GIRRAWEEN HIGH SCHOOL
MATHEMATICS EXTENSION 2 TASK 1
DECEMBER 2009

Instructions:
* Start each new question on a new piece of paper.
* Use only one side of the paper.
* Show all necessary working.
* Marks may be deducted for careless or badly arranged work.
* Approved Scientific calculators may be used.

Time allowed 90 minutes

Question 1. 8 Marks **MARKS**

Given $z = 2\text{-}3i$ and $w = \text{-}1 + 4i$ find in the form $a + ib$

(a) $z + w$ **1**
(b) zw **2**
(c) $\overline{zw}$ **1**
(d) z^2 **2**
(e) $\dfrac{1}{w}$ **2**

Question 2. 16 Marks

Given that $z = \sqrt{3} - i$ and $w = -1 + i$

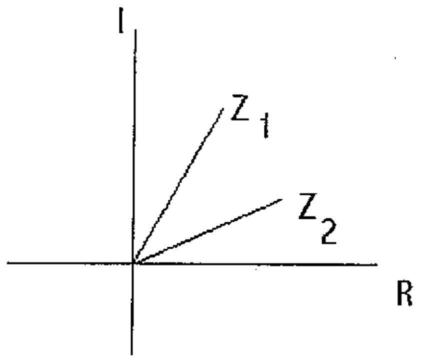
(a) (i) Express both z and w in modulus argument form **4**
(ii) Express $\dfrac{z^3}{w^4}$ in the form $a + ib$ **3**

(b) (i) Find all real numbers x and y such that $\sqrt{8+6i} = x + iy$ **3**
(ii) Hence or otherwise solve for z: $z^3 + (1+3i)z^2 - 4z = 0$ **3**

(c) Find the fourth roots of $2\sqrt{3} + 2i$ you can leave the answer in mod/arg form **3**

Question 3. 6 Marks

Copy the diagram below into your answer book, z_1 and z_2 represent unit vectors



(a) Draw the vectors representing

(i) $z_1 + z_2$ **1**

(ii) $\dfrac{1}{z_1}$ **1**

(iii) $\overline{z_2}$ **1**

(iv) $-z_2$ **1**

(v) $z_1 - z_2$ **1**

(vi) $(z_1)^2$ **1**

Question 4 20 Marks

(a) Sketch on separate Argand diagrams

(i) $|z-(2+i)| \le 2$ **2**

(ii) $|z-(2+i)| \le |z-2|$ **2**

(iii) $\arg(z-2i) = \dfrac{\pi}{3}$ **2**

(iv) $|z^2 - (\bar{z})^2| \ge 4$ **3**

(b) Express the following loci in Cartesian form

(i) $|z-(2+i)| = 3$ **2**

(ii) $|z-2| = |z+2i|$ **2**

(iii) $\arg\left(\dfrac{z-2}{z+2}\right) = \dfrac{\pi}{4}$ (Hint all details required) **4**

(c) The point P on the Argand plane represents the complex number z where z satisfies $\dfrac{1}{z} + \dfrac{1}{\bar{z}} = 1$ Find the locus of P as z varies. **3**

Question 5. 11 Marks

Let $z=\cos\theta+i\sin\theta$ by expanding z^4 using De Moivre and binomial find expressions for

(a) $\cos 4\theta$ **2**

(b) $\sin 4\theta$ **2**

(c) $\tan 4\theta$ in terms of $\tan^n\theta$ **2**

(d) Prove that $z^n+\dfrac{1}{z^n}=2\cos n\theta$ **1**

(e) $z^n-\dfrac{1}{z^n}=2i\sin n\theta$ **1**

(f) Show that $\cos^6\theta=\dfrac{1}{32}(\cos 6\theta+6\cos 4\theta+15\cos 2\theta+10)$ **3**

Question 6. 14 Marks

(a) (i) Prove that $\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}=\sin\theta+i\cos\theta$ **3**

(ii) Hence or otherwise deduce that

$(1+\sin\frac{\pi}{5}+i\cos\frac{\pi}{5})^5+i(1+\sin\frac{\pi}{5}-i\cos\frac{\pi}{5})^5=0$ **3**

(b) Let $w=\cos\dfrac{2\pi}{5}+i\sin\dfrac{2\pi}{5}$

(i) Show that w^k is a solution to $z^5-1=0$ where k is an integer **2**

(ii) Explain why $1+w+w^2+w^3+w^4=0$ **2**

(iii) Hence or otherwise find the value of $(w-1)(1+2w+3w^2+4w^3+5w^4)$ **2**

(iv) Given that S = $\dfrac{w}{1-w^2}+\dfrac{w^2}{1-w^4}+\dfrac{w^3}{1-w}+\dfrac{w^4}{1-w^3}$ Show that $S=0$ **2**

Question 7. 14 Marks

(a) If n is a positive integer prove that $\left(\dfrac{1+i\tan\theta}{1-i\tan\theta}\right)^n=\dfrac{1+i\tan n\theta}{1-i\tan n\theta}$ **3**

(b) (i) Solve the equation $z^6-1=0$ **3**

(ii) Hence factorise z^6-1 into linear and quadratic factors with real coefficients **4**

(iii) Sketch the roots of $z^6-1=0$ onto a Argand diagram. **2**

(iv) Find the area of the regular polygon formed by connecting the roots. **2**

Question 8. 9 Marks

(a) Determine the greatest and least values of arg (z), where $|z-4i|=2$ **3**

(b) If $z=\cos\theta+i\sin\theta$ prove that $\dfrac{2}{1+z}=1-i\tan\dfrac{\theta}{2}$ **3**

(c) OABC is a square $\overrightarrow{OA}$ is represented by the vector $3+i$. Find the vectors representing $\overrightarrow{OB}$ and $\overrightarrow{OC}$ **3**

Question 9. 9 Marks

Using the fact that $|z|^2=z\bar{z}$

(a) Show that $|z_1+z_2|^2+|z_1-z_2|^2=2|z_1|^2+2|z_2|^2$ **2**

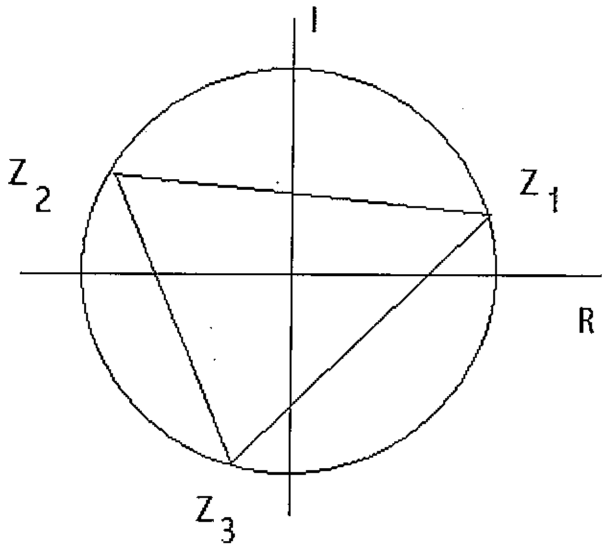
(b) (i) Express $1+\cos\theta+i\sin\theta$ in modulus argument form. **3**

(ii) Find $(1+\cos\theta+i\sin\theta)^2$ in modulus argument form. **1**

(c) (i) Express $\sin\theta+i\cos\theta$ in modulus argument form. **2**

(ii) Hence find $\sqrt{\sin\theta+i\cos\theta}$ in modulus argument form. **1**

Question 10. 7 Marks



Z_1, Z_2 and Z_3 are three points on the unit circle forming an equilateral triangle.

(a) (i) If $Z_1 = \cos\theta+i\sin\theta$ express Z_2 and Z_3 in terms of θ **2**

(ii) Prove that $Z_1+Z_2+Z_3=0$ **2**

(iii) Prove that $Z_1^2+Z_2^2+Z_3^2=Z_1Z_2+Z_2Z_3+Z_3Z_1$ **3**

HSC TASK 1 EXT 2

DEC 2009.

Q1 $z = 2-3i \quad \omega = -1+4i$

(a) $z+\omega = 1+i$ (1)

(b) $z\omega = (2-3i)(-1+4i)$

$= -2+12+3i+8i$

$= 10+11i$ (2)

(c) $\overline{z\omega} = 10-11i$ (1)

(d) $z^2 = 4-9-6i$

$= -5-6i$ (2)

(e) $\dfrac{1}{\omega} = \dfrac{\overline{\omega}}{\omega\overline{\omega}} = \dfrac{-1-4i}{1+16}$

$= \dfrac{-1}{17} - \dfrac{4i}{17}$ (2)

Question 2.

$z = \sqrt{3}-i \quad \omega = -1+i$

(a)(i) $|z| = \sqrt{(\sqrt{3})^2+(-1)^2} = 2$

$\arg z = \tan^{-1}\dfrac{-1}{\sqrt{3}} = -\pi/6$

$|\omega| = \sqrt{(-1)^2+1^2} = \sqrt{2}$

$\arg\omega = \tan^{-1}\dfrac{1}{-1} = \dfrac{3\pi}{4}$

$z = 2\,\text{cis}(-\pi/6) \quad \omega = \sqrt{2}\,\text{cis}\,3\pi/4$ (4)

(ii) $\dfrac{z^3}{\omega^4} = \dfrac{8\,\text{cis}\,-\pi/2}{4\,\text{cis}\,3\pi}$ $\quad \text{cis}\,3\pi \equiv \text{cis}\,\pi$

$= 2\,\text{cis}\,-3\pi/2$

$= 2\,\text{cis}\,\pi/2$

$= 2i$ (3)

(b) $\sqrt{8+6i} = x+iy$

$\therefore 8+6i = x^2-y^2+2xyi$

$8 = x^2-y^2$ (A)

$6 = 2xy$

$3 = xy$ (B)

$\therefore y = 3/x$

$\therefore 8 = x^2 - \dfrac{9}{x^2}$

$8x^2 = x^4-9$

$0 = x^4-8x^2-9$

$= (x^2-9)(x^2+1)$

$\therefore x = \pm 3.$

$\therefore y = \pm 1$

$\sqrt{8+6i} = \pm(3+i).$ (3)

(ii) $z^3+(1+3i)z^2-4z = 0$

$z\left(z^2-(1+3i)z-4\right) = 0$

$\therefore z = 0$ OR.

$z = \dfrac{(1+3i) \pm \sqrt{(1+3i)^2+16}}{2}$

$z = \dfrac{(1+3i) \pm \sqrt{8+6i}}{2}$

$z = \dfrac{1+3i \pm 3+i}{2}$

$z = 0,\ 2+2i,\ -1+i.$ (3)

(c) $z^4 = 2\sqrt{3}+2i$

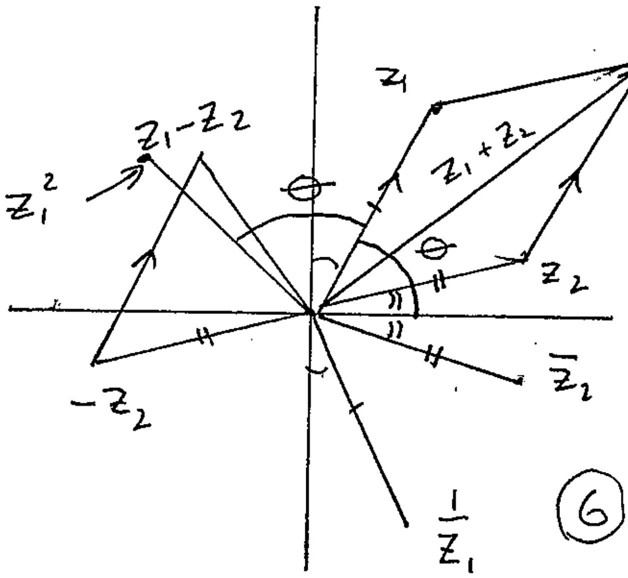
$z^4 = 16\,\text{cis}\,\pi/6$

$\therefore z = 16^{1/4}\,\text{cis}\,\dfrac{2k\pi+\pi/6}{4}$ $\quad k = -2,-1,0,$

$z = 2\,\text{cis}\,\dfrac{-23\pi}{24},\ 2\,\text{cis}\,\dfrac{-11\pi}{24},\ 2\,\text{cis}\,\dfrac{\pi}{24}$

$2\,\text{cis}\,\dfrac{13\pi}{24}.$ (3)

Question 3.



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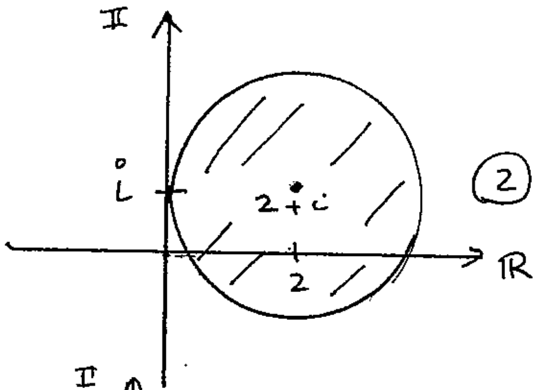
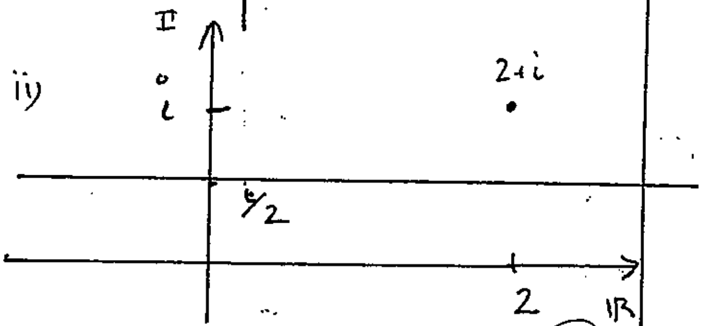
Question 4

(a) (i)



2

(ii)



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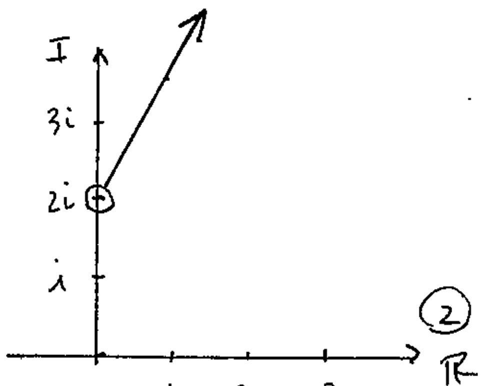
(iii)



2

(iv) $|z^2 - (\bar{z})^2| \geq 4$

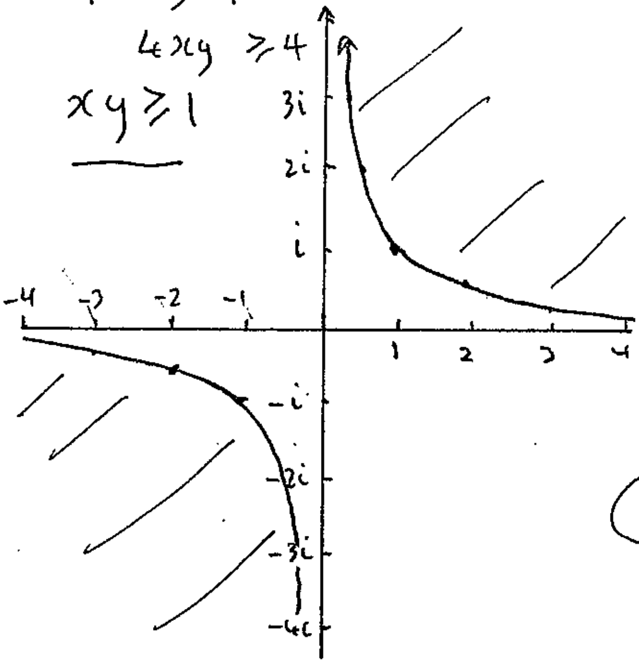
$|(x+iy)^2 - (x-iy)^2| \geq 4$

$|x^2 - y^2 + 2xyi - (x^2 - y^2 - 2xyi)| \geq 4$

$|4xyi| \geq 4$

$4xy \geq 4$

$xy \geq 1$

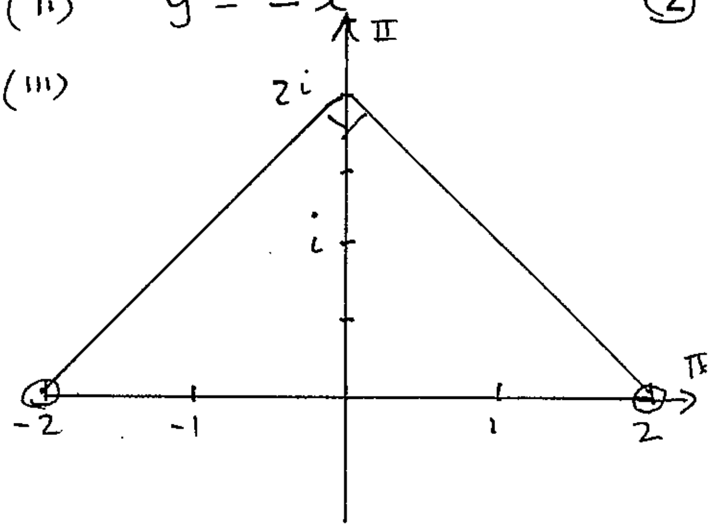


3

(b) (i) $(x-2)^2 + (y-1)^2 = 9.$ 2

(ii) $y = -x$ 2

(iii)



$x^2 + (y-2)^2 = 8$

$y > 0$

circle centre $(0+2i)$

radius $\sqrt{8}$

above real axis. 4.

(c) $\frac{1}{z} + \frac{1}{\bar{z}} = 1$

$\frac{1}{x+iy} + \frac{1}{x-iy} = 1$

$\frac{x-iy+x+iy}{x^2+y^2} = 1$

$\frac{2x}{x^2+y^2} = 1$

$$2x = x^2 + y^2$$
$$0 = x^2 - 2x + y^2$$
$$1 = x^2 - 2x + 1 + y^2$$
$$(x-1)^2 + y^2 = 1$$

Circle centre $(1,0)$ $\{1+0i\}$

RADIUS $\underline{1}$. (3)

Question 5.

(a) $Z = \cos\theta + i\sin\theta$

$Z^4 = \cos4\theta + i\sin4\theta$ (DE MOIVRE)

$Z^4 = \cos^4\theta + 4\cos^3\theta\sin\theta i - 6\cos^2\theta\sin^2\theta - 4\cos\theta\sin^3\theta i + \sin^4\theta$.

(BINOMIAL)

EQUATING REALS.

$\cos4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ (2)

EQUATING IMAGINARY

(b) $\sin4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ (2)

c) $\tan4\theta = \dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$

$(\div\cos^4\theta)$

$= \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ (2)

(d) $Z^n + \dfrac{1}{Z^n} = 2\cos n\theta$

$(\cos\theta + i\sin\theta)^n + \dfrac{1}{(\cos\theta + i\sin\theta)^n}$

$\cos n\theta + i\sin n\theta + \cos -n\theta + i\sin -n\theta$

DEMOIVRE

But cos is an even fn

sin is an odd fn

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$ (1)

Similarly

(e) $\cos n\theta + i\sin n\theta - (\cos n\theta - i\sin$

$= 2i\sin n\theta$ (1)

(f) $(2\cos\theta)^6 = \left(Z + \frac{1}{Z}\right)^6$

$64\cos^6\theta = Z^6 + 6Z^4 + 15Z^2 + 20 + \dfrac{15}{Z^2} + \dfrac{6}{Z^4} + \dfrac{1}{Z^6}$

(BINOMIAL)

$64\cos^6\theta = Z^6 + \frac{1}{Z^6} + 6\left(Z^4 + \frac{1}{Z^4}\right) + 15\left(Z^2 + \frac{1}{Z^2}\right) + 20$

$64\cos^6\theta = 2\cos6\theta + 12\cos4\theta + 30\cos2\theta + 20$

$\cos^6\theta = \dfrac{1}{32}\{\cos6\theta + 6\cos4\theta + 15\cos2\theta + 10$ (3)

Question 6.

(a) (i)

$\dfrac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta} \times \dfrac{1+\sin\theta + i\cos\theta}{1+\sin\theta + i\cos\theta}$

$= \dfrac{(1+\sin\theta)^2 - \cos^2\theta + 2(1+\sin\theta)i\cos\theta}{(1+\sin\theta)^2 + \cos^2\theta}$

$= \dfrac{(1+\sin\theta)^2 - (1-\sin^2\theta) + 2(1+\sin\theta)i\cos\theta}{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}$

$= \dfrac{(1+\sin\theta)^2 - (1+\sin\theta)(1-\sin\theta) + 2(1+\sin\theta)\cos\theta}{2(1+\sin\theta)}$

$= \dfrac{(1+\sin\theta) - (1-\sin\theta) + 2\cos\theta i}{2\cancel{x}}$ (3)

$= \dfrac{2\sin\theta + 2\cos\theta i}{2} = \sin\theta + \cos\theta i$

Question 6 continued.

$$\left(1+\sin\tfrac{\pi}{5}+i\cos\tfrac{\pi}{5}\right)^5+i\left(1+\sin\tfrac{\pi}{5}-i\cos\tfrac{\pi}{5}\right)^5=0$$

$$\div\ \left(1+\sin\tfrac{\pi}{5}-i\cos\tfrac{\pi}{5}\right)^5$$

$$\left(\frac{1+\sin\frac{\pi}{5}+i\cos\frac{\pi}{5}}{1+\sin\frac{\pi}{5}-i\cos\frac{\pi}{5}}\right)^5+i=0$$

$$\left(\sin\tfrac{\pi}{5}+i\cos\tfrac{\pi}{5}\right)^5+i=0$$

$$\left[\cos\left(\tfrac{\pi}{2}-\tfrac{\pi}{5}\right)+i\sin\left(\tfrac{\pi}{2}-\tfrac{\pi}{5}\right)\right]^5+i=0$$

$$\left(\cos\tfrac{3\pi}{2}+i\sin\tfrac{3\pi}{2}\right)+i=0$$

De MOIVRE

$$\cos-\tfrac{\pi}{2}+i\sin-\tfrac{\pi}{2}+i=0$$

$$0-i+i=0 \qquad (3)$$

b) (i) $\omega^k=\cos\frac{2k\pi}{5}+i\sin\frac{2k\pi}{5}$

DE MOIVRE

$$z^5-1=\left(\cos\tfrac{2k\pi}{5}+i\sin\tfrac{2k\pi}{5}\right)^5-1$$
$$=(\cos 2k\pi+i\sin 2k\pi)-1$$
$$=1+0i-1$$
$$=0 \qquad (2)$$

$$z^5-1=(z-1)(z^4+z^3+z^2+z+1)$$

IF ω IS A COMPLEX ROOT

$$(\omega-1)(\omega^4+\omega^3+\omega^2+\omega+1)=0$$

But $\omega-1\neq 0$ (COMPLEX)

$$\therefore\ \omega^4+\omega^3+\omega^2+\omega+1=0 \qquad (2)$$

(iii) $(\omega-1)(1+2\omega+3\omega^2+4\omega^3+5\omega^4)$

$$=\omega+2\omega^2+3\omega^3+4\omega^4+5\omega^5-1$$
$$-2\omega-3\omega^2-4\omega^3-5\omega^4$$

$$=-\omega-\omega^2-\omega^3-\omega^4-1\quad+5\omega^5$$
$$= 0+5(1) \quad = 5 \quad (2)$$

$$S=\frac{\omega}{1-\omega^2}+\frac{--}{1-\omega^4}+\frac{}{1-\omega}+\frac{}{1-\omega}$$

$$S=\frac{\omega}{1-\omega^2}\cdot\frac{\omega^3}{\omega^3}+\frac{\omega^2}{1-\omega^4}\cdot\frac{\omega}{\omega}+\frac{\omega^3}{1-\omega}+\frac{\omega^4}{1-\omega^3}$$

$$S=\frac{\omega^4}{\omega^3-1}+\frac{\omega^3}{\omega-1}+\frac{\omega^3}{1-\omega}+\frac{\omega^4}{1-\omega^3}$$

$$S=\frac{-\omega^4}{\cancel{\omega^3-1}\ 1-\omega^3}-\frac{\omega^3}{1-\omega}+\frac{\omega^3}{1-\omega}+\frac{\omega^4}{1-\omega^3}$$

$$S=0. \qquad (2)$$

Question 7.

(a)
$$\left(\frac{1+i\tan\theta}{1-i\tan\theta}\right)^n=\left[\frac{\frac{\cos\theta+i\sin\theta}{\cos\theta}}{\frac{\cos\theta-i\sin\theta}{\cos\theta}}\right]^n$$

$$=\left[\frac{\frac{\cos\theta+i\sin\theta}{\cos\theta}}{\frac{\cos-\theta+i\sin-\theta}{\cos\theta}}\right]^n=\frac{\cos n\theta+i\sin n\theta}{\cos-n\theta+i\sin-n\theta}$$

$\left\{\begin{array}{l}\cos\theta \text{ is an even fn}\\ \sin\theta \text{ is an odd fn}\end{array}\right\}$ $=\dfrac{\cos n\theta+i\sin n\theta}{\cos n\theta-i\sin n\theta}$

$$=\frac{\cos n\theta(1+i\tan n\theta)}{\cos n\theta(1-i\tan n\theta)}$$

$$=\frac{1+i\tan n\theta}{1-i\tan n\theta} \qquad (3)$$

(b) $z^6-1=0$

$$z^6=1$$
$$z^6=\operatorname{cis}2k\pi$$
$$z=\operatorname{cis}\frac{2k\pi}{6} \qquad k=-3,-2,-1,0,$$

$$z=\operatorname{cis}-\pi,\ \operatorname{cis}-\tfrac{2\pi}{3},\ \operatorname{cis}-\tfrac{\pi}{3},\ \operatorname{cis}0$$
$$\operatorname{cis}\tfrac{\pi}{3},\ \operatorname{cis}\tfrac{2\pi}{3} \qquad (3)$$

$$z=-1,\ 1,\ \operatorname{cis}\tfrac{\pi}{3},\ \operatorname{cis}-\tfrac{\pi}{3},\ \operatorname{cis}\tfrac{2\pi}{3},\ \operatorname{cis}$$

(ii) $z^6-1 = (z-1)(z+1)(z-\text{cis}\,\pi/3)$
$(z-\text{cis}\,{-\pi/3})(z-\text{cis}\,\frac{2\pi}{3})(z-\text{cis}\,{-\frac{2\pi}{3}})$

Consider

$(z-\text{cis}\,\pi/3)(z-\text{cis}\,{-\pi/3})$

$= [z-(\cos\pi/3 + i\sin\pi/3)][z-(\cos{-\pi/3}+i\sin{-\pi/3})]$

$= [z-(\cos\pi/3 + i\sin\pi/3)][z-(\cos\pi/3 - i\sin\pi/3)]$

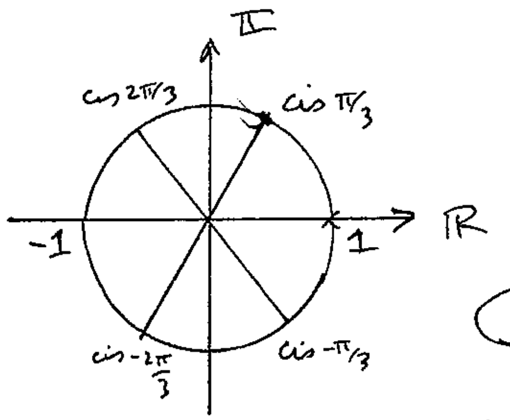
$= z^2 - 2\cos\pi/3\, z + (\cos\pi/3 + i\sin\pi/3)(\cos\pi/3 - i\sin\pi/3)$

$= z^2 - 2\cos\pi/3\, z + \cos^2\pi/3 + \sin^2\pi/3$

$= z^2 - 2\cos\pi/3\, z + 1$

$\therefore\ z^6 - 1 = (z-1)(z+1)$ (4)

$(z^2 - 2\cos\pi/3\, z+1)(z^2 - 2\cos 2\pi/3\, z + 1)$



(2)

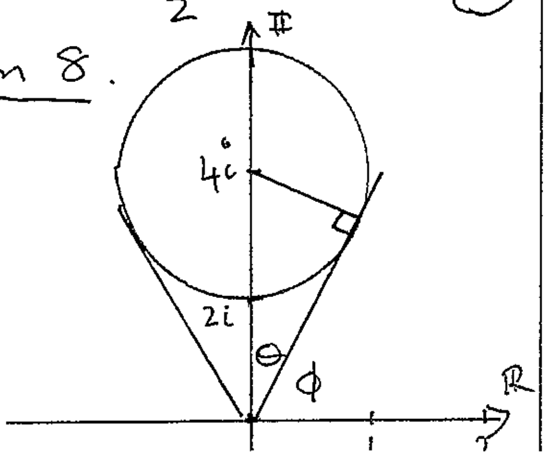
(iv) Area $= 6 \times \frac{1}{2} \times 1 \times 1 \times \sin 60$

$= 3 \times \sqrt{3}/2$

$= \frac{3\sqrt{3}}{2}\ u^2.$ (2)

Question 8.

(a)



for least arg ϕ, tangent to circle

$\therefore\ \sin\theta = 2/4$

$\sin\theta = 1/2$

$\theta = 30°$

$\phi = 60°\ (\pi/3)$ (3)

$\therefore$ MAX ARG $180-60=120$

$(2\pi/3)$

(b) $\dfrac{2}{1+z} = \dfrac{2}{1+\cos\theta + i\sin\theta}$

$= \dfrac{2}{1+\cos\theta+i\sin\theta} \times \dfrac{1+\cos\theta - i\sin\theta}{1+\cos\theta - i\sin\theta}$

$= \dfrac{2(1+\cos\theta - i\sin\theta)}{(1+\cos\theta)^2 + \sin^2\theta}$

$= \dfrac{\cancel{2}(1+\cos\theta - i\sin\theta)}{\cancel{2}(1+\cos\theta)}$

$= 1 - \dfrac{i\sin\theta}{1+\cos\theta}$

$= 1 - \dfrac{i(\cancel{2}\sin\theta/2\cos\theta/2)}{\cancel{1}+\cancel{2}\cos^2\theta/2 - \cancel{1}}$

$= 1 - i\tan\theta/2.$ (3)

(c) $\vec{OA} = 3+i$

$\therefore\ \vec{OC} = -1+3i\ (i\,OA)$

$\vec{OB} = \vec{OA} + \vec{OC}$

$= (1+i)(3+i)$

$= 2+4i$ (3)

Question 9

$$|Z_1+Z_2|^2 = (Z_1+Z_2)(\overline{Z_1+Z_2})$$
$$=(Z_1+Z_2)(\bar Z_1+\bar Z_2)$$
$$=Z_1\bar Z_1 + Z_1\bar Z_2 + Z_2\bar Z_1 + Z_2\bar Z_2$$
$$=|Z_1|^2 + Z_1\bar Z_2 + Z_2\bar Z_1 + |Z_2|^2$$
$$|Z_1-Z_2|^2 = (Z_1-Z_2)(\overline{Z_1-Z_2})$$
$$=(Z_1-Z_2)(\bar Z_1-\bar Z_2)$$
$$=Z_1\bar Z_1 - Z_1\bar Z_2 - Z_2\bar Z_1 + Z_2\bar Z_2$$
$$=|Z_1|^2 - Z_1\bar Z_2 - Z_2\bar Z_1 + |Z_2|^2$$

$$\therefore |Z_1+Z_2|^2 + |Z_1-Z_2|^2 = 2\{|Z_1|^2+|Z_2|^2\}. \quad (2)$$

(b) $1+\cos\theta + i\sin\theta = Z$

(i) $|Z| = \sqrt{(1+\cos\theta)^2+(\sin\theta)^2}$
$$= \sqrt{2+2\cos\theta}$$
$$= \sqrt{2+2(2\cos^2\theta/2-1)}$$
$(\cos 2A = 2\cos^2 A - 1)$
$$= \sqrt{4\cos^2\theta/2}$$
$$= 2\cos\theta/2$$

$$\arg Z = \tan^{-1}\frac{\sin\theta}{1+\cos\theta}$$
$$= \frac{2\sin\theta/2\cos\theta/2}{\cancel{1}+2\cos^2\theta/2\,\cancel{-1}} \quad (3)$$
$$= \tan^{-1}(\tan\theta/2) = \frac{\theta}{2}$$

$$Z = 2\cos\frac{\theta}{2}(\text{cis}\,\theta/2)$$

(ii) $\therefore Z^2 = 4\cos^2\theta/2\,(\text{cis}\,\theta)$
$$= 4\cos^2\theta/2\,(\cos\theta+i\sin\theta) \quad (1$$

(c) (i) $\sin\theta + i\cos\theta = Z$
$$Z = \cos(\pi/2-\theta) + i\sin(\pi/2-\theta)$$
$$|Z| = 1 \qquad \arg = (\pi/2-\theta) \quad (2)$$

(ii) $\sqrt{\sin\theta+i\cos\theta} = \pm\,\text{cis}\,\frac{\pi/2-\theta}{2}$
$$= \pm\,\text{cis}(\pi/4-\theta/2) \quad (1$$

Question 10.

(i) $Z_1 = \cos\theta + i\sin\theta$
$$Z_2 = \text{cis}\,\tfrac{2\pi}{3}\,(Z_1)$$
$$Z_3 = \text{cis}\,-2\pi/3\,(Z_1) \quad (2)$$

ROTATION IN ARGAND

(ii) $Z_1+Z_2+Z_3 = (1+\text{cis}\,2\pi/3+\text{cis}\,-2\pi/3)Z$

Now $1+\text{cis}\,2\pi/3+\text{cis}\,-2\pi/3 = 1+\omega+\omega^2$

ω is cube root of unity

$\therefore 1+\omega+\omega^2 = 0 \quad (2)$

OR $1+\cos 2\pi/3 + i\sin 2\pi/3 + \cos -2\pi/3 + i\sin -2\pi/3$
$$= 1 + \tfrac{1}{2} + \tfrac{\sqrt3}{2}i - \tfrac12 - \tfrac{\sqrt3}{2}i = 0.$$

(iii) $Z_2Z_3 = \text{cis}\,\tfrac{2\pi}{3}(Z_1)\cdot\text{cis}\,-\tfrac{2\pi}{3}(Z_1)$
$$= Z_1^2$$

similarly $Z_1Z_3 = Z_2^2 \quad Z_1Z_2 = Z_3^2$

$$\therefore Z_1^2+Z_2^2+Z_3^2 = Z_1Z_2+Z_1Z_3+Z_2Z_3$$
(3)