GIRRAWEEN HIGH SCHOOL

MATHEMATICS EXTENSION 2

TASK 1 2013 December 2012: COMPLEX NUMBERS

ANSWERS COVER SHEET

Name: ______________________________

Teacher: ______________________________

FINAL MARK

	MARK	E2	E3	E4	E5	E6	E7	E8
1 Multiple Choice	/5		√					
2	/21		√					
	/21							
3	/14		√					
	/14							
4	/14		√					
	/14							
5	/13		√					
	/13							
6	/16		√					
	/16							
7	/14		√					
	14							
TOTAL	/97		/97					

HSC Outcomes **Mathematics Extension 2**

E1 appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems.

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings.

E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections.

E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials.

E5 uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces, resisted motion and circular motion

E6 combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions.

E7 uses the techniques of slicing and cylindrical shells to determine volumes.

E8 applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems.

E9 communicates abstract ideas and relationships using appropriate notation and logical argument.

GIRRAWEEN HIGH SCHOOL

TASK 1 2013 (December 2012)

MATHEMATICS

EXTENSION 2

Complex Numbers

Time allowed – 100 Minutes

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Each question attempted is to be returned on a *separate* piece of paper clearly marked Question 1 , Question 2 , etc. Each piece of paper must show your name.
- You may ask for extra pieces of paper if you need them.
- For Multiple choice: Circle the correct answer on your examination paper.

Question 1 (Multiple Choice) Circle the correct answer on the Examination Paper

(a) The value of i^{27} is:

(A) i (B) -1 (C) $-i$ (D) 1

(b) If $z = 2 + i$ and $w = 3 - 2i$ then $z\overline{w} =$

(A) $8 + 7i$ (B) $4 + 7i$ (C) $8 - 7i$ (D) $4 - 7i$

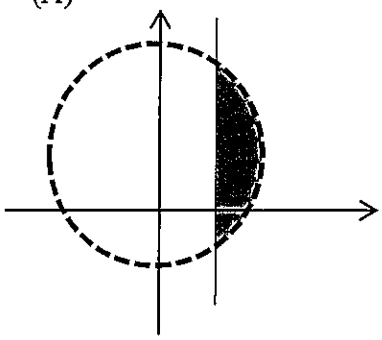
(c) When expressed in modulus/ argument form, $3 - i\sqrt{3} =$

(A) $2\sqrt{3}(cos\frac{-\pi}{6} + isin\frac{-\pi}{6})$ (B) $2\sqrt{3}(cos\frac{-\pi}{3} + isin\frac{-\pi}{3})$ (C) $2\sqrt{3}(cos\frac{-5\pi}{6} + isin\frac{-5\pi}{6})$

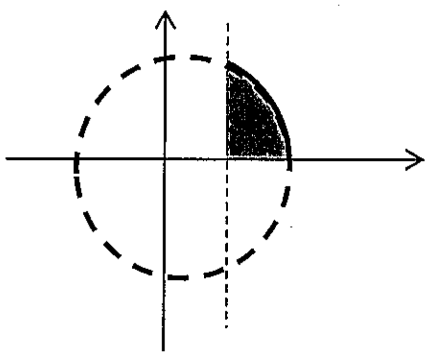
(D) $2\sqrt{3}(cos\frac{-2\pi}{3} + isin\frac{-2\pi}{3})$

(d) The region in the complex plane defined by $|z - i| \leq 2$ and $Re(z) > 1$ is represented by:

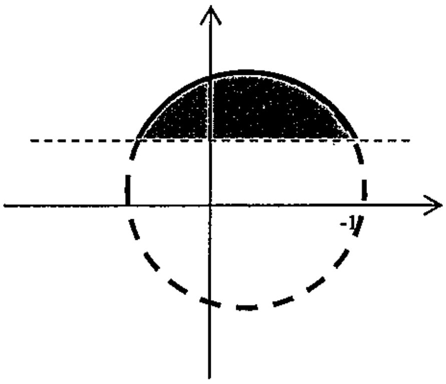
(A)



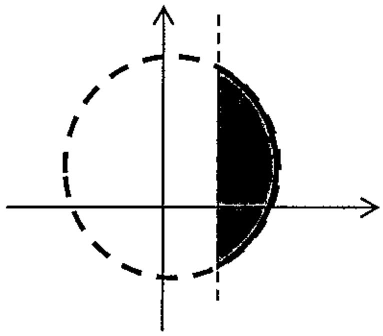
(B)



(C)



(D)



(e) In Cartesian form, $\frac{11-7i}{3-5i} =$

(A) $2 - 76i$ (B) $2 + i$ (C) $\frac{2-76i}{34}$ (D) $\frac{2+i}{34}$

For Question 2 onward show all workings on the blank paper provided:

Question 2 ***(21 Marks)***

Marks

(a) (i) Find $\frac{-1+i}{1+i\sqrt{3}}$ in Cartesian form. **2**

(ii) Convert both $-1+i$ and $1+i\sqrt{3}$ to Modulus/argument form. **4**

(iii) Using the answers to (i) and (ii) find the value of $cos\,\frac{5\pi}{12}$. **2**

(b) (i) If $x+iy=\sqrt{5-12i}$ find the exact value of x and y. **5**

(ii) Hence solve the equation $z^2-5z+(5+3i)=0$ **3**

(c) Use DeMoivre's theorem to find $(-1+i\sqrt{3})^5$ in Cartesian form. **3**

(d) Find all three cube roots of $-4\sqrt{2}+4i\sqrt{2}$. Leave your answers in modulus/ argument form. **2**

Question 3 ***(14 Marks)***

(a) Sketch each of the following loci on separate Argand diagrams:

(i) $z\bar{z}=2\times Re(z)$ **3**

(ii) $Arg(z+2-i)=-\frac{\pi}{4}$ **2**

(iii) $|z+1-2i|=|z-1|$ **2**

(iv) $|z-i|=Im(z+i)$ **3**

(b) Sketch and shade the region satisfied by $|z-2i|\le 2$ and $\frac{\pi}{4}<Arg\,z\le\frac{\pi}{2}$ on an Argand diagram. **4**

Examination continues on the next page

Question 4 *(14 Marks)* **Marks**

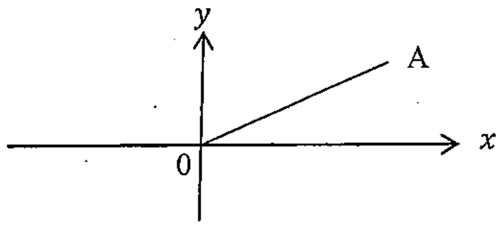
(a) z is an arbitrary point on the Argand diagram such that $\overrightarrow{OA} = z$.

(see diagram below). Copy the diagram on to your writing paper and draw in

(i) $\overrightarrow{OB}$ so that $\overrightarrow{OB} = iz$ **1**

(ii) $\overrightarrow{OC}$ so that $\overrightarrow{OC} = z \times (cos\frac{-\pi}{3} + isin\frac{-\pi}{3})$ **1**

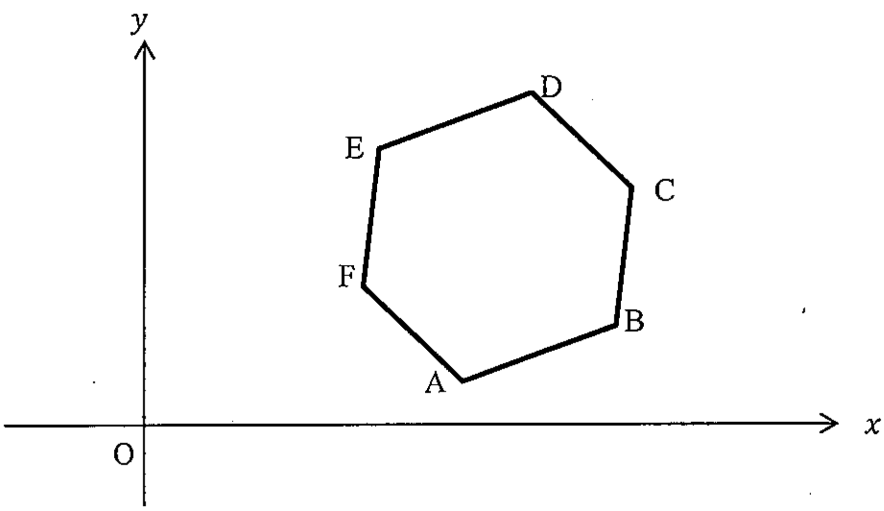
(iii) $\overrightarrow{OD}$ so that $\overrightarrow{OD} = -z$ **1**



(b) On the diagram below, ABCDEF is a regular hexagon. If $\overrightarrow{OA} = z_1$ and $\overrightarrow{OB} = z_2$

(i) Express $\overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{AD}$ in terms of z_1. and z_2. **3**

(ii) Show that $\overrightarrow{AB} \times \overrightarrow{BC} \times \overrightarrow{CD} = (z_1 - z_2)^3$ **2**



Question 4 continues on the next page

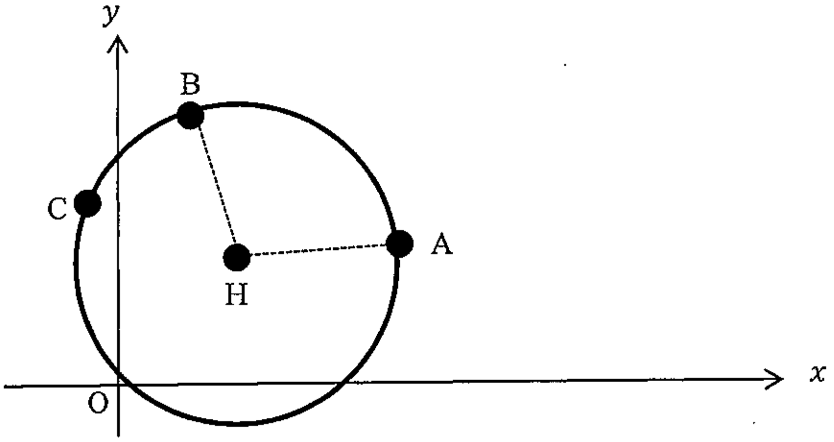
Question 4 (continued) **Marks**

(c) On the diagram below, A, O and B are at the circumference of a circle with centre H. If $\overrightarrow{OA} = z_1, \overrightarrow{OH} = z_2$ and $\overrightarrow{OB} = z_3$

(i) Show that $Arg\left\{\frac{z_3-z_2}{z_1-z_2}\right\} = Arg\left\{\frac{z_3}{z_1}\right\}^2$ **4**

(ii) C is a point on the circle in the major arc AB. If $\overrightarrow{OC} = z_4$ **2**

show that $Arg\left\{1-\frac{z_1}{z_4}\right\} = Arg\left\{1-\frac{z_1}{z_3}\right\}$



Question 5 *(13 Marks)*

(a) The graph of $Arg\left(\frac{z+2}{z-4}\right) = \frac{\pi}{3}$ represents part of a circle. *Note that this is not a semicircle.*

Draw the graph of the part of the circle represented by $Arg\left(\frac{z+2}{z-4}\right) = \frac{\pi}{3}$ **6**

and find the centre and radius of the circle and its Cartesian equation.

(b) (i) If $z = cos\boldsymbol{\theta} + \mathrm{i}\sin\boldsymbol{\theta}$ show that $z - \frac{1}{z} = 2i\ sin\boldsymbol{\theta}$ and that **4**

$z^n - \frac{1}{z^n} = 2i \sin n\boldsymbol{\theta}$.

(i) Hence express $sin^5\boldsymbol{\theta}$ in terms of $sin5\boldsymbol{\theta}, sin3\boldsymbol{\theta}$ and $sin\boldsymbol{\theta}$. **3**

Question 6 *(16 Marks)* **Marks**

(a) (i) Use DeMoivre's Theorem to find the formula for $cos6\boldsymbol{\theta}$ **2**

(You may leave your answer in terms of mixtures of$cos\theta$ and $sin\theta$.)

(ii) Hence show that

$$cos6\theta = (cos^2\theta - sin^2\theta)(cos^4\theta - 14cos^2\theta sin^2\theta + sin^4\theta)$$ **2**

(iii) Hence using that $cos\frac{\pi}{2} = 0$ show that $cos\frac{\pi}{12}$ is a root of the equation $16x^4 - 16x^2 + 1 = 0$ **2**

(iv) Hence find the exact value of $cos\frac{\pi}{12}$. **3**

(b)(i) Resolve $z^6 + 1$into real quadratic factors. **3**

(ii) Hence show that **4**

$$cos3\boldsymbol{\theta} = \mathbf{4cos}\theta(\boldsymbol{cos^2\theta} - \boldsymbol{cos^2}\tfrac{\pi}{6})$$

Question 7 *(14 Marks)*

(a) (i) Use that $z^{10} - 1 = (z^2)^5 - 1$ to factorise $z^{10} - 1$. **1**

(ii) Hence or otherwise find the roots of the equation

$z^8 + z^6 + z^4 + z^2 + 1 = 0.$ **1**

(iii) Hence show that $cos\frac{\pi}{5}$ is a root of the equation

$2cos4\theta + 2cos2\theta + 1 = 0.$ **2**

(You may assume that if $z = cos\theta + isin\theta$ then $z^n + \frac{1}{z^n} = 2cosn\theta$)

(iv) Hence find the exact value of $cos\frac{\pi}{5}$. **4**

(b) Let w be the root of $z^{10} - 1 = 0$ with the smallest positive argument.

(i) Show that $w^2, w^3, w^4, w^5, w^6, w^7, w^8$ and w^9 are the other non real roots of $z^{10} - 1 = 0$. **1**

(ii) By expanding $(w + w^9)(w^2 + w^8)(w^3 + w^7)(w^4 + w^6)$ **5**
and using that w is a root of $z^8 + z^6 + z^4 + z^2 + 1 = 0$ show that
$16cos^2\frac{\pi}{5}cos^2\frac{2\pi}{5} = 1.$

End of examination

Solutions to Complex Numbers test p. 1

Dec. 2012 for 2013 MSC

(1) (a) C (b) B (c) A (d) D (e) B

Q.2 = 21

(2)(a)(i) $\dfrac{-1+i}{1+i\sqrt{3}} \times \dfrac{(1-i\sqrt{3})}{(1-i\sqrt{3})}$ 1

2

$= \dfrac{(\sqrt{3}-1)+i(\sqrt{3}+1)}{4}$ 1

(ii) $-1+i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$1+i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ 4

(iii) Using mod/arg form,

$\dfrac{-1+i}{1+i\sqrt{3}} = \dfrac{1}{\sqrt{2}}\cos\dfrac{5\pi}{12} + \dfrac{i}{\sqrt{2}}\sin\dfrac{5\pi}{12}$ 1

Using (i) & equating reals:

$\dfrac{\sqrt{3}-1}{4} = \dfrac{1}{\sqrt{2}}\cos\dfrac{5\pi}{12}.$ 2

$\dfrac{\sqrt{6}-\sqrt{2}}{4} = \cos\dfrac{5\pi}{12}.$ 1

(b) $(x+iy)^2 = 5-12i$, x, y real.

$(x^2-y^2)+2ixy = 5-12i$ 1

Equating reals: $x^2-y^2=5$ (1)

Equating imaginaries $2xy = -12$

$y = \dfrac{-6}{x}$ (2) 1

Sub. (2) in (1): $x^2 - \left(\dfrac{-6}{x}\right)^2 = 5$ 1

$x^2 - \dfrac{36}{x^2} = 5$

PTO →

Q.(2)(b)(i) [continued]:

$x^4 - 36 = 5x^2$

$x^4 - 5x^2 - 36 = 0$

$(x^2 - 9)(x^2 + 4) = 0$

$x^2 = 9$ [as x is real, $x^2 \neq -4$] 1

$\therefore x = \pm 3.$

As $y = \frac{-6}{x}$, $y = \mp 2$. 1 $\underline{5}$

$\therefore \sqrt{5 - 12i} = \pm(3 - 2i).$

(ii) $z^2 - 5z + (5 + 3i) = 0$

$z = \dfrac{5 \pm \sqrt{(-5)^2 - 4 \times 1 \times (5 + 3i)}}{2}$ 1

$= \dfrac{5 \pm \sqrt{5 - 12i}}{2}$

$= \dfrac{5 \pm (3 - 2i)}{2}$ [using (i)] 1

$\underline{3}$

$z = \dfrac{5 + (3 - 2i)}{2}$ or $z = \dfrac{5 - (3 - 2i)}{2}$

$\underline{z = 4 - i \text{ or } z = 1 + i}$ 1

(c) $(-1 + i\sqrt{3})^5$

$= 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^5$ 1

$= 32\left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right)$ 1

$= 32\left(\cos -\frac{2\pi}{3} + i\sin -\frac{2\pi}{3}\right)$ $\underline{3}$

$= 32\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= \underline{-16 - 16i\sqrt{3}}$ 1.

Solutions to Complex N^{os} p.3

Q.(2)(d) $-4\sqrt{2}+4i\sqrt{2}$

$= 64\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ 1

$\underline{2}$

$\therefore \sqrt[3]{-4\sqrt{2}+4i\sqrt{2}}$

$= 4\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)+\frac{2k\pi}{3}, \quad k=0,1,2.$ 1

$= 4\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right), 4\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right), 4\left(\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12}\right)$

$\Rightarrow$ can also be $4\left(\cos\frac{-5\pi}{12}+i\sin\frac{-5\pi}{12}\right)$

Q.(3)(a)(i) Let $z = x+iy$

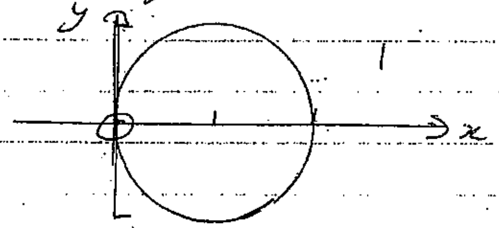
$z\bar{z} = 2\times Re(z)$

$(x+iy)(x-iy) = 2x$ 1

$x^2+y^2 = 2x$

$(x^2-2x+1)+y^2 = 1.$

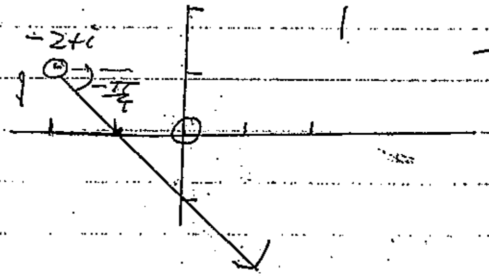
$(x-1)^2+y^2 = 1.$ 1 $\underline{3}$



1

(ii) $Arg(z+2-i) = -\frac{\pi}{4}.$

1 $\underline{2}$



(iii) $|z+1-2i| = |z-1|$

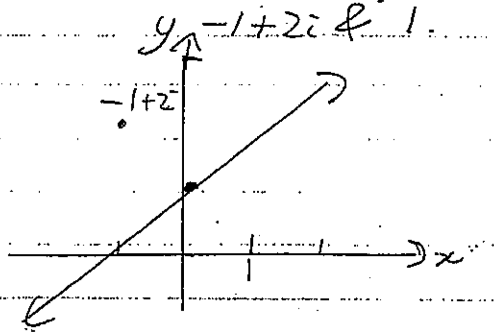
$\sqrt{(x+1)^2+(y-2)^2} = \sqrt{(x-1)^2+y^2}$ 1

$\cancel{x^2}+2x+1+\cancel{y^2}-4y+4 = \cancel{x^2}-2x+1+\cancel{y^2}$

$4x+4 = 4y$ 1 $\underline{2}$

$x+1 = y$ or $y = x+1.$

[Note: We could also perpendicular bisector of line between $-1+2i$ & 1.



1

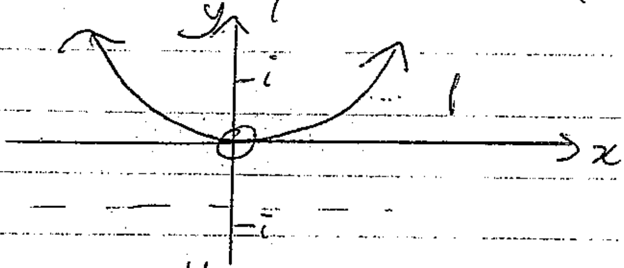
Q.(3)(a)(iv) $|z-i| = \text{Im}(z+i)$

$\sqrt{x^2+(y-1)^2} = y+1$ 1

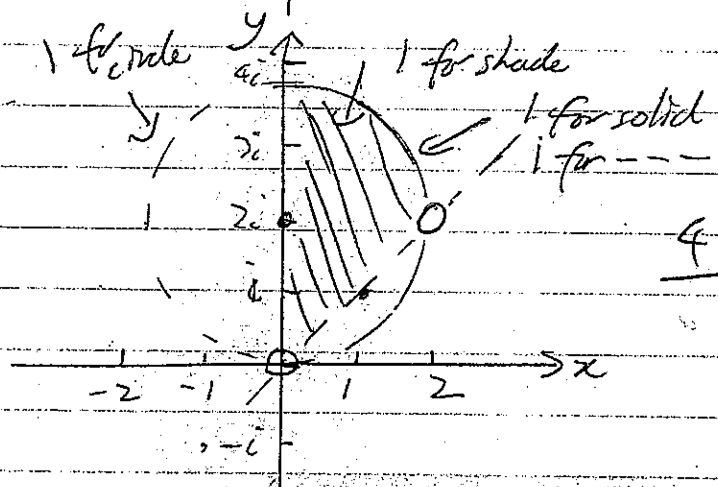
$x^2+\cancel{y^2}-2y+1 = \cancel{y^2}+2y+1$ 3

$x^2 = 4y$. 1

Note: We could also have noted that this was a parabola with focus at i & directrix at $y = -1$. [so total Im $= -i$].

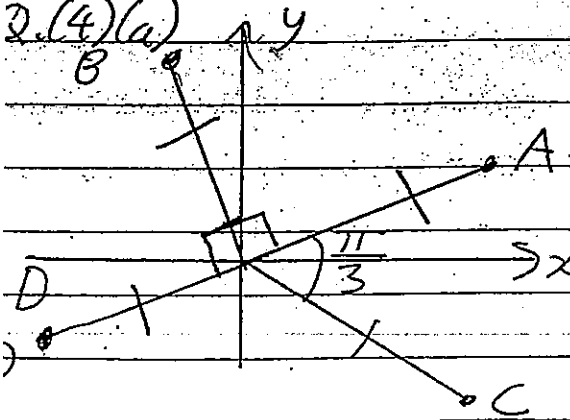


(b)



4

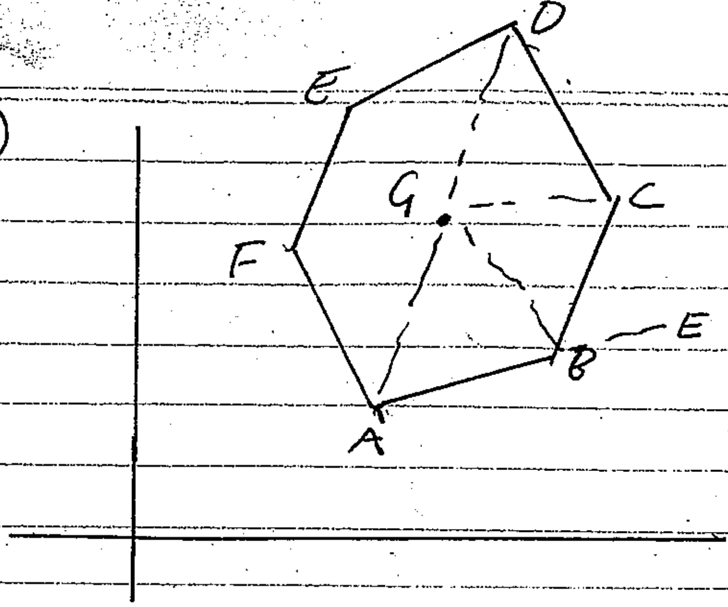
Q.(4)(a)



3

p.5 Solutions
2013 MSC
(Dec 2012)
Complex N^{os}.

(4)(b)



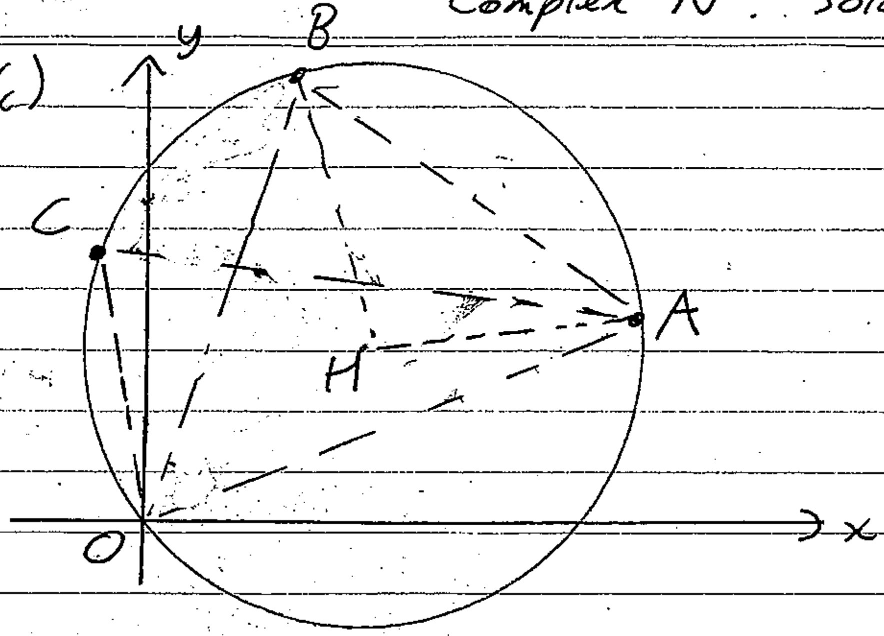
(i) $\vec{AB} = z_2 - z_1$

$\vec{BC}$: Let $\vec{BE}$ continue $\vec{AB}$
(be $\vec{AB}$ produced)
$\angle CBE = \frac{\pi}{3}$ (exterior $\angle$ regular hexagon)
$\therefore \vec{BC} = cis\frac{\pi}{3} \times \vec{AB}$
$= cis\frac{\pi}{3}(z_2 - z_1)$

$\vec{AD}$: Let G be midpoint of AD.
As AG bisects $\angle FAB$ & BG bisects $\angle ABC$
$\angle GAB = \angle GBA = \frac{\pi}{3}$
$\therefore \triangle AGB$ is equilateral & $AB = BG = AG$.
$\therefore \vec{AG} = cis\frac{\pi}{3} \times \vec{AB}$
$\vec{AD} = 2cis\frac{\pi}{3} \times \vec{AB}$
$= 2cis\frac{\pi}{3}(z_2 - z_1)$

(ii) $\vec{AB} \times \vec{BC} \times \vec{CD}$
$= (z_2 - z_1) \times cis\frac{\pi}{3}(z_2 - z_1) \times cis\frac{2\pi}{3}(z_2 - z_1)$
$= cis\,\pi \times (z_2 - z_1)^3$
$= -(z_2 - z_1)^3$
$= (z_1 - z_2)^3$

(4)(c)



(i) $z_3 - z_2 =$
$z_1 - z_2 = \overrightarrow{HA}$

$\angle BHA = \text{Arg}(z_3 - z_2) - \text{Arg}(z_1 - z_2)$

$\angle BHA = \text{Arg}\left(\dfrac{z_3 - z_2}{z_1 - z_2}\right)$

Similarly as $\overrightarrow{OB} = z_3$, $\overrightarrow{OA} = z_1$
$\angle BOA = \text{Arg}\left(\dfrac{z_3}{z_1}\right)$

As $\angle BOA = 2 \times \angle BHA$ [$\angle$ at centre $= 2 \times \angle$ at circumference on same arc].

$\text{Arg}\left(\dfrac{z_3 - z_2}{z_1 - z_2}\right) = 2\,\text{Arg}\left(\dfrac{z_3}{z_1}\right)$

$\text{Arg}\left(\dfrac{z_3 - z_2}{z_1 - z_2}\right) = \text{Arg}\left[\left(\dfrac{z_3}{z_1}\right)^2\right]$ [by De Moivre].

(ii)

$\therefore \angle OCA = \text{Arg}\left(\dfrac{\overrightarrow{AC}}{\overrightarrow{OC}}\right)$

$= \text{Arg}\left(\dfrac{z_4 - z_1}{z_4}\right)$

$\overrightarrow{AB} = z_3 - z_1$

$\therefore \angle OBA = \text{Arg}\left(\dfrac{z_3 - z_1}{z_3}\right)$

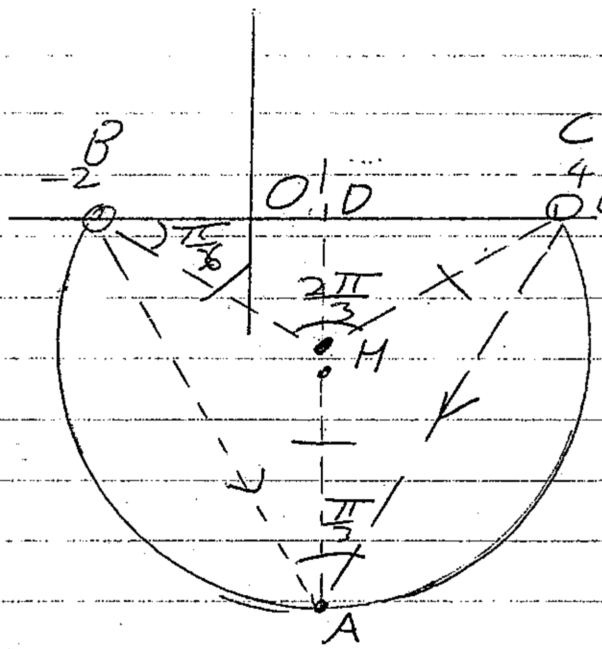
PTO $\Rightarrow$

(4)(c)(ii) As $\angle OCA = \angle OBA$ [$\angle$'s on same arc].

$$\text{Arg}\left(\frac{z_4 - z_1}{z_4}\right) = \text{Arg}\left(\frac{z_3 - z_1}{z_3}\right) \quad 1$$

$$\text{Arg}\left(1 - \frac{z_1}{z_4}\right) = \text{Arg}\left(1 - \frac{z_1}{z_3}\right) \quad 1 \qquad \underline{\underline{2}}$$

(5)(a)



1 for open

1 for correct part of circle [side].

$\underline{6}$

$\vec{OA} = z$

Let $B = -2$, $D = 1$, $C = 4$, $H =$ circle centre.

By symmetry H is on $x = 1$.

$\angle BHC = \frac{2\pi}{3}$ [$\angle$ at centre $= 2\times\angle$ at circumference].

As $BM = CM = AM$ (circle radii)

$\angle MBD = \frac{\pi}{6}$ [$\angle$'s opposite $=$ sides in isosceles $\triangle HBC$]. 1

$DH = 3\tan\frac{\pi}{6}$

$= \sqrt{3}$ 1

$H = 1 - i\sqrt{3}$.

$(BH)^2 = 3^2 + (\sqrt{3})^2$ 1

$= 12$.

Circle is $(x-1)^2 + (y+\sqrt{3})^2 = 12$, $y < 0$. 1

Q. (5)(b)(i) $z = \cos\theta + i\sin\theta$

$\therefore \frac{1}{z} = \cos(-\theta) + i\sin(-\theta)$ [by DeMoivre] 1

$= \cos\theta - i\sin\theta$ [as cos even, sin odd]. 1

$\therefore z - \frac{1}{z} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)$ 1

$= 2i\sin\theta.$

4

Similarly $z^n - \frac{1}{z^n} = \cos n\theta + i\sin n\theta - [\cos(-n\theta) + i\sin(-n\theta)]$

[by DeMoivre]

$= \cos n\theta + i\sin n\theta - [\cos n\theta - i\sin n\theta]$

[as cos even, sin odd] 1

$= 2i\sin n\theta.$

(ii) Hence if $z = \cos\theta + i\sin\theta$,

$\left(z - \frac{1}{z}\right)^5 = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$

$(2i\sin\theta)^5 = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$ 1

$32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$ [using (i)] 1

$\div$ BS by $32i$

$\sin^5\theta = \frac{\sin 5\theta}{16} - \frac{5\sin 3\theta}{16} + \frac{10\sin\theta}{16}$ 3

$\sin^5\theta = \frac{1}{16}\left[\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right].$ 1

Q.(6) By DeMoivre (a)(i)

$(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$

$\cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20i\cos^3\theta\sin^3\theta$ 1

$+15\cos^2\theta\sin^4\theta + 6i\cos\theta\sin^5\theta - \sin^6\theta = \cos 6\theta + i\sin 6\theta$

Equating real parts 1 $\underline{2}$

Hence $\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$

(ii) $\cos 6\theta = \cos^6\theta - \sin^6\theta - 15\cos^2\theta\sin^2\theta(\cos^2\theta - \sin^2\theta)$

$= (\cos^2\theta - \sin^2\theta)(\cos^4\theta + \cos^2\theta\sin^2\theta + \sin^4\theta) - 15\cos^2\theta\sin^2\theta(\cos^2\theta - \sin^2\theta)$

$= (\cos^2\theta - \sin^2\theta)(\cos^4\theta - 14\cos^2\theta\sin^2\theta + \sin^4\theta)$ $\underline{1}$

(iii) As $\cos\frac{\pi}{2} = 0$

$\cos\frac{\pi}{12}$ is a solution to $\cos 6\theta = 0$

As $\cos^2\frac{\pi}{12} \neq \sin^2\frac{\pi}{12}$,

$\cos\frac{\pi}{12}$ must be a solution to

$\cos^4\theta - 14\sin^2\theta\cos^2\theta + \sin^4\theta = 0$ [from (ii)] 1

$\cos^4\theta - 14\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 = 0$

$\cos^4\theta - 14\cos^2\theta + 14\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta = 0$

$16\cos^4\theta - 16\cos^2\theta + 1 = 0$ 1 $\underline{2}$

(iv) $\cos^2\frac{\pi}{12} = \dfrac{16 \pm \sqrt{16^2 - 4\times 16\times 1}}{2\times 16}$

$= \dfrac{16+8\sqrt{3}}{32}$ or $\dfrac{16-8\sqrt{3}}{32}$

$\cos^2\frac{\pi}{12} = \dfrac{2+\sqrt{3}}{4}$ or $\cos^2\frac{\pi}{12} = \dfrac{2-\sqrt{3}}{4}$. 1

Note: As $\cos\frac{\pi}{12} > \cos\frac{\pi}{6}$ & $\cos^2\frac{\pi}{6} = \frac{3}{4}$,

$\cos^2\frac{\pi}{12} = \dfrac{2+\sqrt{3}}{4}$ 1

$\cos\frac{\pi}{12} = \dfrac{\sqrt{2+\sqrt{3}}}{2}$. 1

Q.(6)(b)(i) Roots of $z^6+1=0$ are 3

$z = \cos\frac{\pi}{6}+i\sin\frac{\pi}{6}, \cos\frac{\pi}{2}+i\sin\frac{\pi}{2}, \cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}, \cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6},$ 1

$\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}, \cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}$

Hence $z^6+1 = \left(z-\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right)\left(z-\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)\right)(z-i)(z+i)$

$\left(z-\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)\right)\left(z-\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)\right)$ 1

Noting that $\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6} = \cos\frac{-\pi}{6}+i\sin\frac{-\pi}{6}$

$= \cos\frac{\pi}{6}-i\sin\frac{\pi}{6}$ (as cos even, sin odd).

Similarly $\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6} = \cos\frac{5\pi}{6}-i\sin\frac{5\pi}{6}$.

$z^6+1 = \left(z^2-2z\cos\frac{\pi}{6}+1\right)(z^2+1)\left(z^2-2z\cos\frac{5\pi}{6}+1\right)$ *Note: 1 Full marks for (i) to be.

As $\cos\frac{5\pi}{6} = -\cos\frac{\pi}{6}$

$z^6+1 = \left(z^2-2z\cos\frac{\pi}{6}+1\right)\left(z^2+2z\cos\frac{\pi}{6}+1\right)(z^2+1)$ 1

$\div z^3$ $\div z^3$

$z^3+\frac{1}{z^3} = \left(z+\frac{1}{z}-2\cos\frac{\pi}{6}\right)\left(z+\frac{1}{z}+2\cos\frac{\pi}{6}\right)\left(z+\frac{1}{z}\right)$ 1

If $z=\cos\theta+i\sin\theta$,

$z^n+\frac{1}{z^n} = \cos n\theta+i\sin n\theta+\cos(-n\theta)+i\sin(-n\theta)$ (by DeMoivre)

$= \cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta$ (as cos even, sin odd)

$= 2\cos n\theta$

$\therefore 2\cos 3\theta = \left(2\cos\theta-2\cos\frac{\pi}{6}\right)\left(2\cos\theta+2\cos\frac{\pi}{6}\right)2\cos\theta$ 1

$\cos 3\theta = 4\cos\theta\left(\cos\theta-\cos\frac{\pi}{6}\right)\left(\cos\theta+\cos\frac{\pi}{6}\right)$

$\cos 3\theta = 4\cos\theta\left(\cos^2\theta-\cos^2\frac{\pi}{6}\right)$ 1 4

as required.

(7)(a)(i) $z^{10}-1 = (z^2)^5 - 1$
$= (z^2-1)(z^8+z^6+z^4+z^2+1)$ 1

(ii) Hence roots of $z^8+z^6+z^4+z^2+1=0$ are
$z=\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$, $z=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}$, $z=\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}$,
$z=\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}$, $z=\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}$, $z=\cos\frac{7\pi}{5}+i\sin\frac{7\pi}{5}$,
$z=\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}$, $z=\cos\frac{9\pi}{5}+i\sin\frac{9\pi}{5}$. 1

(iii) $z^8+z^6+z^4+z^2+1 = 0$
$\div$ BS by z^4 & rearranging

$\left(z^4+\frac{1}{z^4}\right)+\left(z^2+\frac{1}{z^2}\right)+1 = 0$ 1

As $z=\cos\theta+i\sin\theta$, $z^n+\frac{1}{z^n}=2\cos n\theta$

$2\cos 4\theta+2\cos 2\theta+1 = 0$ if $\theta=\frac{\pi}{5}, \frac{2\pi}{5}$ etc. 2

1

$\therefore 2\cos 4\theta+2\cos 2\theta+1 = 0$ has $\cos\frac{\pi}{5}$ as a root.

(iv) Could be done using $4\theta = 2\times 2\theta$
Hence $\cos 4\theta = 2\cos^2 2\theta - 1$ [See next page] →
& solving for $\cos\frac{2\pi}{5}$. then using $\cos\frac{2\pi}{5}=2\cos^2\frac{\pi}{5}-1$.
However, $\cos\frac{4\pi}{5} = -\cos\frac{\pi}{5}$.

So as $2\cos\frac{4\pi}{5}+2\cos\frac{2\pi}{5}+1 = 0$
$-2\cos\frac{\pi}{5}+2\left(2\cos^2\frac{\pi}{5}-1\right)+1 = 0$ 1
$4\cos^2\frac{\pi}{5}-2\cos\frac{\pi}{5}-1 = 0$ 1
$\cos\frac{\pi}{5} = \frac{2\pm\sqrt{2^2-4\times 4\times -1}}{2\times 4}$ 4
$= \frac{2\pm 2\sqrt{5}}{8}$
$\cos\frac{\pi}{5} = \frac{1\pm\sqrt{5}}{4}$ 1

As $\frac{\pi}{5}$ is in Q_1, $\cos\frac{\pi}{5}$ is positive.
$\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$ 1

Q. (7)(a)(iv) alternative:

$2\cos 4\theta + 2\cos 2\theta + 1 = 0$

As $4\theta = 2 \times 2\theta$

$\cos 4\theta = 2\cos^2(2\theta) - 1$

Hence equation becomes:

$2[2\cos^2 2\theta - 1] + 2\cos 2\theta + 1 = 0$

$4\cos^2 2\theta + 2\cos 2\theta - 1 = 0$

Hence $\cos 2\theta = \dfrac{-2 \pm \sqrt{2^2 - 4 \times 4 \times -1}}{2 \times 4}$

$\cos 2\theta = \dfrac{-1 \pm \sqrt{5}}{4}.$

As we are looking for $\cos\frac{2\pi}{5}$ at this stage, $\cos\theta$ must be positive.

Hence $\cos\frac{2\pi}{5} = \dfrac{-1+\sqrt{5}}{4}.$

As $\cos\frac{2\pi}{5} = \cos\left[2 \times \frac{\pi}{5}\right]$

$= 2\cos^2\left(\frac{\pi}{5}\right) - 1$

$2\cos^2\frac{\pi}{5} - 1 = \dfrac{-1+\sqrt{5}}{4}.$

$\cos^2\frac{\pi}{5} = \dfrac{3+\sqrt{5}}{8}$

$\cos\frac{\pi}{5} = \sqrt{\dfrac{3+\sqrt{5}}{8}}$ [as $\frac{\pi}{5}$ is in Q_1, $\cos\frac{\pi}{5}$ is negative].

To show $\sqrt{\dfrac{3+\sqrt{5}}{8}} = \dfrac{1+\sqrt{5}}{4}$

$\left(\dfrac{1+\sqrt{5}}{4}\right)^2 = \dfrac{1+2\sqrt{5}+5}{16} = \dfrac{6+2\sqrt{5}}{16} = \dfrac{3+\sqrt{5}}{8}.$

Either $\cos\frac{\pi}{5} = \sqrt{\dfrac{3+\sqrt{5}}{8}}$ or $\cos\frac{\pi}{5} = \dfrac{1+\sqrt{5}}{4}$ are acceptable.

Q. (7)(b) As $w=\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$.

(i)

w^5, w^{10} are REAL roots.

Testing $w^2, w^3, w^4, w^6, w^7, w^8, w^9$

$(w^2)^{10} = w^{20}=1$

$(w^3)^{10} = w^{30}=1$

$(w^4)^{10} = w^{40}=1$

$(w^6)^{10} = w^{60}=1$ 1

$(w^7)^{10} = w^{70}=1$

$(w^8)^{10} = w^{80}=1$

$(w^9)^{10} = w^{90}=1$.

$w^2, w^3, w^4, w^6, w^7, w^8, w^9$ are other non real roots.

(ii) $(w+w^9)(w^2+w^8)(w^3+w^7)(w^4+w^6)$

$=(w^3+w^9+w^{11}+w^{17})(w^7+w^9+w^{11}+w^{13})$

$=w^{10}+w^{12}+w^{14}+w^{16}+w^{16}+w^{18}+w^{20}+w^{22}+w^{18}+w^{20}+w^{22}+w^{24}$ 1

$+w+w^{26}+w^{28}+w^{30}$

$=\cancel{1}+\cancel{w^2}+\cancel{w^4}+\cancel{w^6}+\cancel{w^6}+w^8+\cancel{1}+w^2+w^8+\cancel{1}+w^2+\cancel{w^4}$

$+w^4+w^6+w^8+\cancel{1}$

$=4+3(w^2+w^4+w^6+w^8)$ 1

$=1+3(1+w^2+w^4+w^6+w^8)$

As $w=\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$ is a root of $z^8+z^6+z^4+z^2+1=0$

$1+w^2+w^4+w^6+w^8=0$

Hence $(w+w^9)(w^2+w^8)(w^3+w^7)(w^4+w^6)=1$. 1

Using $w+w^9=2\cos\frac{\pi}{5}$, $w^2+w^8=2\cos\frac{2\pi}{5}$, $w^3+w^7=2\cos\frac{3\pi}{5}$, $w^4+w^6=2\cos\frac{4\pi}{5}$

$2\cos\frac{\pi}{5}\times 2\cos\frac{2\pi}{5}\times 2\cos\frac{3\pi}{5}\times 2\cos\frac{4\pi}{5}=1$ 1

As $\cos\frac{3\pi}{5}=-\cos\frac{2\pi}{5}$, $\cos\frac{4\pi}{5}=-\cos\frac{\pi}{5}$ 5

$2\cos\frac{\pi}{5}\times 2\cos\frac{2\pi}{5}\times -2\cos\frac{2\pi}{5}\times -2\cos\frac{\pi}{5}=1$

$16\cos^2\frac{\pi}{5}\cos^2\frac{2\pi}{5}=1$. 1