GIRRAWEEN HIGH SCHOOL

MATHEMATICS EXTENSION 2

TASK 1 2012 – COMPLEX NUMBERS

ANSWERS COVER SHEET

FINAL MARK

Name:______________________________

Teacher:____________________________

	MARK	E1	E2	E3	E4	E5	E6	E7	E8	E9
1	/7	√		√						√
	/7									
2	/9	√		√						√
	/9									
3	/18	√		√						√
	/18									
4	/11	√		√						√
	/11									
5	/16	√		√						√
	/16									
6	/18	√		√						√
	/18									
7	/13	√		√						√
	/13									
8	/15	√		√						√
	/15									
9	/9	√		√						√
	/9									
TOTAL	/116	/116				/116				/116

HSC Outcomes **Mathematics Extension 2**

E1 appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems.

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings.

E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections.

E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials.

E5 uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces, resisted motion and circular motion

E6 combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions.

E7 uses the techniques of slicing and cylindrical shells to determine volumes.

E8 applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems.

E9 communicates abstract ideas and relationships using appropriate notation and logical argument.

Girraween High School

Year 12

Mathematics Extension 2

Task 1 December 2011 – Complex Numbers

Instructions:

*Attempt all questions.

*Write all answers on your own paper. Remember to start each of Questions 1, 2, 3, 4, 5 and 6 on a separate sheet of paper.

*Show all necessary working.

*Marks may be deducted for careless or badly arranged work.

*Time allowed: 100 minutes

	Marks
Question 1 ***(7 Marks)***	
Given that $z = 5 - 3i$ and $w = 4 + i$ find	
(a) $2z + 3w$	1
(b) zw	1
(c) $w\bar{z}$	2
(d) z^2	1
(e) $\frac{z}{w}$	2

Question 2 ***(9 Marks)***

Given that $z = i\sqrt{3} - 1$ and $w = 2 + 2i$

(a) Find zw in Cartesian $(x + iy)$ form	1
(b) Convert z and w to modulus/ argument form.	4
(c) Hence find zw in modulus/argument form and use your result to find the exact value of $\cos\frac{11\pi}{12}$.	4

Question 3 ***(18 Marks)*** **Marks**

(a) (i) Convert $-2 + 2i\sqrt{3}$ to molulus/ argument form. **2**

(ii) Hence find the value of $(-2 + 2i\sqrt{3})^7$ in Cartesian form. **4**

(b) (i) Find all real numbers x and y such that $(x + iy)^2 = 3 - 4i$. **5**

(ii) Hence solve the quadratic equation $z^2 + (4 + 3i)z + (1 + 7i) = 0$ **3**

(c) Find the five 5^{th} roots of $16\sqrt{3} - 16i$. Leave your answers in modulus/ argument form. **4**

Question 4 ***(11 Marks)***

Sketch these regions on separate Argand diagrams:

(a) $|z - (3 + 4i)| < 5$ **3**

(b) $-\frac{\pi}{6} < Arg(z - 2) \leq \frac{\pi}{2}$ **4**

(c) $2 \leq |z| < 3$ and $Arg z < -\frac{\pi}{2}$ **4**

Question (5) ***(16 Marks)***

(a) Find the Cartesian equations for the following loci:

(i) $|z - 2i + 3| = 6$ **2**

(ii) $|z + 4i| = |z - 3|$ **3**

Question 5 continues on the next page

Question 5 (continued) **Marks**

(b) z is a complex number so that $\overrightarrow{OA} = z$. Copy the Argand diagram

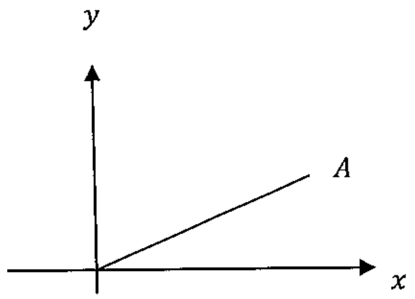
below on to your working paper and draw in

(i) $\overrightarrow{OB} = z + 1$ **1**

(ii) $\overrightarrow{OC} = z - i$ **1**

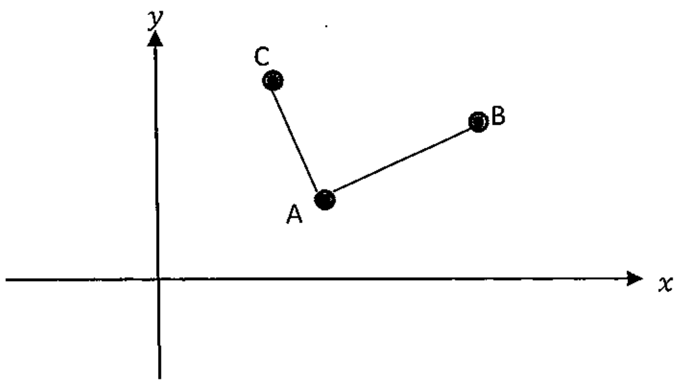
(iii) $\overrightarrow{OD} = iz$ **1**

(iv) $\overrightarrow{OE} = -z$ **1**



(c) A, B, C and D are points on an Argand diagram such that AB = AC

and $\angle BAC = \frac{\pi}{2}$. (See diagram below). If $\overrightarrow{OA} = z_1$ and $\overrightarrow{OB} = z_2$,



(i) Find an expression for $\overrightarrow{OC}$ in terms of z_1 and z_2. **1**

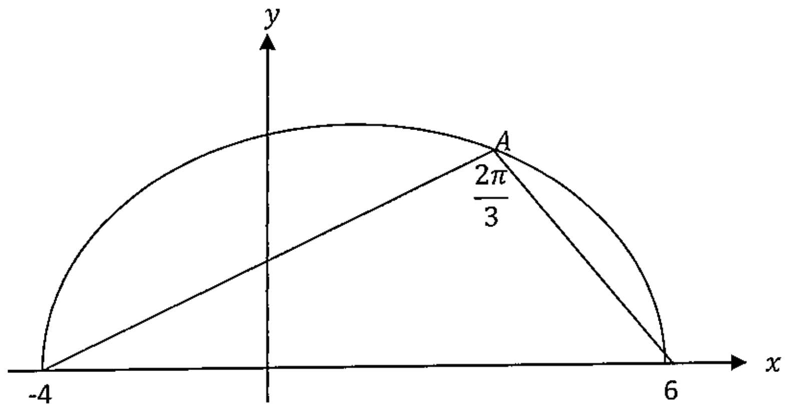
(ii) Find an expression for $\overrightarrow{OD}$ in terms of z_1 and z_2 if $ABDC$ is a square. **1**

(iii) If ABCD is a square show that $AD \times BC = 2|z_2 - z_1|^2$ and that $\frac{\overrightarrow{BC}}{\overrightarrow{AD}}$ is entirely imaginary. **5**

Question 6 *(18 Marks)* **Marks**

(a) The locus $Arg\left(\frac{z-6}{z+4}\right) = \frac{2\pi}{3}$ represents part of a circle (see below.) **5**

If $\overrightarrow{OA} = z$ find the centre and radius of the circle and the Cartesian equation for the locus of z. (Note that this is NOT a semicircle.)



(b) If $z = \cos\theta + i\sin\theta$

(i) Show that $z - \frac{1}{z} = 2i\sin\theta$. **2**

(ii) Show that $z^n + \frac{1}{z^n} = 2cosn\theta$ **2**

(iii) Hence express $sin^6\theta$ in terms of $\cos 6\theta$, $\cos 4\theta$ and $\cos 2\theta$. **4**

(c) If $w = x + iy$ and $w = \frac{z-2i}{1-z}$

(i) Show that $z = \frac{x+i(y+2)}{(x+1)+iy}$ **2**

(ii) Find the locus of w if $|z| = 2$. **3**

Question 7 *(13 Marks)*

(a) If w is the non real root of $z^7 - 1 = 0$ with the smallest possible positive argument:

(i) Show that $w = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ and that w^2, w^3, w^4, w^5 and w^6 are the other non real roots of $z^7 - 1 = 0$. **3**

(ii) Show that $w + w^2 + w^3 + w^4 + w^5 + w^6 = -1$ **1**

(iii) By expanding $(w + w^6)(w^2 + w^5)(w^3 + w^4)$ show that $(w + w^6)(w^2 + w^5)(w^3 + w^4) = 1$. Hence show that $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \frac{1}{8}$. **4**

(b) Resolve $z^8 + 1$ into real quadratic factors. Hence show that **5**

$$cos4\theta = 8(cos\theta - cos\frac{\pi}{8})(cos\theta + cos\frac{\pi}{8})(cos\theta - cos\frac{3\pi}{8})(cos\theta + cos\frac{3\pi}{8})$$

(You do NOT need to prove $z^n + \frac{1}{z^n} = 2cosn\theta$ if $|z| = 1$ this time.)

Examination continues on the next page

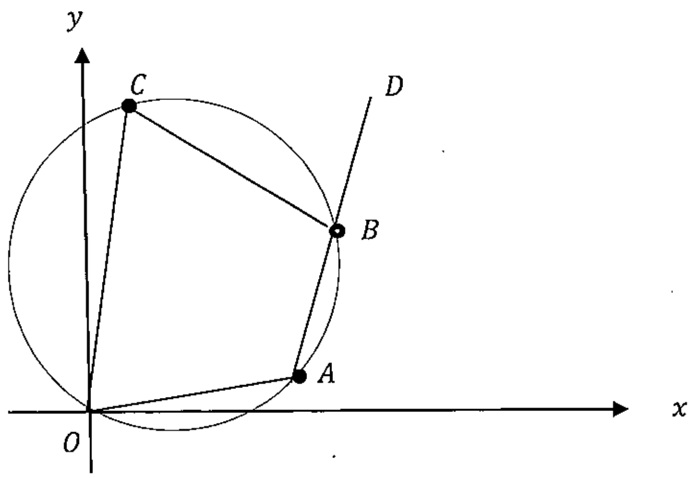
Question 8 *(15 Marks)* **Marks**

(a) (i) By expanding $(cos\theta + isin\theta)^5$ show that $tan5\theta = \frac{5tan\theta - 10tan^3\theta + tan^5\theta}{1 - 10tan^2\theta + 5tan^4\theta}$ **3**

(ii)Hence show that $tan\frac{\pi}{5}$ is a solution of $z^4 - 10z^2 + 5 = 0$ **3**

(iii)Hence find the exact value of $tan\frac{\pi}{5}$. **4**

(b) z_1, z_2 and z_3 are represented by the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively.
If O, A, B and C are the vertices of a circle and AB is extended to D (see below)x



(i) Show that $Arg\left(\frac{z_3}{z_1}\right) = Arg\left(\frac{z_3 - z_2}{z_2 - z_1}\right)$ **3**

(ii) Show that if $\overrightarrow{OE} = \frac{1}{z_1}, \overrightarrow{OF} = \frac{1}{z_2}, \overrightarrow{OG} = \frac{1}{z_3}$, then $E, F,$ and G are in a straight line. **2**

Question 9 *(9 Marks)*

(i) Show that $\frac{1+cos\theta+isin\theta}{1-cos\theta-isin\theta} = \frac{isin\theta}{1-cos\theta}$. **2**

(ii) Hence or otherwise show that $\frac{1+cos\theta+isin\theta}{1-cos\theta-isin\theta} = icot\left(\frac{\theta}{2}\right)$ **2**

(iii) Hence or otherwise show that if $z = icot\left(\frac{\theta}{2}\right)$ then $\frac{z-1}{z+1} = cos\theta + isin\theta$. **2**

(iv) Hence solve the equation $\left(\frac{z-1}{z+1}\right)^4 = -1$. **3**

END OF EXAMINATION

Solutions Dec. 2011 Complex N°5

Q.(1)(a) $2z + 3w$

$= 2(5-7i) + 3(4+i)$

$= 10 - 6i + 12 + 3i$

$= 22 - 3i.$

(b) zw

$= (5-3i)(4+i)$

$= 20 + 5i - 12i + 3$

$= 23 - 7i$

(c) $w\bar{z}$

$= (4+i)(5+3i)$

$= 20 + 12i + 5i - 3$

$= 17 + 17i$ or $17(1+i)$

(d) z^2

$= (5-3i)^2$

$= 25 - 30i - 9$

$= 16 - 30i$

(e) $\frac{z}{w}$

$= \frac{(5-3i)}{(4+i)} \times \frac{4-i}{(4-i)}$

$= \frac{20 - 5i - 12i - 3}{17}$

$= \frac{17(1-i)}{17}$

$= 1 - i.$

Q.(2)(a)

zw

$= (-1 + i\sqrt{3})(2 + 2i)$

$= -2 - 2i + 2i\sqrt{3} - 2\sqrt{3}$

$= -2 - 2\sqrt{3} + i(2\sqrt{3} - 2).$

(b) $|z| = \sqrt{1^2 + (\sqrt{3})^2}$

$= 2$

$\text{Arg } z = \tan^{-1}(-\sqrt{3})$ [but Q_2] = Quadrant 2.

$= \frac{2\pi}{3}$

$\therefore z = 2 \text{cis } \frac{2\pi}{3}$

$|w| = \sqrt{2^2 + 2^2}$

$= 2\sqrt{2}.$

$\text{Arg } w = \tan^{-1}\left(\frac{2}{2}\right)$

$= \frac{\pi}{4}$

$w = 2\sqrt{2} \text{ cis } \frac{\pi}{4}.$

(c) $\therefore$ zw

$= 2 \text{cis } \frac{2\pi}{3} \times 2\sqrt{2} \text{ cis } \frac{\pi}{4}$

$= 4\sqrt{2}\left(\cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12}\right)$

(2)(c) (continued).

Equating real parts of answers to (2)(a) & (c)

$4\sqrt{2} \cos \frac{11\pi}{12} = -2 - 2\sqrt{3}$

$\therefore \cos \frac{11\pi}{12} = \frac{-2 - 2\sqrt{3}}{4\sqrt{2}}$

$\cos \frac{11\pi}{12} = -\frac{(\sqrt{2} + \sqrt{6})}{4}.$

(3)(a)(i) $-2 + 2i\sqrt{3}$

$= 4 \text{ cis } \frac{2\pi}{3}.$

(ii) $\therefore$ By De Moivre's theorem,

$(-2 + 2i\sqrt{3})^7$

$= \left(4 \text{ cis } \frac{2\pi}{3}\right)^7$

$= 4^7 \text{ cis } \frac{14\pi}{3}$ Note: $\cos \frac{14\pi}{3} = \cos \frac{2\pi}{3}$, $\sin \frac{14\pi}{3} = \sin \frac{2\pi}{3}$.

$= 16384\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$

$= -8192 + 8192i\sqrt{3}.$

Q.(3)(b)(i) $(x+iy)^2 = 3-4i$

$\therefore (x^2-y^2)+2ixy = 3-4i$.

$\therefore x^2-y^2 = 3$ (1) Equating reals

$2xy = -4$ (2) Equating imaginaries

Using (2): $y = -\frac{2}{x}$.

Sub. (2) in (1): $x^2 - \left(-\frac{2}{x}\right)^2 = 3$

$x^2 - \frac{4}{x^2} = 3$ $\times$ BS by x^2

$x^4 - 4 = 3x^2$

$x^4 - 3x^2 - 4 = 0$

$(x^2-4)(x^2+1) = 0$

As x is real, $x = \pm 2$.

As $y = -\frac{2}{x}$, $y = \mp 1$.

$\therefore$ The numbers are $x = 2, y = -1$

& $x = -2, y = 1$

$\therefore$ The 2 square roots of $3-4i$ are

$2-i$ & $-2+i$.

(ii) Hence as the solutions to

$z^2 + (4+3i)z + (1+7i) = 0$ are

$z = \frac{-4-3i \pm \sqrt{(4+3i)^2 - 4\times1\times(1+7i)}}{2}$

$= \frac{-4-3i \pm \sqrt{16+24i-9-4-28i}}{2}$

$= \frac{-4-3i \pm \sqrt{3-4i}}{2}$

$= \frac{-4-3i \pm (2-i)}{2}$ [from (i)]

$= \frac{-4-3i+2-i}{2}$ or $\frac{-4-3i-2+i}{2}$ $z = \frac{-2-4i}{2}$ or $z = \frac{-6-2i}{2}$

$z = -1-2i$ or $z = -3-i$

Q.(3)(c) $16\sqrt{3} - 16i = 32\left(\cos -\frac{\pi}{6} + i\sin -\frac{\pi}{6}\right)$

$\therefore$ The 5th roots are $2\left(\cos \frac{-\pi}{30} + i\sin \frac{-\pi}{30}\right)$

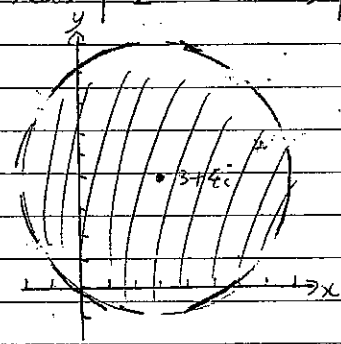
$2\left(\cos \frac{11\pi}{30} + i\sin \frac{11\pi}{30}\right)$

$2\left(\cos \frac{23\pi}{30} + i\sin \frac{23\pi}{30}\right)$

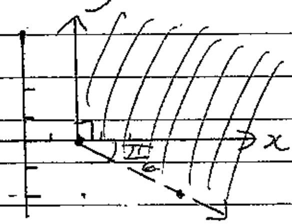
$2\left(\cos \frac{7\pi}{6} + i\sin \frac{7\pi}{6}\right)$

$2\left(\cos \frac{47\pi}{30} + i\sin \frac{47\pi}{30}\right)$

Q.(4)(a) $|z-(3+4i)| < 5$.

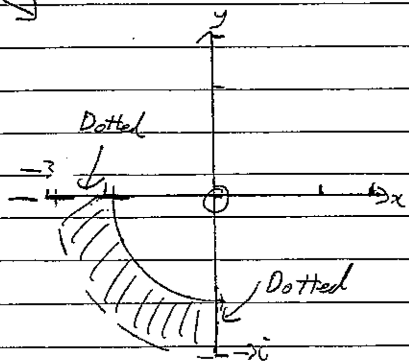


(b) $-\frac{\pi}{6} < \text{Arg}(z-2) \leq \frac{\pi}{2}$



(c) $2 \leq |z| < 3$

and $\text{Arg}\, z < -\frac{\pi}{2}$



Q.(5)(a)(i) $|z-2i+3| = 6$

If $z = x+iy$ then $(x+3)^2+(y-2)^2 = 36$

(ii) $|z+4i| = |z-3|$

Either: If $z = x+iy$

$$\sqrt{x^2+(y+4)^2} = \sqrt{(x-3)^2+y^2}$$

$$x^2+y^2+8y+16 = x^2-6x+9+y^2$$

$$-x^2-y^2-16$$

$$8y = -6x-7$$

$$6x+8y+7 = 0$$

OR

it will be the perpendicular bisector of the line between 3 and $-4i$.

$\frac{3}{}$ $\ \ $ $\frac{3}{2}-2i$ $\ \ $ $-4i$

m of bisector $= -\frac{3}{4}$

Passes through $\frac{3}{2}-2i$ $\ \ -6$ $\ \ -6x-\frac{i}{2}$

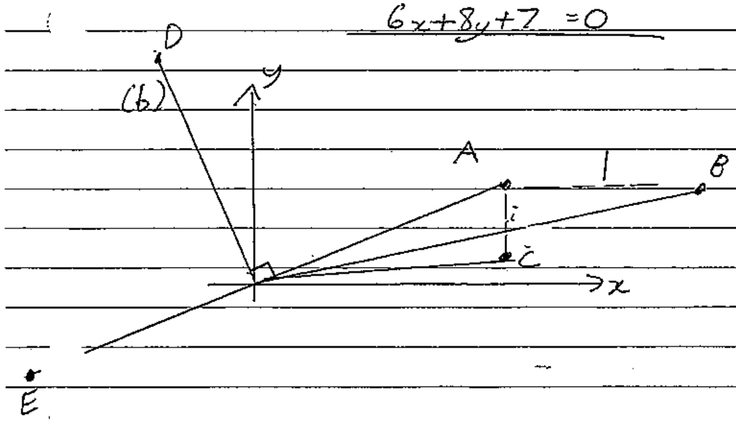
$$\therefore y+2 = -\frac{3}{4}\left(x-\frac{3}{2}\right) \quad \times 8$$

$$8y+16 = -6x+9$$

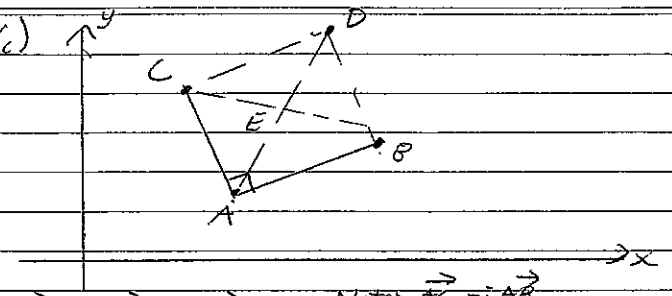
$$+6x-9$$

$$6x+8y+7 = 0$$

(b)



Q.(5)(c)



Note: $\vec{AC} = i\vec{AB}$
$= i(z_2-z_1)$

(i) $\vec{OC} = \vec{OA} + \vec{AC}$

$= z_1 + iz_2 - iz_1$

$\vec{OC} = z_1(1-i) + iz_2$.

(ii) $\vec{OD} = \vec{OB} + \vec{BD}$

$= z_2 + iz_2 - iz_1$

$= z_2(1+i) - iz_1$.

(iii) If ABCD is a square then $AD = \sqrt{(AB)^2+(BD)^2}$
[by Pythagoras]

$= \sqrt{2(AB)^2}$ [as BD = AB
– equal sides in square]

$= AB\sqrt{2}$.

Similarly, $BC = AB\sqrt{2}$.

Hence $AD \times BC$

$= AB\sqrt{2} \times AB\sqrt{2}$

$= 2(AB)^2$

$= 2|z_2-z_1|^2$ as $AB = |z_2-z_1|$.

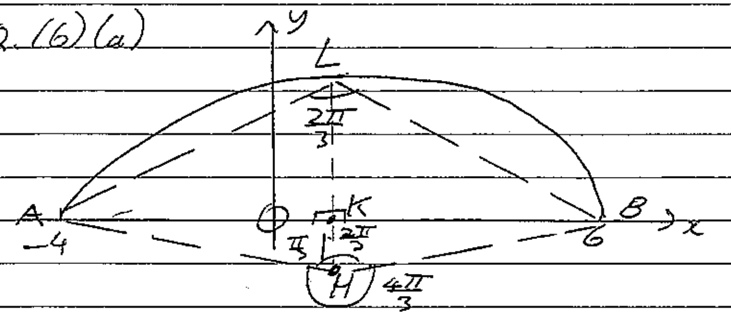
Also $\frac{\vec{BC}}{\vec{AD}}$: Letting E be point of intersection of $\vec{BC}$ & $\vec{AD}$

$\angle CED = 90°$ [diagonals of square ⊥].

But $\angle CED = Arg\left(\frac{\vec{BC}}{\vec{AD}}\right)$

$\therefore Arg\left(\frac{\vec{BC}}{\vec{AD}}\right) = 90°$

Q. (6)(a)



Marks

Note: Centre of circle will be on line $x = 1$ (by symmetry).

If H is circle centre, $A = -4$ & $B = 6$

then $\angle AHB$ (reflex) $= \frac{4\pi}{3}$ [$\angle$ at centre $= 2\times \angle$ at circumference on same arc].

$\therefore \angle AHB$ (obtuse) $= \frac{2\pi}{3}$ [revolution]

Letting $1 = K$,

& point at $x = 1$ on edge of circle $= L$

$\angle AHK = \frac{\pi}{3}$ (by symmetry).

$\therefore \angle KAH = \frac{\pi}{6}$ ($\angle$ sum $\triangle KAH$).

$\therefore KH = 5\tan\frac{\pi}{3}$

$= \frac{5}{\sqrt{3}}$.

$\therefore$ & $AH = \sqrt{5^2 + \left(\frac{5}{\sqrt{3}}\right)^2}$

$= \frac{10\sqrt{3}}{3}$.

$\therefore$ Locus of $z = (x-1)^2 + \left(y + \frac{5\sqrt{3}}{3}\right)^2 = \frac{100}{3}, y > 0$.

Q. (6)(b)(i) $z - \frac{1}{z}$

$= (\cos\theta + i\sin\theta) - (\cos\theta + i\sin -\theta)$ [by DeMoivre]

$= (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)$ as $\cos\theta$ even, $\sin\theta$ odd

$= 2i\sin\theta$.

(ii) $z^n + \frac{1}{z^n}$

$= (\cos n\theta + i\sin n\theta) + (\cos(-n\theta) + \sin(-n\theta))$ [by DeMoivre]

$= (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta)$ [as cos even, sin odd]

$= 2\cos n\theta$

(iii) If $z = \cos\theta + i\sin\theta$

$\left(z - \frac{1}{z}\right)^6$

$= (2i\sin\theta)^6$ [using (i)].

$= -64\sin^6\theta$. (1)

As $\left(z - \frac{1}{z}\right)^6$

$= z^6 - 6z^4 + 15z^2 - 20 + \frac{15}{z^2} - \frac{6}{z^4} + \frac{1}{z^6}$

$= \left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$

$= 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$ [using (ii)] (2)

Equating (1) & (2)

$-64\sin^6\theta = 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$

$\therefore \sin^6\theta = -\frac{\cos 6\theta}{32} + \frac{3\cos 4\theta}{16} - \frac{15\cos 2\theta}{32} + \frac{5}{16}$

Q.(6)(c)(i) $w = \dfrac{z-2i}{1-z}$

$\therefore w - wz = z - 2i$ 1

$w + 2i = z + wz$

$w + 2i = z(1+w)$

$\dfrac{w+2i}{1+w} = z$ 1

As $w = x + iy$,

$z = \dfrac{x+(y+2)i}{(x+1)+iy}$

(ii) If $|z| = 2$ then $\dfrac{|x+(y+2)i|}{|(x+1)+iy|} = 2$

Squaring BS:

$\therefore \dfrac{x^2+(y+2)^2}{(x+1)^2+y^2} = 4$ 1

$x^2+y^2+4y+4 = 4x^2+8x+4+4y^2$

$0 = 3x^2+8x+3y^2-4y$

$0 = x^2+\frac{8}{3}x+y^2-\frac{4}{3}y$ 1

$\frac{20}{9} = (x^2+\frac{8}{3}x+\frac{16}{9})+(y^2-\frac{4}{3}y+\frac{4}{9})$

$\therefore$ Locus of w is

$(x+\frac{4}{3})^2+(y-\frac{2}{3})^2 = \frac{20}{9}$ 1

Q.(7)(a)(i) If w is a root of $z^7 - 1 = 0$

Roots $= cis\,0, cis\frac{2\pi}{7}, cis\frac{4\pi}{7}, cis\frac{6\pi}{7}, cis\frac{8\pi}{7}, cis\frac{10\pi}{7}, cis\frac{12\pi}{7}$ 1

Clearly, $cis\frac{2\pi}{7}$ $(\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7})$ is non real root with smallest possible positive argument. 1

$w^2 = (cis\frac{2\pi}{7})^2 = cis\frac{4\pi}{7}$ which is a root of $z^7-1=0$

$w^3 = (cis\frac{2\pi}{7})^3 = cis\frac{6\pi}{7}$ " " "

$w^4 = (cis\frac{2\pi}{7})^4 = cis\frac{8\pi}{7}$ " " " 1

$w^5 = (cis\frac{2\pi}{7})^5 = cis\frac{10\pi}{7}$ " " "

$w^6 = (cis\frac{2\pi}{7})^6 = cis\frac{12\pi}{7}$ " " "

Hence w^2, w^3, w^4, w^5 & w^6 are the other non real roots.

(ii) By sum of roots of $z^7-1=0$

$[\alpha+\beta+\ldots = -\frac{b}{a}]$

$1+w+w^2+w^3+w^4+w^5+w^6 = 0$ 1

-1

Hence $w+w^2+w^3+w^4+w^5+w^6 = -1$

(iii) $(w+w^6)(w^2+w^5)(w^3+w^4)$

$= (w^3+w^6+w^8+w^{11})(w^3+w^4)$

$= w^6+w^7+w^9+w^{10}+w^{11}+w^{12}+w^{14}+w^{15}$ 1

$= w^6+1+w^2+w^3+w^4+w^5+1+w$

$= 1+(1+w+w^2+w^3+w^4+w^5+w^6)$ 1

$= 1$ $[$as $1+w+w^2+w^3+w^4+w^5+w^6=0$ from (ii)$]$ (1)

Also as $w+w^6$

$= (\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}+\cos\frac{12\pi}{7}+i\sin\frac{12\pi}{7})$

$= (\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}+\cos\frac{2\pi}{7}-i\sin\frac{2\pi}{7})$

$= 2\cos\frac{2\pi}{7}$ (2)

Similarly, $w^2+w^5 = 2\cos\frac{4\pi}{7}$

$[= -2\cos\frac{3\pi}{7}]$ (3)

& $w^3+w^4 = 2\cos\frac{6\pi}{7}$

$[= -2\cos\frac{\pi}{7}]$ (4) PTO →

Q.(7)(a)(iii) [continued]

$\therefore$ As $(w+w^6)(w^2+w^5)(w^3+w^4)=1$

$$2\cos\frac{2\pi}{7}\times -2\cos\frac{3\pi}{7}\times -2\cos\frac{\pi}{7}=1$$

$$8\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\cos\frac{\pi}{7}=1$$

$$\underline{\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}=\frac{1}{8}}$$

(7)(b) Roots of $z^8+1=0$ are

$z=\text{cis}\,\frac{\pi}{8}+\frac{2k\pi}{8}$, $k=0,1,2,\ldots 7$.

$$\therefore z^8+1=\left(z-\text{cis}\frac{\pi}{8}\right)\left(z-\text{cis}\frac{15\pi}{8}\right)\left(z-\text{cis}\frac{3\pi}{8}\right)\left(z-\text{cis}\frac{13\pi}{8}\right)$$
$$\left(z-\text{cis}\frac{5\pi}{8}\right)\left(z-\text{cis}\frac{11\pi}{8}\right)\left(z-\text{cis}\frac{7\pi}{8}\right)\left(z-\text{cis}\frac{9\pi}{8}\right)$$
$$=\left(z^2-2z\cos\frac{\pi}{8}+1\right)\left(z^2-2z\cos\frac{3\pi}{8}+1\right)\left(z^2-2z\cos\frac{5\pi}{8}+1\right)$$
$$\left(z^2-2z\cos\frac{7\pi}{8}+1\right) \quad (1)$$

*Note: As $\left(z-\text{cis}\frac{\pi}{8}\right)\left(z-\text{cis}\frac{15\pi}{8}\right)$

$$=z^2-z\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}+\cos\frac{\pi}{8}-i\sin\frac{\pi}{8}\right)+1$$
$$=z^2-2z\cos\frac{\pi}{8}+1.$$

Similarly, $\left(z-\text{cis}\frac{3\pi}{8}\right)\left(z-\text{cis}\frac{13\pi}{8}\right)=z^2-2z\cos\frac{3\pi}{8}+1$

$$\left(z-\text{cis}\frac{5\pi}{8}\right)\left(z-\text{cis}\frac{11\pi}{8}\right)=z^2-2z\cos\frac{5\pi}{8}+1$$
$$\left(z-\text{cis}\frac{7\pi}{8}\right)\left(z-\text{cis}\frac{9\pi}{8}\right)=z^2-2z\cos\frac{7\pi}{8}+1.$$

Continuing with (1): as $\cos\frac{5\pi}{8}=-\cos\frac{3\pi}{8}$ & $\cos\frac{7\pi}{8}=-\cos\frac{\pi}{8}$

$$z^8+1=\left(z^2-2z\cos\frac{\pi}{8}+1\right)\left(z^2-2z\cos\frac{3\pi}{8}+1\right)\left(z^2+2z\cos\frac{3\pi}{8}+1\right)$$
$$\left(z^2+2z\cos\frac{\pi}{8}+1\right)$$

$\div$ BS by z^4:

$$z^4+\frac{1}{z^4}=\left(z+\frac{1}{z}-2\cos\frac{\pi}{8}\right)\left(z+\frac{1}{z}-2\cos\frac{3\pi}{8}\right)\left(z+\frac{1}{z}+2\cos\frac{3\pi}{8}\right)$$
$$\left(z+\frac{1}{z}+2\cos\frac{\pi}{8}\right)$$

If $z=\cos\theta+i\sin\theta$, $z^4+\frac{1}{z^4}=2\cos 4\theta$ & $z+\frac{1}{z}=2\cos\theta$

Hence $2\cos 4\theta=\left(2\cos\theta-2\cos\frac{\pi}{8}\right)\left(2\cos\theta-2\cos\frac{3\pi}{8}\right)\left(2\cos\theta+2\cos\frac{3\pi}{8}\right)$
$$\left(2\cos\theta+2\cos\frac{\pi}{8}\right)$$

$$\underline{\cos 4\theta=8\left(\cos\theta-\cos\frac{\pi}{8}\right)\left(\cos\theta-\cos\frac{3\pi}{8}\right)\left(\cos\theta+\cos\frac{3\pi}{8}\right)\left(\cos\theta+\cos\frac{\pi}{8}\right)}$$

Q.(8)(a)(i)

$$(\cos\theta+i\sin\theta)^5=\cos^5\theta+5i\cos^4\theta\sin\theta-10\cos^3\theta\sin^2\theta-10i\cos^2\theta\sin^3\theta$$
$$+5\cos\theta\sin^4\theta+i\sin^5\theta.$$

$$\therefore \cos 5\theta+i\sin 5\theta=\cos^5\theta+5i\cos^4\theta\sin\theta-10\cos^3\theta\sin^2\theta-10i\cos^2\theta\sin^3\theta$$
$$+5\cos\theta\sin^4\theta-i\sin^5\theta \quad \text{[by DeMoivre]}.$$

$$\therefore \tan 5\theta=\frac{\sin 5\theta}{\cos 5\theta}$$
$$=\frac{5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta \;\div\cos^5\theta}{\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta\;\div\cos^5\theta}$$

$$\tan 5\theta=\frac{5\tan\theta-10\tan^3\theta+\tan^5\theta}{1-10\tan^2\theta+5\tan^4\theta}.$$

(8)(a)(ii) Hence as $\tan\pi=\tan\left(5\times\frac{\pi}{5}\right)=0$

$\tan\frac{\pi}{5}$ is a solution to $\tan 5\theta=0$

i.e. $\frac{5\tan\theta-10\tan^3\theta+\tan^5\theta}{1-10\tan^2\theta+5\tan^4\theta}=0$ $\times$ BS by $1-10\tan^2\theta+5\tan^4\theta$

$$\tan^5\theta-10\tan^3\theta+5\tan\theta=0$$
$$\tan\theta\left(\tan^4\theta-10\tan^2\theta+5\right)=0$$

As $\tan\frac{\pi}{5}\neq 0$, $\tan\frac{\pi}{5}$ is a solution of

$$\tan^4\theta-10\tan^2\theta+5=0$$

Letting $z=\tan\theta$, $z^4-10z^2+5=0$

(iii) Hence solving $z^4-10z^2+5=0$ [as a quadratic equation in z^2]

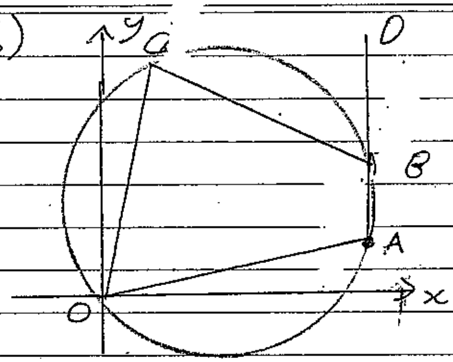
$$z^2=\frac{10\pm\sqrt{10^2-4\times1\times5}}{2\times1}$$
$$=\frac{10\pm\sqrt{80}}{2}$$
$$=\frac{10\pm4\sqrt{5}}{2}$$
$$z^2=5\pm2\sqrt{5}.$$

$\therefore z^2=5+2\sqrt{5}$ or $z^2=5-2\sqrt{5}$.

As $\frac{\pi}{5}<\frac{\pi}{4}$, $0<\tan\frac{\pi}{5}<1$.

$$\therefore \tan^2\left(\frac{\pi}{5}\right)=5-2\sqrt{5}.$$
$$\underline{\tan\frac{\pi}{5}=\sqrt{5-2\sqrt{5}}.}$$

Q. (8)(b)



(i) $\angle AOC = Arg\left(\frac{z_3}{z_1}\right)$ 1

$\angle CBD = Arg\left(\frac{z_3 - z_2}{z_2 - z_1}\right)$ 1

$\angle AOC = \angle CBD$ [exterior $\angle$ of cyclic quadrilateral] 1

$\therefore Arg\left(\frac{z_3}{z_1}\right) = Arg\left(\frac{z_3 - z_2}{z_2 - z_1}\right)$

(ii) If $\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}$ are in a straight line then

$Arg\left(\frac{1}{z_3} - \frac{1}{z_2}\right) = Arg\left(\frac{1}{z_2} - \frac{1}{z_1}\right)$

To prove this,

we know $Arg\left(\frac{z_3}{z_1}\right) = Arg\left(\frac{z_3 - z_2}{z_2 - z_1}\right)$

$\therefore Arg\left(\frac{z_3}{z_1} \times \frac{z_2 - z_1}{z_2 \cancel{z_3}}\right) = Arg\left(\frac{z_3 - z_2}{z_2 - z_1} \times \frac{(z_2 - z_1)}{z_2 z_3}\right)$ 1

as we are multiplying by the same complex number we are changing the arguments by the same amount.

$Arg\left(\frac{z_2 - z_1}{z_1 z_2}\right) = Arg\left(\frac{z_3 - z_2}{z_2 z_3}\right)$

$Arg\left(\frac{1}{z_1} - \frac{1}{z_2}\right) = Arg\left(\frac{1}{z_2} - \frac{1}{z_3}\right)$ 1

$\therefore \frac{1}{z_1}, \frac{1}{z_2}$ & $\frac{1}{z_3}$ are in the same straight line.

Q. (9)(i) $\frac{1 + \cos\theta + i\sin\theta}{1 - \cos\theta - i\sin\theta} \times \frac{(1 - \cos\theta) + i\sin\theta}{(1 - \cos\theta) + i\sin\theta}$ 1

$= \frac{(1 - \cos^2\theta) + i\sin\theta + i\sin\theta\cos\theta + i\sin\theta - i\sin\theta\cos\theta - \sin^2\theta}{(1 - \cos\theta)^2 + \sin^2\theta}$

$= \frac{\sin^2\theta + 2i\sin\theta - \sin^2\theta}{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}$

$= \frac{2i\sin\theta}{2 - 2\cos\theta}$

$= \frac{i\sin\theta}{1 - \cos\theta}$ 1

(ii) Using $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$
& $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$

$\frac{i\sin\theta}{1 - \cos\theta}$

$= \frac{2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1 - (1 - 2\sin^2\frac{\theta}{2})}$ 1

$= \frac{2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$

$= \frac{i\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$

$= i\cot\frac{\theta}{2}$. 1

PTO →

[Alternative (9)(ii)].

Otherwise for (9)(ii)

$$\frac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta}$$

As $\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right)-1$
$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$
& $\cos\theta$ also $= 1-2\sin^2\left(\frac{\theta}{2}\right)$

$$= \frac{1+2\cos^2\left(\frac{\theta}{2}\right)-1+2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1-\left(1-2\sin^2\frac{\theta}{2}\right)-2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{2\cos^2\left(\frac{\theta}{2}\right)+2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\left(\frac{\theta}{2}\right)-2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}\right)} \begin{matrix}\times \sin\frac{\theta}{2}+i\cos\frac{\theta}{2}\\ \times \sin\frac{\theta}{2}+i\cos\frac{\theta}{2}\end{matrix}$$

$$= \frac{\cot\frac{\theta}{2}\times\left(\cos\frac{\theta}{2}\sin\frac{\theta}{2}+i\cos^2\frac{\theta}{2}+i\sin^2\frac{\theta}{2}\ \cancel{4}-\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}}$$

$$= i\cot\frac{\theta}{2}$$

OR using t formulae, $t = \tan\frac{\theta}{2}$

$\cos\theta = \frac{1-t^2}{1+t^2}$

$\sin\theta = \frac{2t}{1+t^2}$

$$\frac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta}$$

$$= \frac{1+\frac{1-t^2}{1+t^2}+\frac{2it}{1+t^2}}{1-\frac{1-t^2}{1+t^2}-\frac{2it}{1+t^2}} \begin{matrix}\times 1+t^2\\ \times 1+t^2\end{matrix}$$

$$= \frac{1+t^2+1-t^2+2it}{1+t^2-1+t^2-2it}$$

$$= \frac{2+2it}{2t^2-2it} \begin{matrix}\times (t^2+it)\\ \times (t^2+it)\end{matrix}$$

$$= \frac{1+it}{t^2-it} \times (t^2+it)$$

$$= \frac{t^2+it+it^3-t^2}{t^4+t^2}$$

$$= \frac{it(1+t^2)}{t^2(1+t^2)}$$

$$= \frac{i}{t}$$

$$= \frac{i}{\tan\left(\frac{\theta}{2}\right)}$$

$$= i\cot\left(\frac{\theta}{2}\right)$$

(9) (iii) Hence $\frac{z-1}{z+1}$

$$= \frac{\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta} - \frac{(1-\cos\theta-i\sin\theta)}{1-\cos\theta-i\sin\theta}}{\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}+\frac{(1-\cos\theta-i\sin\theta)}{1-\cos\theta-i\sin\theta}}$$

$$= \frac{\left(\frac{2\cos\theta+2i\sin\theta}{1-\cos\theta-i\sin\theta}\right)}{\frac{2}{(1-\cos\theta-i\sin\theta)}} \begin{matrix}\times(1-\cos\theta-i\sin\theta)\\ \times(1-\cos\theta-i\sin\theta)\end{matrix}$$

$$= \frac{2\cos\theta+2i\sin\theta}{2}$$

$$= \cos\theta + i\sin\theta.$$

(iv) Hence solving $\left(\frac{z-1}{z+1}\right)^4 = -1.$

$z = i\cot\left(\frac{\theta}{2}\right)$ where $(\cos\theta+i\sin\theta)^4 = -1$

i.e. $\cos 4\theta + i\sin 4\theta = \cos\pi + i\sin\pi$

Hence $4\theta = \pi$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Will accept as final answers. $\left[\ \&\ \ z = i\cot\frac{\pi}{8}, i\cot\frac{3\pi}{8}, i\cot\frac{5\pi}{8}, i\cot\frac{7\pi}{8}.\right]$

As $\cot\frac{7\pi}{8} = -\cot\frac{\pi}{8}$ & $\cot\frac{5\pi}{8} = -\cot\frac{3\pi}{8}$

$$z = \pm i\cot\frac{\pi}{8}, \pm i\cot\frac{3\pi}{8}.$$