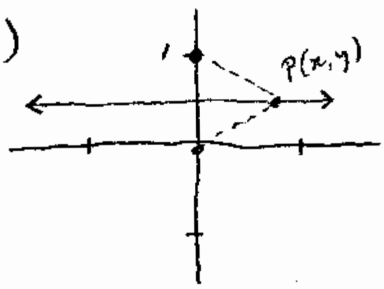
since $\text{Re}\, z = x$, $z + \frac{|z|^2}{z} = 2Rz$ ✓

c) (i)



✓ ✓

(ii)



✓ ✓

d) $z = x + yi$ [3]

$(x+yi)(1+i) = x + xi + yi - y = (x-y) + i(x+y)$

$\therefore (x-y) + i(x+y) = 2 + 6i$ ✓

$x - y = 2$

$x + y = 6$ $\quad \therefore 2x = 8, \quad x = 4, \quad y = 2$ ✓

$\therefore \quad z = 4 + 2i$ ✓

Question 2

a) $p(2-i) + \frac{q}{2-i} \times \frac{2+i}{2+i} = 2p - ip + \frac{2q+iq}{4+1}$ ✓ [3]

$= \frac{10p - 5pi + 2q + iq}{5} = 1 \quad \therefore 10p + 2q + i(-5p+q) = 5$ ✓

$\therefore 10p + 2q = 5$ and $-5p + q = 0$

$$\begin{array}{r} 10p + 2q = 5 \\ -10p + 2q = 0 \\ \hline \end{array} \qquad 4q = 5, \quad q = 5/4 \checkmark$$

$$p = 1/4$$

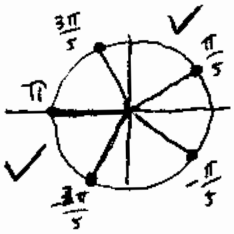
[3]

b) i) $z^5 = -1$ if $z = r(\cos\theta + i\sin\theta)$, $z^5 = r^5(\cos\theta + i\sin\theta)^5$

$= r^5(\cos 5\theta + i\sin 5\theta) = -1$, since $|-1| = 1$ $r^5 = 1$, $r = 1$

$\therefore \cos 5\theta + i\sin 5\theta = -1 \quad \therefore \cos 5\theta = -1, \quad 5\theta = \pi + 2k\pi$ ✓

$\theta = \frac{\pi + 2k\pi}{5} = \pi/5, \frac{3\pi}{5}, -\pi/5, \pi, -3\pi/5$ ✓



$\therefore$ Roots are $-1, \operatorname{cis}\pi/5, \operatorname{cis}(-\pi/5), \operatorname{cis}\left(\frac{3\pi}{5}\right), \operatorname{cis}\left(-\frac{3\pi}{5}\right)$ ✓

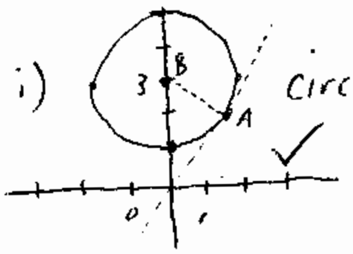
ii) Sum of roots are [2]

$-1 + \cos\frac{\pi}{5} + i\sin\frac{\pi}{5} + \cos\left(-\frac{\pi}{5}\right) + i\sin\left(-\frac{\pi}{5}\right) + \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5} + \cos\left(-\frac{3\pi}{5}\right) + i\sin\left(-\frac{3\pi}{5}\right)$

$= -1 + \cos\frac{\pi}{5} + i\sin\pi/5 + \cos\frac{\pi}{5} - i\sin\frac{\pi}{5} + \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5} + \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5} = 0$

$= -1 + 2\cos\frac{\pi}{5} + 2\cos\frac{3\pi}{5} = 0 \quad \therefore \cos\pi/5 + \cos\frac{3\pi}{5} = 1/2$ ✓✓

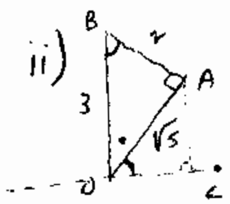
c) i)



Circle radius = 2 centre = (0,3) [2]

✓ $x^2 + (y-3)^2 = 4$ ✓

ii)



Minimum arg when z is tangent OA ✓ [3]

$\angle COA = \angle OBA$ (both $90 - \angle AOB$)

$\angle OBA = \cos^{-1} 2/3 \quad \therefore \angle COA = \arg z = \cos^{-1} 2/3$ ✓

$|z| = \sqrt{5}$ ✓

ion 3

$(2\cos\theta)^3 = (z^1 + z^{-1})^3 = z^3 + 3z + 3z^{-1} + z^{-3}$ [2]

$8\cos^3\theta = (z^3 + z^{-3}) + 3(z + z^{-1}) = 2\cos 3\theta + 6\cos\theta$ ✓

$\therefore \cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$ ✓

a) ii) $\frac{1}{4}\int_0^{\frac{\pi}{3}} \cos 3\theta + 3\cos\theta \, d\theta = \frac{1}{4}\left[\frac{1}{3}\sin 3\theta + 3\sin\theta\right]_0^{\frac{\pi}{3}}$ ✓✓ [3]

$= \frac{1}{4}\left[(0 + 3\sin\frac{\pi}{3}) - (0+0)\right] = \frac{1}{4}\left(\frac{3\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{8}$ ✓

$-\frac{1}{2}$ for computational error.

b) (i) If $\text{Im}(z) > 0$ and $z = a + bi$ then $b > 0$ [2]

$\frac{1}{z} = \frac{1}{a+bi} \times \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2}$ ✓

$\therefore \text{Im}\left(\frac{1}{z}\right) = \frac{-b}{a^2+b^2}$ since $b > 0$ and $a^2 + b^2 > 0$

$\text{Im}\left(\frac{1}{z}\right) < 0$ ✓

No penalty for not stating $a^2+b^2>0$

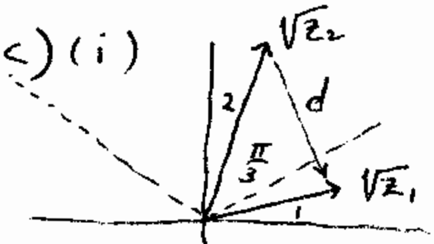
No penalty for $\text{Im}\left(\frac{a+bi}{a^2+b^2}\right) = \frac{-bi}{a^2+b^2}$

(ii) If $z = r\,\text{cis}\,\theta$, $|z| = r$ and $\frac{1}{|z|} = \frac{1}{r}$ [2]

$\frac{1}{z} = \frac{1}{r}\text{cis}(-\theta)$, $\left|\frac{1}{z}\right| = \frac{1}{r}$ $\therefore \left|\frac{1}{z}\right| = \frac{1}{|z|}$ ✓✓

$-\frac{1}{2}$ for minor transcript errors

c) (i)



$\arg\frac{z_2}{z_1} = \arg z_2 - \arg z_1 = \frac{2\pi}{3}$ [3]

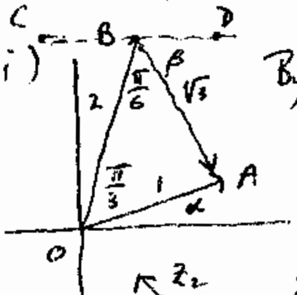
$\arg\sqrt{z_2} = \frac{1}{2}\arg z_2$, $\arg\sqrt{z_1} = \frac{1}{2}\arg z_1$ ✓

$\therefore \arg\sqrt{z_2} - \arg\sqrt{z_1} = \frac{1}{2}(\arg z_2 - \arg z_1) = \frac{\pi}{3}$

Using cosine rule $d^2 = 2^2 + 1^2 - 2(2)(1)\cos\frac{\pi}{3} = 5 - 4\cos\frac{\pi}{3} = 3$ ✓

$\therefore |\sqrt{z_1} - \sqrt{z_2}| = \sqrt{3}$ ✓

ii)



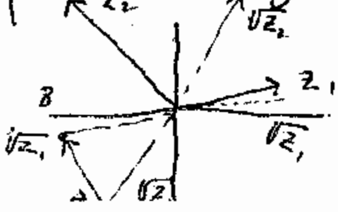
By Pythagoras $\angle OAB = \frac{\pi}{2}$ $\therefore \angle OBA = \frac{\pi}{6}$ ✓ [2]

$\angle OBC = \frac{\pi}{3} + \alpha$ $\therefore \beta = \pi - \left(\frac{\pi}{3} + \alpha + \frac{\pi}{6}\right) = \frac{\pi}{2} - \alpha$

Since $\angle DBA$ is principal argument of $(\sqrt{z_1} - \sqrt{z_2})$

$\arg(\sqrt{z_1} - \sqrt{z_2}) = \alpha - \frac{\pi}{2}$ ✓

iii)



AB would be other answer [1]

✓