GIRRAWEEN HIGH SCHOOL

YEAR 11 MATHEMATICS EXTENSION 2 TASK 1

Time allowed: 110 minutes | December 2010

Instructions:

- Show all necessary working.
- Marks may be deducted for careless or badly arranged work.
- Approved scientific calculators may be used.

Question 1 (12 marks)

(a) Evaluate: 4

(i) i^{25} (ii) i^{-37} (iii) $\sum_{r=0}^{r=1000} (i)^r$

(b) If $z_1 = -2+3i$ and $z_2 = 3-4i$, express the following in the form $a+ib$. 8

(i) $z_1 + z_2$ (ii) $z_2 - z_1$ (iii) $\frac{z_1 z_2}{\overline{z_2}}$

Question 2 (13 marks)

(a) Express in modulus argument form where argument is in radians. 6

(i) $z = -1-i$ (ii) $z = -3+i\sqrt{3}$

(b) Evaluate each of the following in Cartesian form

(i) $[3(\cos 40° + i\sin 40°)][4(\cos 80° + i\sin 80°)]$ 3

(ii) $\dfrac{(2cis15°)^7}{(4cis45°)^3}$ 4

Question 3 (12 marks)

(a) (i) Find all real numbers x and y such that $(x+iy)^2 = -5-12i$. 4

(ii) Hence solve the equation $z^2 - 4z + (9+12i) = 0$ 4

(b) Solve the equation $z^4 = 8(\sqrt{3}+i)$. Write answers in modulus argument form. 4

Question 4 (27 marks)

(a) By letting $z = \cos\theta + i\sin\theta$ and expanding z^6 find expressions for

(i) $\sin 6\theta$ 5

(ii) $\cos 6\theta$ 4

(iii) $\tan 6\theta$ in terms of $\tan\theta$ 4

(b) Prove that

(i) $z^n + \dfrac{1}{z^n} = 2n\cos n\theta$ (ii) $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$ 4

(c) Using the result/s of (b)

(i) Prove that $\sin^6\theta = \dfrac{-1}{32}\left(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\right)$ 5

(ii) Find all complex numbers z such that $\dfrac{(z^2 - \frac{1}{z^2}) \times i}{(z^2 + \frac{1}{z^2})} = -\sqrt{3}$. 5

Question 5 (17 marks)

(i) Find the seventh roots of unity. Show the roots on an Argand diagram. 6

(ii) Resolve $z^7 - 1$ into real quadratic and real linear factors. 4

(iii) Find the sum of the roots. 2

(iv) Using (iii) above and the fact that $\cos(360-\theta) = \cos\theta$, deduce that

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$$ 5

Question 6 (17 marks)

(a) If ω is a complex cube root of unity

(i) Show that $(1-\omega-\omega^2)(1-\omega+\omega^2)(1+\omega-\omega^2)=8$ 4

(ii) Form a quadratic equation with roots $a\omega+b\omega^2,\ a\omega^2+b\omega$. 4

(b) If $x+iy=\dfrac{a+ib}{c+id}$, prove that $x^2+y^2=\dfrac{a^2+b^2}{c^2+d^2}$ 4

(c) Prove that $\left(\dfrac{1+\cos\theta+i\sin\theta}{1+\cos\theta-i\sin\theta}\right)^n=\cos n\theta+i\sin n\theta$ 5

(Hint: Express $1+\cos\theta$ and $\sin\theta$ in terms of $\dfrac{\theta}{2}$)

Question 7 (26 marks)

(a) If $z=3+2i$, plot on an Argand diagram 5

(i) z (ii) $-z$ (iii) $\bar{z}$

(iv) iz (v)) $z+2i$

(b) Sketch these on separate Argand diagrams.

(i) $|z+2+3i|<2$ 2

(ii) $1\le|z-1|<3$ 3

(iii) $\dfrac{\pi}{6}\le\arg z<\dfrac{\pi}{3}$ 3

(iv) $\arg(z-1-2i)=\dfrac{\pi}{4}$. Find the equation of the Locus. 5

(v) $2\le|z|\le3$ and $\operatorname{Im}(z)\ge1$ 3

(c) Prove that for any two complex numbers z and ω

$|z+\omega|^2+|z-\omega|^2=2[\,|z|^2+|\omega|^2\,]$. Give a geometrical interpretation

of this equation. 5

Question 8 (14 marks)

(a) Plot the points $z = i$ and $\omega = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ on an Argand diagram. Deduce from the diagram that $\tan\frac{3\pi}{8} = \sqrt{2} + 1$.(Hint: $|z| = |\omega| = 1$, $\arg z = \frac{\pi}{2}$, $\arg\omega = \frac{\pi}{4}$) 4

(b) Find the locus of z :

(i) If $\arg\frac{z-3}{z-1} = \frac{\pi}{4}$ (Find the centre and radius of the corresponding circle.

Show working and write reasons) 6

(ii) If $\left|\frac{z-1}{z+2}\right| = 2$ (Find the algebraic equation and describe it) 4

Question 9 (14 marks)

Let $x = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ and $b = x + \frac{1}{x}$, b > 0.

(i) Show that $x^4 + x^3 + x^2 + x + 1 = 0$. 3

(ii) Use (i) to show that $b^2 + b - 1 = 0$. 2

(iii) Show that $x^2 - bx + 1 = 0$ and hence $x = \frac{1}{2}(b + i\sqrt{4 - b^2})$ 4

(iv) Hence show that $\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$ and $\sin\frac{2\pi}{5} = \frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}$. 5

Extension 2 Task 1, 2010 Solutions

Question 1 (12 marks)

(a) (i) $i^{25} = i^{24} \times i$

$= (i^4)^6 \times i$ (1)

$= i$

(ii) $i^{-37} = \dfrac{1}{i^{37}}$

$= \dfrac{1}{i^{36} \times i}$ (1)

$= \dfrac{1}{(i^4)^9 \times i}$

$= \dfrac{1}{i} = \dfrac{i}{i^2}$

$= -i$

(iii) $\displaystyle\sum_{r=0}^{1000} (i)^r$

$= i^0 + i^1 + i^2 + i^3 + \dots + i^{997} + i^{998} + i^{999} + i^{1000}$ (2)

$= 1 + (i - 1 - i + 1) + (i - 1 - i + 1) + \dots + (i - 1 - i + 1)$

$= 1$

(iii) $z_1 z_2 = (-2+3i)(3-4i)$

$= -6 + 8i + 9i - 12i^2$

$= -6 + 17i + 12 = 6 + 17i$

$\dfrac{z_1 z_2}{\overline{z_2}} = \dfrac{6+17i}{3+4i}$

$= \dfrac{(6+17i)}{3+4i} \times \dfrac{(3-4i)}{3-4i}$ (4)

$= \dfrac{18 - 24i + 51i - 68i^2}{(3)^2 - (4i)^2}$

$= \dfrac{86+27i}{9+16} = \dfrac{86}{25} + \dfrac{27}{25} i$

(b) $z_1 = -2+3i \quad z_2 = 3-4i$

(i) $z_1 + z_2 = -2 + 3i + 3 - 4i$

$= (-2+3) + i(3-4)$ (2)

$= 1 - i$

(ii) $z_2 - z_1 = 3 - 4i - (-2+3i)$

$= 3 - 4i + 2 - 3i = 5 - 7i$ (2)

Question 2 (13 marks)

(a) (i) $z = -1 - i$

$r = \sqrt{1+1} = \sqrt{2}$

$\tan\alpha = \dfrac{1}{1} = 1$

$\alpha = \dfrac{\pi}{4}$ (3)

$\theta = -\pi + \alpha$

$= -\pi + \dfrac{\pi}{4}$

$= \dfrac{-4\pi + \pi}{4}$

$= \dfrac{-3\pi}{4}$

$-1 - i = \sqrt{2}\, \text{cis}\left(\dfrac{-3\pi}{4}\right)$

(ii) $z = -3 + i\sqrt{3}$

$r = \sqrt{9+3} = \sqrt{12}$

$= \sqrt{4 \times 3} = 2\sqrt{3}$

$\tan\alpha = \dfrac{\sqrt{3}}{3} = \dfrac{1}{\sqrt{3}}$

$\alpha = \dfrac{\pi}{6}$ (3)

$\theta = \pi - \alpha = \pi - \dfrac{\pi}{6}$

$= \dfrac{6\pi - \pi}{6} = \dfrac{5\pi}{6}$

$-3 + i\sqrt{3} = 2\sqrt{3}\, \text{cis}\left(\dfrac{5\pi}{6}\right)$

(b) (i) $= 12(\cos 120 + i\sin 120)$

$= 12\left(\dfrac{-1}{2} + i \times \dfrac{\sqrt{3}}{2}\right)$

$= -6 + i6\sqrt{3}$ (3)

(ii) $\dfrac{(2\,\text{cis}\,15^\circ)^7}{(4\,\text{cis}\,45^\circ)^3} = \dfrac{2^7\,\text{cis}\,105^\circ}{64\,\text{cis}\,135^\circ}$

$= 2\,\text{cis}\,(105 - 135)$

$= 2\,\text{cis}\,(-30)$ (4)

$= 2(\cos(-30) + i\sin(-30))$

$= 2(\cos 30 - i\sin 30)$

$= 2\left(\dfrac{\sqrt{3}}{2} - i \times \dfrac{1}{2}\right) = \sqrt{3} - i$

Question 3 (12 marks)

(a) (i) $(x+iy)^2 = -5-12i$

$x^2+2ixy-y^2 = -5-12i$

$x^2-y^2=-5$ —— (1)

$2xy=-12$

$xy=-6$ —— (2)

From (2) $y=\frac{-6}{x}$

substitute in (1)

$x^2-\left(\frac{-6}{x}\right)^2=-5$ (4)

$x^2-\frac{36}{x^2}=-5$

$x^4-36=-5x^2$

$x^4+5x^2-36=0$

$(x^2+9)(x^2-4)=0$

$x^2=-9$ or $x^2=4$

$x=\pm 2$

when $x=2$, $y=\frac{-6}{2}=-3$

when $x=-2$, $y=\frac{-6}{-2}=3$

The square roots of $-5-12i$ are $2-3i$ and $-2+3i$

$\underline{\underline{\pm(2-3i)}}$

(ii) $z^2-4z+(9+12i)=0$

$z=\frac{4\pm\sqrt{16-4\times1\times(9+12i)}}{2}$

$=\frac{4\pm\sqrt{16-36-48i}}{2}$

$=\frac{4\pm\sqrt{-20-48i}}{2}=\frac{4\pm\sqrt{4(-5-12i)}}{2}$

$=\frac{4\pm2\sqrt{-5-12i}}{2}$ (4)

$=\frac{4\pm2(2-3i)}{2}$

$=\frac{4+2(2-3i)}{2}$ or $\frac{4-2(2-3i)}{2}$

$=\frac{4+4-6i}{2}$ or $\frac{4-4+6i}{2}$

$=\frac{8-6i}{2}$ or $\frac{6i}{2}$

$=\underline{\underline{4-3i \text{ or } 3i}}$

(b) $z^4=8(\sqrt{3}+i)$ $\qquad z=\left[8(\sqrt{3}+i)\right]^{\frac{1}{4}}$

$\underline{\sqrt{3}+i}$ $\qquad r=\sqrt{3+1}=2$

$\tan\alpha=\frac{1}{\sqrt{3}}$ $\qquad \alpha=\frac{\pi}{6}$

$\theta=\frac{\pi}{6}$

$\sqrt{3}+i=2\operatorname{cis}\frac{\pi}{6}$

$8(\sqrt{3}+i)=16\operatorname{cis}\frac{\pi}{6}$

$8(\sqrt{3}+i)=16\operatorname{cis}\left(2k\pi+\frac{\pi}{6}\right)$ where k is an integer

$=16\operatorname{cis}\frac{12k\pi+\pi}{6}$

$z=\left[16\operatorname{cis}\frac{12k\pi+\pi}{6}\right]^{\frac{1}{4}}$

$=2\operatorname{cis}\frac{12k\pi+\pi}{24}$ $\quad k=0,1,2,3$

$\underline{k=0}$

$z_1=\underline{\underline{2\operatorname{cis}\frac{\pi}{24}}}$

$\underline{k=1}$

$z_2=\underline{\underline{2\operatorname{cis}\frac{13\pi}{24}}}$ (4)

$\underline{k=2}$

$z_3=2\operatorname{cis}\frac{25\pi}{24}$

$=2\operatorname{cis}-\left(2\pi-\frac{25\pi}{24}\right)$

$=2\operatorname{cis}-\left(\frac{48\pi-25\pi}{24}\right)$

$=2\operatorname{cis}-\left(\frac{23\pi}{24}\right)$

$=\underline{\underline{2\operatorname{cis}-\frac{23\pi}{24}}}$

$\underline{k=3}$

$z_4=2\operatorname{cis}\frac{37\pi}{24}$

$=2\operatorname{cis}-\left(2\pi-\frac{37\pi}{24}\right)$

$=2\operatorname{cis}-\left(\frac{48\pi-37\pi}{24}\right)$

$=2\operatorname{cis}-\left(\frac{11\pi}{24}\right)$

$=\underline{\underline{2\operatorname{cis}\frac{-11\pi}{24}}}$

Question 4 (27 marks)

(a) $z^6 = (\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$ by De Moivre's theorem

Using binomial theorem we have

$(\cos\theta + i\sin\theta)^6 = \cos^6\theta + {}^6C_1 \cos^5\theta (i\sin\theta)$

$+ {}^6C_2 \cos^4\theta (i\sin\theta)^2 + {}^6C_3 \cos^3\theta (i\sin\theta)^3$

$+ {}^6C_4 \cos^2\theta (i\sin\theta)^4 + {}^6C_5 \cos\theta (i\sin\theta)^5 + (i\sin\theta)^6$

$= \cos^6\theta + 6\cos^5\theta \times i\sin\theta + 15\cos^4\theta \times i^2\sin^2\theta$

$+ 20\cos^3\theta \times i^3\sin^3\theta + 15\cos^2\theta \times i^4\sin^4\theta$

$+ 6\cos\theta \times i^5\sin^5\theta + i^6\sin^6\theta$

$= \cos^6\theta + 6\cos^5\theta \times i\sin\theta - 15\cos^4\theta\sin^2\theta$ (5)

$- i\,20\cos^3\theta\sin^3\theta + 15\cos^2\theta\sin^4\theta + i\,6\cos\theta\sin^5\theta$

$+ -\sin^6\theta$

Equating real and imaginary parts

i) $\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$

ii) $\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$ (4)

(iii) $\tan 6\theta = \dfrac{\sin 6\theta}{\cos\theta}$

$= \dfrac{6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta}{\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta}$

Divide both numerator and denominator by $\cos^6\theta$ we get

$= \dfrac{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}$ (4)

(b) (i) $z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta$

$\frac{1}{z^n} = z^{-n} = (\cos\theta + i\sin\theta)^{-n}$

$= \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$

$z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= \underline{\underline{2\cos n\theta}}$ (2)

$z^n - \frac{1}{z^n} = \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$

$= \underline{\underline{2i\sin n\theta}}$ (2)

(c)(i) $(2i\sin\theta)^6 = \left(z-\frac{1}{z}\right)^6$

$= z^6 - {}^6C_1 z^5 \times \frac{1}{z} + {}^6C_2 z^4\left(\frac{1}{z}\right)^2 - {}^6C_3 z^3\left(\frac{1}{z}\right)^3 + {}^6C_4 z^2\left(\frac{1}{z}\right)^4 - {}^6C_5 z\left(\frac{1}{z}\right)^5 + \left(\frac{1}{z}\right)^6$

$= z^6 - 6z^4 + 15z^2 - 20 + \frac{15}{z^2} - \frac{6}{z^4} + \frac{1}{z^6}$

$= \left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$

$= 2\cos 6\theta - 6\times 2\cos 4\theta + 15\times 2\cos 2\theta - 20$ (5)

$= 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$

$64\times -1\sin^6\theta = 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$

$\sin^6\theta = \frac{2(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)}{-64}$

$= \frac{-1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$

(ii) $z^2 - \frac{1}{z^2} = 2i\sin 2\theta$ and $z^2 + \frac{1}{z^2} = 2\cos 2\theta$

$\frac{\left(z^2 - \frac{1}{z^2}\right)\times i}{\left(z^2 + \frac{1}{z^2}\right)} = -\sqrt{3}$ can be written as

$\frac{(2i\sin 2\theta)\times i}{2\cos 2\theta} = -\sqrt{3}$

ie $\frac{-2\sin 2\theta}{2\cos 2\theta} = -\sqrt{3}$

$\tan 2\theta = \sqrt{3}$, $0 \le \theta \le 360°$

Let $u = 2\theta$ $\quad 0 \le u \le 720°$

$\tan u = \sqrt{3}$

$u = \frac{\pi}{3}, \frac{\pi}{3}+\pi, \frac{\pi}{3}+2\pi, \frac{\pi}{3}+3\pi$

$= \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}$ (5)

$\theta = \frac{u}{2} = \frac{\pi}{6}, \frac{4\pi}{6}, \frac{7\pi}{6}, \frac{10\pi}{6}$

$= \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

$z = \cos\theta + i\sin\theta$

$= \text{cis}\frac{\pi}{6}, \text{cis}\frac{2\pi}{3}, \text{cis}\frac{7\pi}{6}, \text{cis}\frac{5\pi}{3}$

Question 5 (17 marks)

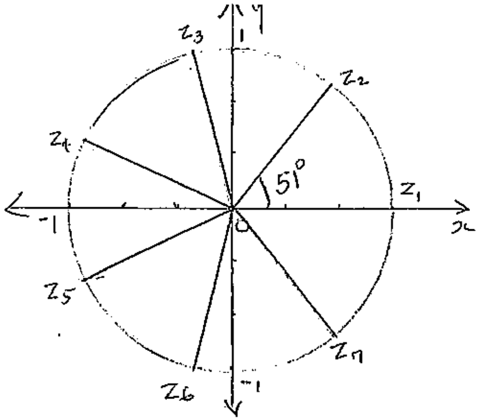
(i) $z^7 = 1 = \cos 2k\pi + i\sin 2k\pi$

$z = (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{7}}$

$= \cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}$

$k = 0, 1, 2, 3, 4, 5, 6$ (6)

$k=0$	$k=3$
$z_1 = \cos 0 + i\sin 0 = 1$	$z_4 = \cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7}$
$k=1$	$k=4$
$z_2 = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$	$z_5 = \cos\frac{8\pi}{7} + i\sin\frac{8\pi}{7}$
	$z_6 = \text{cis}\frac{10\pi}{7}$
$k=2$	
$z_3 = \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}$	$z_7 = \text{cis}\frac{12\pi}{7}$



(ii) $z^7 - 1 = (z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)(z-z_6)(z-z_7)$

$= (z-1)[(z-z_2)(z-z_7)][(z-z_3)(z-z_6)][(z-z_4)(z-z_5)]$

$= (z-1)[z^2 - z(z_2+z_7) + z_2z_7][z^2 - z(z_3+z_6) + z_3z_6]$

$[z^2 - z(z_4+z_5) + z_4z_5]$

$z_2+z_7 = z_2+\overline{z_2} = 2\cos\frac{2\pi}{7}$

$z_3+z_6 = z_3+\overline{z_3} = 2\cos\frac{4\pi}{7}$

$z_4+z_5 = z_4+\overline{z_4} = 2\cos\frac{6\pi}{7}$

$z_2z_7 = z_2\overline{z_2} = |z_2|^2 = 1$

$z_3z_6 = z_3\overline{z_3} = |z_3|^2 = 1$

$z_4z_5 = z_4\overline{z_4} = |z_4|^2 = 1$

$z^7 - 1 = (z-1)\left(z^2 - 2\cos\frac{2\pi}{7}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{7}z + 1\right)$ (4)

$\left(z^2 - 2\cos\frac{6\pi}{7} + 1\right)$

(iii) $z_1+z_2+z_3+z_4+z_5+z_6+z_7 =$

$= \left(1+\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}+\cos\frac{8\pi}{7}+\cos\frac{10\pi}{7}+\cos\frac{12\pi}{7}\right)$

$+i\left(\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{6\pi}{7}+\sin\frac{8\pi}{7}+\sin\frac{10\pi}{7}+\sin\frac{12\pi}{7}\right)$

$= -\frac{\text{coefficient of } z^6}{\text{coefficient of } z^7} = \frac{-0}{1} = 0$ (2)

(iv) A complex number equal to zero means its real part and imaginary part are separately equal to zero

From (iii) we have

$1+\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}+\cos\frac{8\pi}{7}+\cos\frac{10\pi}{7}+\cos\frac{12\pi}{7} =$

$\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7} = 1$

$\because \cos\frac{8\pi}{7} = \cos 2\pi - \frac{8\pi}{7} = \cos\frac{6\pi}{7}$

$\cos\frac{10\pi}{7} = \cos 2\pi - \frac{10\pi}{7} = \cos\frac{4\pi}{7}$

$\cos\frac{12\pi}{7} = \cos 2\pi - \frac{12\pi}{7} = \cos\frac{2\pi}{7}$

$2\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}\right) = -1$ (5)

$\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7} = \frac{-1}{2}$

Question 6 (17 marks)

(a)(i) $1+\omega+\omega^2=0$

$1-\omega-\omega^2 = 1-(\omega+\omega^2)$

$= 1-(-1) = 2$

$1-\omega+\omega^2 = -\omega-\omega^2-\omega+\omega^2$

$= -2\omega$

$1+\omega-\omega^2 = -\omega^2-\omega^2$

$= -2\omega^2$ (4)

$(1-\omega-\omega^2)(1-\omega+\omega^2)(1+\omega-\omega^2)$

$= 2\times -2\omega \times -2\omega^2$

$= 8\omega^3 = 8$

(ii) Sum of roots

$= a\omega+b\omega^2+a\omega^2+b\omega$

$= a(\omega+\omega^2)+b(\omega+\omega^2)$

$= (\omega+\omega^2)(a+b)$

$= -(a+b)$ (4)

Product of roots

$=(a\omega+b\omega^2)(a\omega^2+b\omega)$

$= a^2\omega^3+ab\omega^2+ab\omega^4+b^2\omega^3$

$= a^2+ab(\omega^2+\omega^4)+b^2$

$= a^2+ab(\omega^2+\omega)+b^2$

a^2-ab+b^2

The quadratic equation whose roots are $a\omega+b\omega^2$ and $a\omega^2+b\omega$ is given by

$z^2+(a+b)z+(a^2-ab+b^2)=0$

(b) $x+iy = \dfrac{a+ib}{c+id}$ —— (1)

Taking conjugates we get

$\overline{x+iy} = \overline{\dfrac{a+ib}{c+id}} = \dfrac{\overline{a+ib}}{\overline{c+id}}$

$x-iy = \dfrac{a-ib}{c-id}$ —— (2)

Multiply (1) and (2)

$(x+iy)(x-iy) = \dfrac{a+ib}{c+id}\times\dfrac{a-ib}{c-id}$

$x^2-(iy)^2 = \dfrac{a^2-(ib)^2}{c^2-(id)^2}$ (4)

$x^2+y^2 = \dfrac{a^2+b^2}{c^2+d^2}$

(c) $LHS = \left(\dfrac{1+\cos\theta+i\sin\theta}{1+\cos\theta-i\sin\theta}\right)^n$

$= \left[\dfrac{2\cos^2\frac{\theta}{2}+i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}-i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]^n$

$= \left[\dfrac{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right)}\right]^n$

$= \left[\dfrac{\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}}\right]^n$ (5)

$= \left[\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right]^n\left[\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right]^{-n}$

$= \left[\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right]^n\left[\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^{-1}\right]^{-n}$

$= \left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^n\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^n$

$= \left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^{2n}$

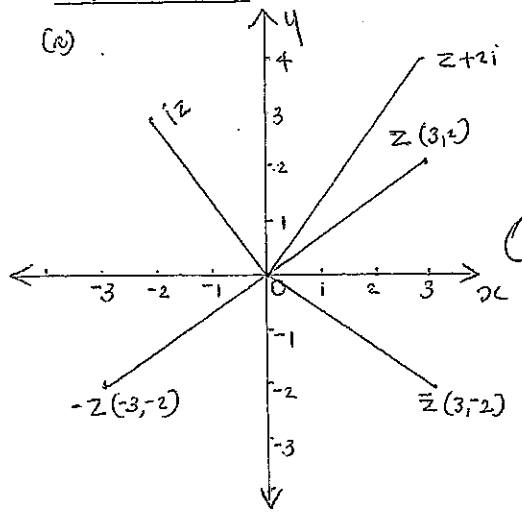
$= \cos\frac{2n\theta}{2}+i\sin\frac{2n\theta}{2}$

$= \cos n\theta+i\sin n\theta$

$= RHS.$

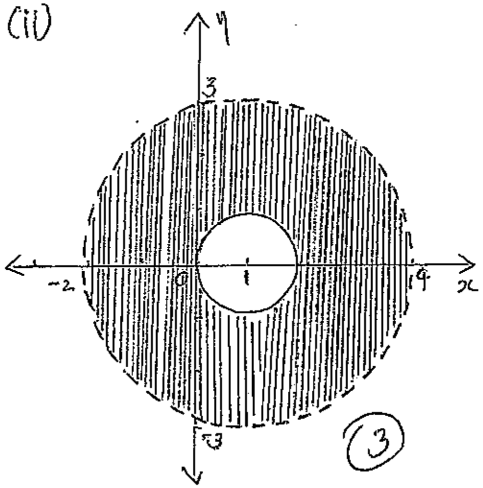
Question 7 (26 marks)

(a)



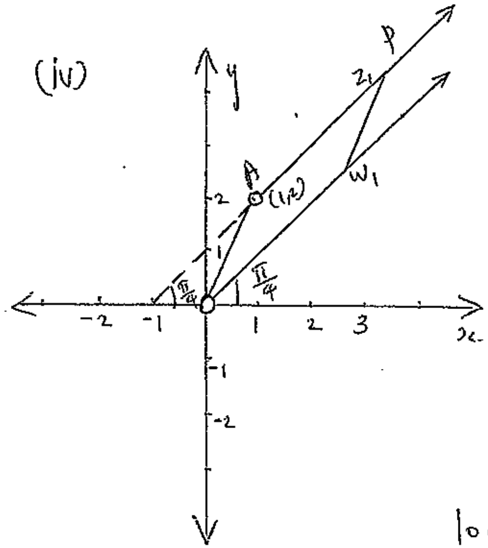
(5)

(ii)



(3)

(iv)



$\arg(z-(1+2i)) = \frac{\pi}{4}$

Gradient of AP = $\tan\frac{\pi}{4} = 1$

Equation of AP

$y - 2 = 1(x-1)$

$y - 2 = x - 1$

$y = x - 1 + 2$

$y = x + 1$

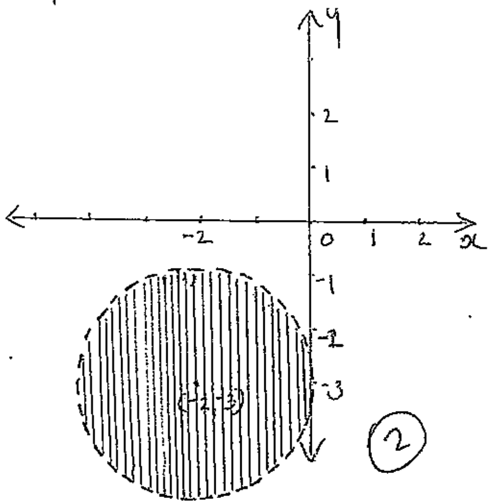
(5)

locus is the ray given by

$y = x+1,\ x > 1,\ y > 2$

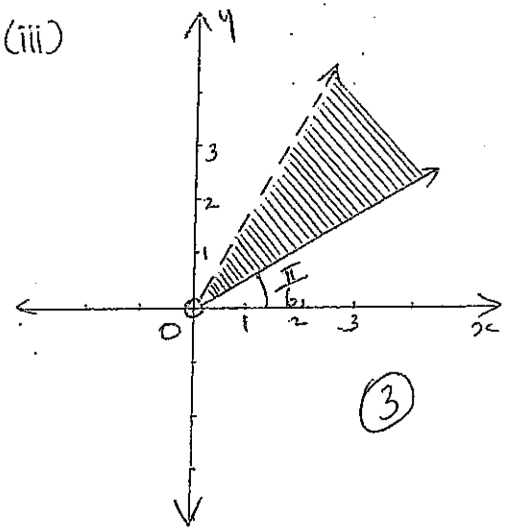
(b)(i) $|z+2+3i| < 2$

$|z-(-2-3i)| < 2$



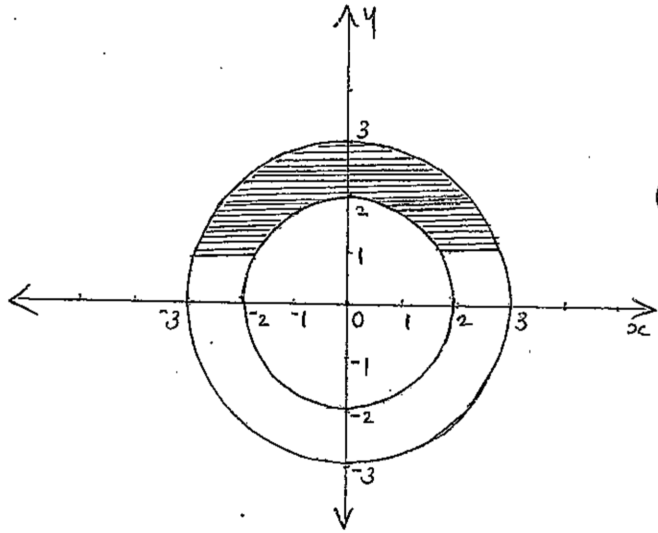
(2)

(iii)



(3)

(v) $2 \le |z| \le 3$ and $\mathrm{Im}(z) \ge 1$



(3)

(c) $|z+w|^2 = (z+w)\overline{(z+w)}$

$= (z+w)(\bar{z}+\bar{w})$

$= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w}$

$= |z|^2 + z\bar{w} + \bar{z}w + |w|^2$ —— (1)

$|z-w|^2 = (z-w)\overline{(z-w)}$

$= (z-w)(\bar{z}-\bar{w})$

$= z\bar{z} - z\bar{w} - \bar{z}w + w\bar{w}$

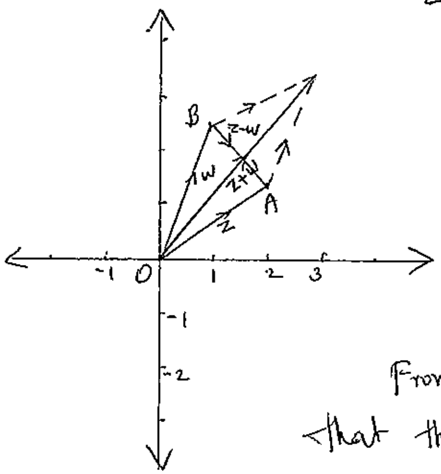
$= |z|^2 - z\bar{w} - \bar{z}w + |w|^2$ —— (2)

5

Adding (1) and (2) we get

$|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$

$= 2(|z|^2 + |w|^2)$

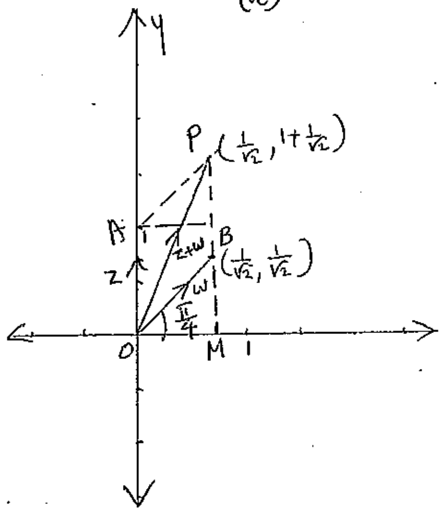


$|z+w|$ and $|z-w|$ represent the lengths of the longer and shorter diagonals of the parallelogram.

From the above we conclude that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the sides.

Question 8 (14 marks)

(a)



ΔAOP is isosceles $(AP = OA = 1)$

$\angle AOP = \dfrac{\pi - (\frac{\pi}{2} + \frac{\pi}{4})}{2} = \dfrac{\pi}{8}$

$\arg(z+w) = \dfrac{\pi}{2} - \dfrac{\pi}{8} = \dfrac{3\pi}{8}$

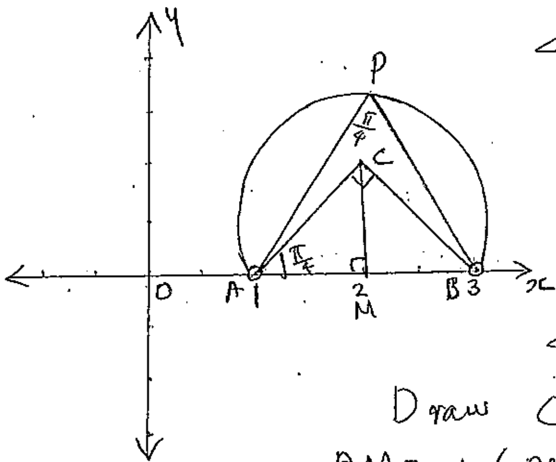
$\tan\dfrac{3\pi}{8} = \dfrac{PM}{OM}$

$= \dfrac{1+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \dfrac{\frac{\sqrt{2}+1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$

4

$= \dfrac{\sqrt{2}+1}{\sqrt{2}} \times \dfrac{\sqrt{2}}{1} = \sqrt{2}+1$

(b) (i) $\arg\left(\dfrac{z-3}{z-1}\right) = \dfrac{\pi}{4}$



$\angle ACB = 90°$ (angle at the centre is twice angle at the circumference.

ΔACB is isosceles

($AC = BC =$ radii)

$\angle CAB = \dfrac{180-90}{2} = 45°$

Draw $CM \perp OX$

$AM = 1$ (perpendicular from the centre of a circle to a chord bisects the chord)

$\tan 45 = \frac{CM}{AM} = \frac{CM}{1}$

$1 = \frac{CM}{1} \quad \therefore CM = 1$

Centre $= (2,1)$ (6)

$AC = \sqrt{AM^2 + MC^2}$ (Pythagoras' theorem)

$= \sqrt{1^2 + 1^2} = \sqrt{2}$

radius $= \sqrt{2}$

locus is the major arc APB of the circle with the centre at $(2,1)$ and radius $\sqrt{2}$, excluding the points A and B.

(ii) $\left|\frac{z-1}{z+2}\right| = 2$

Let $z = x+iy$ (4)

$\frac{|z-1|}{|z+2|} = 2$

$|z-1| = 2|z+2|$

$|x+iy-1| = 2|x+iy+2|$

$|x-1+iy|^2 = 4|x+2+iy|^2$

$(x-1)^2 + y^2 = 4\left[(x+2)^2 + y^2\right]$

$x^2 - 2x + 1 + y^2 = 4\left[x^2 + 4x + 4 + y^2\right]$

$x^2 - 2x + 1 + y^2 = 4x^2 + 16x + 16 + 4y^2$

$\cancel{4x^2} + \cancel{16x} + 16 + \cancel{4y^2} - \cancel{x^2} + \cancel{2x} - 1 - \cancel{y^2} = 0$

$3x^2 + 3y^2 + 18x + 15 = 0$

$x^2 + y^2 + 6x + 5 = 0$

$x^2 + 6x + y^2 = -5$

$x^2 + 6x + 9 + y^2 = -5 + 9$

$(x+3)^2 + y^2 = 4$

Locus is the circle with centre $(-3, 0)$ and radius 2

Question 9 (14 marks)

(i) $x = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$x^5 = \cos 2\pi + i\sin 2\pi = 1$

$x^5 - 1 = 0$ (3)

$(x-1)(x^4 + x^3 + x^2 + x + 1) = 0$

Since $x \neq 1$, we have

$x^4 + x^3 + x^2 + x + 1 = 0$,

(ii) $b^2 + b - 1$

$= \left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) - 1$

$= x^2 + 2 + \frac{1}{x^2} + x + \frac{1}{x} - 1$

$= x^2 + \frac{1}{x^2} + x + \frac{1}{x} + 1$ (2)

$= \frac{x^4 + 1 + x^3 + x + x^2}{x^2}$

$= 0$ (from (i))

(iii) $x^2 - bx + 1$

$= x^2 - \left(x + \frac{1}{x}\right)x + 1$

$= x^2 - x^2 - 1 + 1$

$= 0$

$x = \frac{b \pm \sqrt{b^2 - 4}}{2}$

$= \frac{b \pm \sqrt{-(4-b^2)}}{2}$

$= \frac{b \pm i\sqrt{4-b^2}}{2}$ (4)

$= \frac{b + i\sqrt{4-b^2}}{2}$

($x = \frac{b - i\sqrt{4-b^2}}{2}$ is rejected because the imaginary part of x is positive)

(iv) $b^2 + b - 1 = 0$ from (ii)

$b = \frac{-1 \pm \sqrt{1 - 4\times1\times-1}}{2} = \frac{-1 \pm \sqrt{5}}{2}$

$= \frac{-1+\sqrt{5}}{2}$ ($\because \frac{-1-\sqrt{5}}{2} < 0$ no possible)

we have

$x = \frac{b + i\sqrt{4-b^2}}{2}$

$\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \frac{b}{2} + \frac{i\sqrt{4-b^2}}{2}$

$\cos\frac{2\pi}{5} = \frac{b}{2} = \frac{-1+\sqrt{5}}{4}$ (5)

$\sin\frac{2\pi}{5} = \frac{1}{2}\sqrt{4-b^2}$

$4 - b^2 = 4 - \frac{(\sqrt{5}-1)^2}{4} = 4 - \frac{(5+1-2\sqrt{5})}{4}$

$= 4 - \frac{(6-2\sqrt{5})}{4} = 4 - \frac{(3-\sqrt{5})}{2}$

$= \frac{8-3+\sqrt{5}}{2} = \frac{5+\sqrt{5}}{2}$

$\therefore \sin\frac{2\pi}{5} = \frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}$