FINAL MARK

GIRRAWEEN HIGH SCHOOL
MATHEMATICS EXTENSION 2
HSC ASSESSMENT TASK 1, 2015 (HSC 2016)
ANSWERS COVER SHEET

Name: ______________________________

QUESTION	MARK	E2	E3	E4	E5	E6	E7	E8	E9
Multiple choice	/5		✔						✔
Q6	/18		✔						✔
Q7	/24		✔						✔
Q8	/19		✔						✔
Q9	/20		✔						✔
Q10	/10		✔						✔
Q11	/11		✔						✔
TOTAL									
	/107		**/107**						**/107**

HSC Outcomes **Mathematics Extension 2**

E1 appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems.

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings.

E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections.

E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials.

E5 uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces, resisted motion and circular motion

E6 combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions.

E7 uses the techniques of slicing and cylindrical shells to determine volumes.

E8 applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems.

E9 communicates abstract ideas and relationships using appropriate notation and logical argument.

GIRRAWEEN HIGH SCHOOL

TASK 1

2015

MATHEMATICS

EXTENSION 2

Time allowed – 90 minutes

DIRECTIONS TO CANDIDATES

• Attempt ALL questions.

• All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.

• Board – approved calculators may be used.

• Start each question on a separate page. Each paper must show your name.

Multiple Choice (5 marks) Write the letter corresponding to the correct answer in your answer booklet.

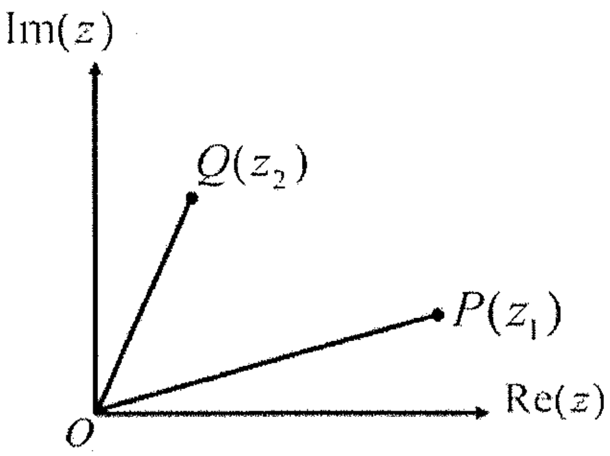
1. Find the conjugate of $\dfrac{1}{3+4i}$

(A) $\dfrac{3}{25}-\dfrac{4}{25}i$ **(B)** $\dfrac{3}{25}+\dfrac{4}{25}i$ **(C)** $-\dfrac{3}{25}-\dfrac{4}{25}i$ **(D)** $-\dfrac{3}{25}+\dfrac{4}{25}i$

2. The cube roots of unity are $1, \omega$ and ω^2. Simplify: $\dfrac{1}{1+\omega}+\dfrac{1}{1+\omega^2}$

(A) 0 **(B)** -1 **(C)** 1 **(D)** 2

3. The points P and Q in the first quadrant represent the complex numbers z_1 and z_2 respectively, as shown in the diagram below. Which statement about the complex number $z_2 - z_1$ is true?



(A) It is represented by the vector QP.

(B) Its principal argument lies between $\dfrac{\pi}{2}$ and π.

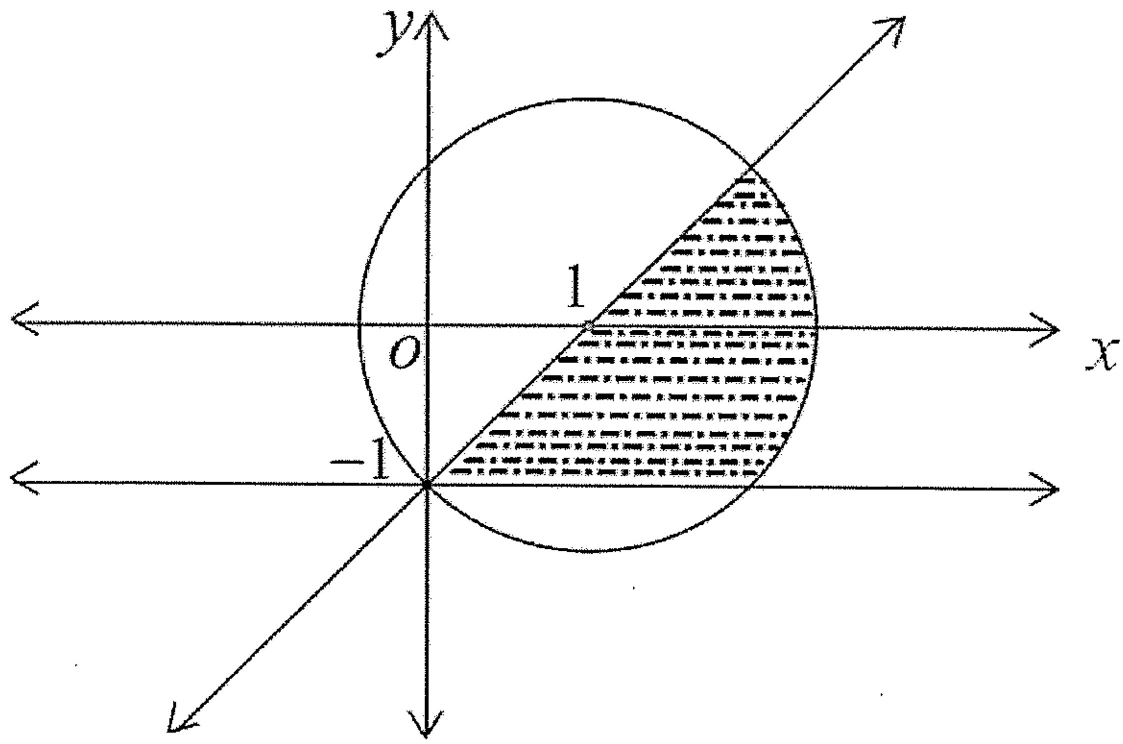
(C) Its real part is positive

(D) Its modulus is greater than $|z_1 + z_2|$.

4. Find $\arg(z^4)$ where $z = -4\sqrt{2} + 4\sqrt{2}i$

(A) π **(B)** 3π **(C)** $-\pi$ **(D)** $\dfrac{3\pi}{4}$

5. Consider the Argand diagram below. Which inequality could define the shaded area?



(A) $|z-1| \leq \sqrt{2}$ and $0 \leq \arg(z-i) \leq \dfrac{\pi}{4}$

(B) $|z-1| \leq \sqrt{2}$ and $0 \leq \arg(z+i) \leq \dfrac{\pi}{4}$

(C) $|z-1| \leq 1$ and $0 \leq \arg(z-i) \leq \dfrac{\pi}{4}$

(D) $|z-1| \leq 1$ and $0 \leq \arg(z+i) \leq \dfrac{\pi}{4}$

Question 6 (18 marks) Marks

(a) If $z=(1-i)(2\sqrt{3}+2i)$

(i) Express z in the form $x+iy$, where x and y are real. 2

(ii) By expressing $1-i$ and $2\sqrt{3}+2i$ in the modulus argument form, show that

$$z=4\sqrt{2}cis\left(-\frac{\pi}{12}\right)$$ 3

(iii) Hence find the exact value of $\tan\frac{\pi}{12}$ 3

(b) (i) Find all pairs of real numbers x and y such that $(x+iy)^2=8+6i$. 3

(ii) Hence solve: $z^2+2(1+2i)z-(11+2i)=0$ 3

(c) Find the values of θ $(0\le\theta\le2\pi)$ such that $\dfrac{3+2i\sin\theta}{1-2i\sin\theta}$ is a real number. 4

Question 7 (24 marks)

(a)(i) Express $\sin 6\theta$ and $\cos 6\theta$ in terms of $\cos\theta$ and $\sin\theta$. 4

(ii) Hence express $\cot 6\theta$ in terms of $\cot\theta$. 3

(b) (i) If $z=\cos\theta+i\sin\theta$, show that $z^n+\dfrac{1}{z^n}=2\cos n\theta$ and $z^n-\dfrac{1}{z^n}=2i\sin n\theta$. 4

(ii) Use De Moivre's theorem to obtain an expression for $\sin^7\theta$ in the form

$a\sin 7\theta+b\sin 5\theta+c\sin 3\theta+d\sin\theta$. What are the values of a,b,c and d. 4

(c) (i) Solve $z^6+1=0$. Show the roots on an Argand diagram. 4

(ii) Resolve z^6+1 into real quadratic factors. Hence show that

$$\cos 3\theta=4\cos\theta\left(\cos^2\theta-\cos^2\frac{\pi}{6}\right)$$ 5

Question 8 (19 marks)

(a) Find all the cube roots of i. Express your answer in the form $x+iy$. 3

(b) **(i)** Find the roots of $z^7=1$. 2

(ii) Show that the roots can be written in the form $1,\omega,\omega^2,\omega^3,\omega^4,\omega^5,\omega^6$ where $\omega = cis\dfrac{2\pi}{7}$. 2

(iii) Show that $1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6=0$. 2

(iv) Form a quadratic equation with roots $\omega+\omega^2+\omega^4$ and $\omega^3+\omega^5+\omega^6$. 3

(v) Hence find the exact value of $\sin\dfrac{2\pi}{7}+\sin\dfrac{3\pi}{7}-\sin\dfrac{\pi}{7}$. 4

(c) Solve the equation $z^5+z^4+z^3+z^2+z+1=0$. 3

Question 9 (20 marks)

(a) Sketch the following loci on separate Argand diagram.

(i) $|z+2-i|=|z-2+i|$ 3

(ii) $\arg(z-1-i)=\dfrac{\pi}{4}$ 3

(iii) $1\le|z-1|\le 2$ 3

(iv) $\arg(z+2)-\arg(z-i)=\pi$ 3

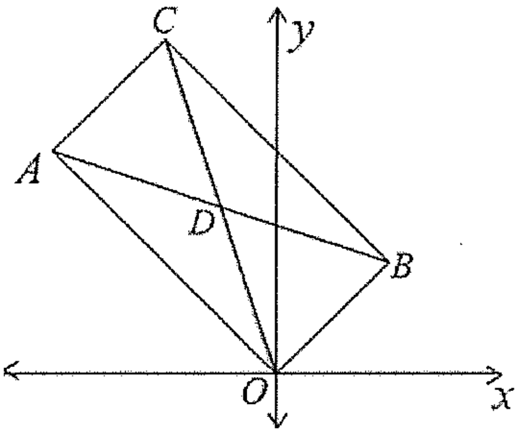
(b) Find the locus of z and sketch on an Argand diagram.

(i) $|z-2|=\text{Re}(z)$ 4

(ii) $\left|z^2-\left(\bar{z}\right)^2\right|\ge 8$ 4

Question 10 (10 marks)

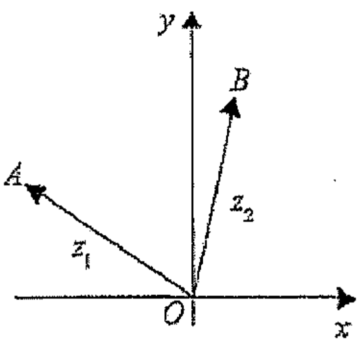
(a) $OACB$ is a rectangle where $OA = 2\times OB$. D is the point of intersection of the diagonals. The point B represents the complex number z.



Find in terms of z, the complex number represented by:

(i) A **(ii)** D 4

(b)



In the Argand diagram, vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the complex numbers

$z_1 = 2\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$ and $z_2 = 2\left(\cos\frac{7\pi}{15} + i\sin\frac{7\pi}{15}\right)$ respectively.

(i) Show that ΔOAB is equilateral. 3

(ii) Express $z_2 - z_1$ in modulus-argument form. 3

Question 11 (11 marks)

(a) Given the equation $\arg\left(\dfrac{z-3}{z+1}\right)=\dfrac{3\pi}{4}$.

(i) Draw $\overrightarrow{z-3}$, $\overrightarrow{z+3}$, $\arg(z-3)$ and $\arg(z+1)$ in an Argand diagram and mark the angle representing $\arg\left(\dfrac{z-3}{z+1}\right)$ giving reasons. 4

(ii) Find the equation of the locus of $\arg\left(\dfrac{z-3}{z+1}\right)=\dfrac{3\pi}{4}$, describe the locus and sketch in the Argand diagram drawn in **(i)**. All reasons must be given. 7

END OF EXAMINATION

2015 Extension 2 Test 1 (2016 HSC)

1. Let $z = \frac{1}{3+4i}$

$= \frac{1}{3+4i} \times \frac{3-4i}{3-4i}$

$= \frac{3-4i}{9-(4i)^2}$

$= \frac{3-4i}{9-16i^2}$

$= \frac{3-4i}{9+16}$

$= \frac{3-4i}{25}$

$= \frac{3}{25} - \frac{4i}{25}$

$\bar{z} = \frac{3}{25} + \frac{4}{25}i$ B

2. $\frac{1}{1+\omega} + \frac{1}{1+\omega^2}$

$= \frac{1+\omega^2+1+\omega}{(1+\omega)(1+\omega^2)}$

$= \frac{1}{1+\omega^2+\omega+\omega^3}$

$= \frac{1}{0+1} = 1$ C

3. B

4. Let $z = -4\sqrt{2} + 4\sqrt{2}i$

$|z| = \sqrt{(4\sqrt{2})^2 + (4\sqrt{2})^2} = \sqrt{32+32} = 8$

$\tan\alpha = \frac{4\sqrt{2}}{4\sqrt{2}} = 1 \quad \alpha = \frac{\pi}{4}$

$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$-4\sqrt{2} + 4\sqrt{2}i = 8\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$z^4 = 8^4(\cos 3\pi + i\sin 3\pi)$

$= 8^4(\cos\pi + i\sin\pi)$ A

5. B

Question 6 (18 marks)

(a)(i) $z = (1-i)(2\sqrt{3}+2i)$ 2

$= 2\sqrt{3} + 2i - i2\sqrt{3} - 2i^2$

$= 2\sqrt{3} + 2i - i2\sqrt{3} + 2$

$= 2 + 2\sqrt{3} + i(2 - 2\sqrt{3})$ — 1

(ii) $z = 1 - i$

$|z| = \sqrt{1+1} = \sqrt{2}$

$\tan\alpha = 1$

$\alpha = \frac{\pi}{4}$

$\theta = \frac{-\pi}{4}$

$1 - i = \sqrt{2}\,\mathrm{cis}\left(\frac{-\pi}{4}\right)$

$w = 2\sqrt{3} + 2i$

$|w| = \sqrt{12+4} = 4$

$\tan\alpha = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$

$\theta = \frac{\pi}{6}$

$2\sqrt{3} + 2i = 4\,\mathrm{cis}\frac{\pi}{6}$

$(1-i)\ (2\sqrt{3}+2i)$

$= \left(\sqrt{2}\ \text{cis}\ \frac{-\pi}{4}\right)\left(4\,\text{cis}\ \frac{\pi}{6}\right)$

$= 4\sqrt{2}\ \text{Cis}\left(\frac{\pi}{6}-\frac{\pi}{4}\right)$

$= 4\sqrt{2}\ \text{Cis}\left(\frac{2\pi-3\pi}{12}\right)$ 3

$= 4\sqrt{2}\ \text{cis}\ \frac{-\pi}{12}$

$= 4\sqrt{2}\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)$

$= 4\sqrt{2}\cos\frac{\pi}{12} - i\,4\sqrt{2}\sin\frac{\pi}{12}$ — (2)

Equating real and imaginary parts of (1) and (2)

$4\sqrt{2}\cos\frac{\pi}{12} = 2+2\sqrt{3}$

$\cos\frac{\pi}{12} = \frac{2+2\sqrt{3}}{4\sqrt{2}}$

$= \frac{1+\sqrt{3}}{2\sqrt{2}}$

$-4\sqrt{2}\sin\frac{\pi}{12} = 2-2\sqrt{3}$

$\sin\frac{\pi}{12} = \frac{2-2\sqrt{3}}{-4\sqrt{2}}$

$= \frac{2\sqrt{3}-2}{4\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

$\tan\frac{\pi}{12} = \frac{\sin\frac{\pi}{12}}{\cos\frac{\pi}{12}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\cancel{2}\sqrt{2}}{1+\sqrt{3}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

3

$$(x+iy)^2 = 8+6i$$

$$x^2+2ixy-y^2 = 8+6i$$

$$x^2-y^2=8$$

$$2xy=6$$

$$xy=3$$

$$y=\frac{3}{x}$$

$$x^2-\frac{9}{x^2}=8$$

$$x^4-9=8x^2$$

$$x^4-8x^2-9=0$$

$-9, 1$ $\quad pq=-9$, $\quad p+q=-8$

$$m^2-8m-9=0 \quad (3)$$

$$(m-9)(m+1)=0$$

$$m=9 \text{ or } m=-1$$

$$x^2=9 \text{ or } x^2=-1$$

$x=\pm 3$ ($\because x$ is real)

when $x=3$, $y=1$

when $x=-3$, $y=-1$

$\therefore$ the square roots are

$3+i$ or $-3-i$

$$\underline{\underline{\pm(3+i)}}$$

(ii)

$$z = \frac{-2(1+2i) \pm \sqrt{4(1+2i)^2-4\times -(11+2i)}}{2}$$

$$= \frac{-2(1+2i) \pm \sqrt{4(1+4i+4i^2)+4(11+2i)}}{2}$$

$$= \frac{-2-4i \pm \sqrt{4+16i-16+44+8i}}{2}$$

$$= \frac{-2-4i \pm \sqrt{32+24i}}{2}$$

$$= \frac{-2-4i \pm \sqrt{4(8+6i)}}{2}$$

$$= \frac{-2-4i \pm 2\sqrt{8+6i}}{2} \quad (3)$$

$$= \frac{-2-4i \pm 2(3+i)}{2}$$

$$= \frac{-2-4i+2(3+i)}{2} \text{ or } \frac{-2-4i-2(3+i)}{2}$$

$$= \frac{-2-4i+6+2i}{2} \text{ or } \frac{-2-4i-6-2i}{2}$$

$$= \frac{4-2i}{2} \text{ or } \frac{-8-6i}{2}$$

$$= \frac{2(2-i)}{2} \text{ or } \frac{2(-4-3i)}{2}$$

$$\underline{\underline{= 2-i \text{ or } -4-3i}}$$

(c) $\frac{3+2i\sin\theta}{1-2i\sin\theta}$

$$= \frac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}$$

$$= \frac{(3+2i\sin\theta)(1+2i\sin\theta)}{1-(2i\sin\theta)^2}$$

$$= \frac{3+6i\sin\theta+2i\sin\theta+4i^2\sin^2\theta}{1-4i^2\sin\theta}$$

$$= \frac{3+8i\sin\theta-4\sin^2\theta}{1+4\sin^2\theta}$$

4

$$= \frac{3-4\sin^2\theta+i8\sin\theta}{1+4\sin^2\theta}$$

$$= \frac{3-4\sin^2\theta}{1+4\sin^2\theta} + \frac{8\sin\theta}{1+4\sin^2\theta}i$$

Since $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is purely real, its imaginary part is 0

$$\frac{8\sin\theta}{1+4\sin^2\theta} = 0$$

$$8\sin\theta = 0$$

$$\sin\theta = 0$$

$$\theta = 0, \pi, 2\pi$$

Question 7 (23 marks)

(1) $(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$ by De Moivre's theorem — (1)

Apply Binomial theorem,

$(\cos\theta + i\sin\theta)^6 = \cos^6\theta + {}^6C_1 \cos^5\theta\, i\sin\theta + {}^6C_2 \cos^4\theta\, i^2\sin^2\theta$

$+ {}^6C_3 \cos^3\theta\, i^3\sin^3\theta + {}^6C_4 \cos^2\theta\, i^4\sin^4\theta + {}^6C_5 \cos\theta\, i^5\sin^5\theta$

$+ {}^6C_6\, i^6\sin^6\theta$

$= \cos^6\theta + 6\cos^5\theta\, i\sin\theta + 15\cos^4\theta \times -1\sin^2\theta - i\,20\cos^3\theta\sin^3\theta$

$+ 15\cos^2\theta\sin^4\theta + 6\cos\theta \times i\sin^5\theta + -1 \times \sin^6\theta$ — (2)

Equating real and imaginary parts of (1) and (2)

$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$

$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$

$\cot 6\theta = \dfrac{\cos 6\theta}{\sin 6\theta}$

$= \dfrac{\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta}{6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta}$

divide both numerator and denominator by $\sin^6\theta$ (3)

$= \dfrac{\cot^6\theta - 15\cot^4\theta + 15\cot^2\theta - 1}{6\cot^5\theta - 20\cot^3\theta + 6\cot\theta}$

(b)(i) $z = \cos\theta + i\sin\theta$

$$z^n = (\cos\theta + i\sin\theta)^n$$

$$= \cos n\theta + i\sin n\theta$$ by De Moivre's theorem

$$\frac{1}{z^n} = z^{-n} = (\cos\theta + i\sin\theta)^{-n}$$

$$= \cos(-n\theta) + i\sin(-n\theta)$$ by De Moivre's

$$= \cos n\theta - i\sin n\theta \quad \left(\because \sin(-\theta) = -\sin\theta, \ \cos(-\theta) = \cos\theta\right)$$

$$z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$

$$= \underline{\underline{2\cos n\theta}}$$ (4)

$$z^n - \frac{1}{z^n} = \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$$

$$= \underline{\underline{2i\sin n\theta}}$$

(ii) $(2i\sin\theta)^7 = \left(z - \frac{1}{z}\right)^7$

$$= z^7 - {}^7C_1 z^6 \times \frac{1}{z} + {}^7C_2 z^5 \times \frac{1}{z^2} - {}^7C_3 z^4 \times \frac{1}{z^3}$$

$$+ {}^7C_4 z^3 \times \frac{1}{z^4} - {}^7C_5 z^2 \times \frac{1}{z^5} + {}^7C_6 z \times \frac{1}{z^6} - {}^7C_7 \frac{1}{z^7}$$

$$= z^7 - 7z^5 + 21z^3 - 35z + 35\frac{1}{z} - 21 \times \frac{1}{z^3} + 7 \times \frac{1}{z^5} - \frac{1}{z^7}$$

$$= \left(z^7 - \frac{1}{z^7}\right) - 7\left(z^5 - \frac{1}{z^5}\right) + 21\left(z^3 - \frac{1}{z^3}\right) - 35\left(z - \frac{1}{z}\right)$$

$$= 2i\sin 7\theta - 7 \times 2i\sin 5\theta + 21 \times 2i\sin 3\theta - 35 \times 2i\sin\theta$$

$2^7 i^7 \sin^7\theta = 2i\sin 7\theta - 14i\sin 5\theta + 42i\sin 3\theta - 70i\sin\theta$

$2^7 \times -i\sin^7\theta = 2i\sin 7\theta - 14i\sin 5\theta + 42i\sin 3\theta - 70i\sin\theta$

$2^6 \times -1\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta$

$-64\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta$

$\sin^7\theta = \frac{-1}{64}\left[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta\right]$

$= \frac{1}{64}\left[-\sin 7\theta + 7\sin 5\theta - 21\sin 3\theta + 35\sin\theta\right]$

$= \frac{-1}{64}\sin 7\theta + \frac{7}{64}\sin 5\theta - \frac{21}{64}\sin 3\theta + \frac{35}{64}\sin\theta$

$a = \frac{-1}{64} \quad b = \frac{7}{64} \quad c = -\frac{21}{64} \quad d = \frac{35}{64}$ 4

(c) $z^6 + 1 = 0$

$z^6 = -1 = \cos\pi + i\sin\pi$

$= \cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi), \; k \in I$

$= \cos(2k+1)\pi + i\sin(2k+1)\pi, \; k \in I$

$z = \left[\cos(2k+1)\pi + i\sin(2k+1)\pi\right]^{\frac{1}{6}}$

$= \cos\frac{(2k+1)\pi}{6} + i\sin\frac{(2k+1)\pi}{6}, \; k = 0, 1, 2, 3, 4, 5$

(by De Moivre's theorem)

when $k=0$,

$z_1 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

when $k=1$,

$z_2 = \cos\frac{3\pi}{6} + i\sin\frac{3\pi}{6}$

$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ α

when $k=2$,

$z_3 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$

when $k=3$,

$z_4 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$

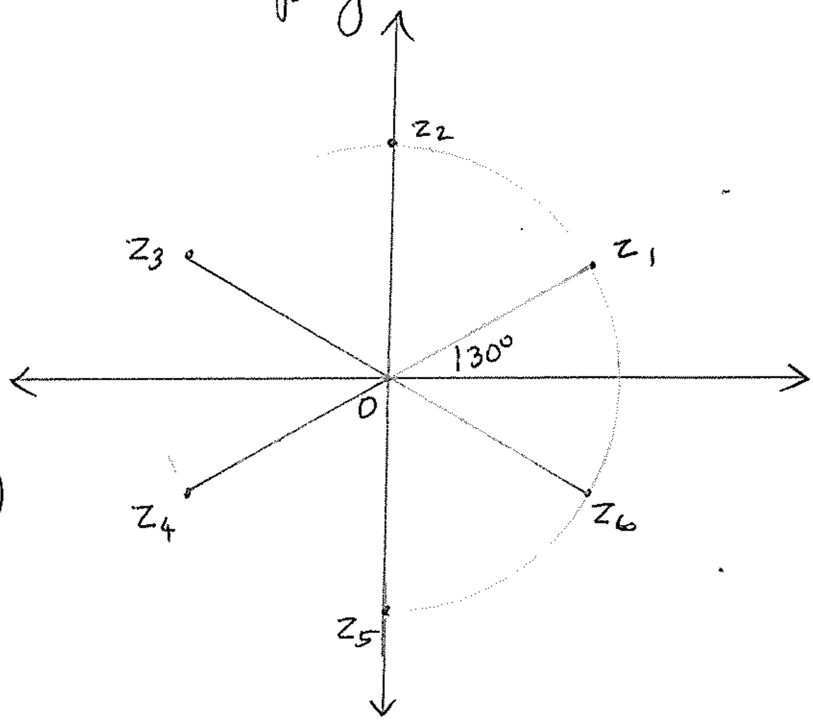
when $k=4$,

$z_5 = \cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}$

$= \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$

when $k=5$,

$z_6 = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$



From the diagram

$z_6 = \overline{z_1}$, $z_5 = \overline{z_2}$, $z_4 = \overline{z_3}$

$z_1 + z_6 = z_1 + \overline{z_1} = 2\cos\frac{\pi}{6}$

$z_1 z_6 = z_1 \overline{z_1} = 1$

$z_3 + z_4 = z_3 + \overline{z_3} = 2\cos\frac{5\pi}{6}$

$z_3 z_4 = z_3 \overline{z_3} = 1$

$$z^6+1=(z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)(z-z_6)$$

$$=(z-z_1)(z-z_6)(z-z_3)(z-z_4)(z-z_2)(z-z_5)$$

$$=(z-z_1)(z-z_6)(z-z_3)(z-z_4)(z-i)(z+i)$$

$$=(z^2+1)\left(z^2-z(z_1+z_6)+z_1z_6\right)\left(z^2-z(z_3+z_4)+z_3z_4\right)$$

$$=(z^2+1)\left(z^2-z\times 2\cos\frac{\pi}{6}+1\right)\left(z^2-z\times 2\cos\frac{5\pi}{6}+1\right)$$

$$z^6+1=(z^2+1)\left(z^2-2\cos\frac{\pi}{6}z+1\right)\left(z^2-2\cos\frac{5\pi}{6}z+1\right)$$

2

divide both sides by z^3

$$\frac{z^6+1}{z^3}=\frac{z^2+1}{z}\times\frac{z^2-2\cos\frac{\pi}{6}z+1}{z}\times\frac{z^2-2\cos\frac{5\pi}{6}z+1}{z}$$

$$z^3+\frac{1}{z^3}=\left(z+\frac{1}{z}\right)\left(z-2\cos\frac{\pi}{6}+\frac{1}{z}\right)\left(z-2\cos\frac{5\pi}{6}+\frac{1}{z}\right)$$

$$2\cos 3\theta=2\cos\theta\left(2\cos\theta-2\cos\frac{\pi}{6}\right)\left(2\cos\theta-2\cos\frac{5\pi}{6}\right)$$

$$2\cos 3\theta=8\cos\theta\left(\cos\theta-\cos\frac{\pi}{6}\right)\left(\cos\theta-\cos\frac{5\pi}{6}\right)$$

$$\cos 3\theta=4\cos\theta\left(\cos\theta-\cos\frac{\pi}{6}\right)\left(\cos\theta-\cos\frac{5\pi}{6}\right)$$

$$=4\left(\cos\theta-\cos\frac{\pi}{2}\right)\left(\cos\theta-\cos\frac{\pi}{6}\right)\left(\cos\theta-\cos\frac{5\pi}{6}\right)$$

$$=4\left(\cos\theta-\cos\frac{\pi}{2}\right)\left(\cos\theta-\cos\frac{\pi}{6}\right)\left(\cos\theta+\cos\frac{\pi}{6}\right)$$

$$=4\left(\cos\theta-\cos\frac{\pi}{2}\right)\left(\cos^2\theta-\cos^2\frac{\pi}{6}\right)$$

5

$\because \cos\frac{5\pi}{6}=-\cos\left(\pi-\frac{5\pi}{6}\right)=-\cos\frac{\pi}{6}$

Question 8 (19 marks)

(a) $z^3 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$= \cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)$

$= \text{Cis}\ \frac{4k\pi + \pi}{2}$

$= \text{Cis}\ \frac{(4k+1)\pi}{2}$

$z = \text{Cis}\ \frac{(4k+1)\pi}{6}$ $k = 0, 1, 2$ by De Moivre's Theorem

when $k=0$,

$z_1 = \text{Cis}\ \frac{\pi}{6}$

$= \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2} + i \times \frac{1}{2} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

when $k=1$

3

$z_2 = \text{Cis}\ \frac{5\pi}{6}$

$= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$

$= -\frac{\sqrt{3}}{2} + i \times \frac{1}{2}$

$= -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

when $k=2$

$z_3 = \text{Cis}\ \frac{9\pi}{6} = \text{Cis}\ \frac{3\pi}{2}$

$= -i$

The cube roots are

$-i, \pm\frac{\sqrt{3}}{2} + \frac{i}{2}$

(b) (i) $z^7 = 1 = \text{Cis}\, 0 = \text{Cis}\, 2k\pi,\ k \in J$

$z = \left[\text{Cis}\, 2k\pi\right]^{\frac{1}{7}}$

$= \text{Cis}\, \frac{2k\pi}{7}$ $k = 0, 1, 2, 3, 4, 5, 6,$ by De Moivre's theorem.

2

Roots are

$1,\ \text{Cis}\,\frac{2\pi}{7},\ \text{Cis}\,\frac{4\pi}{7},\ \text{Cis}\,\frac{6\pi}{7},\ \text{Cis}\,\frac{8\pi}{7},\ \text{Cis}\,\frac{10\pi}{7},\ \text{Cis}\,\frac{12\pi}{7}$

(ii) Let $w = \text{Cis}\,\frac{2\pi}{7}$

$w^2 = \left(\text{Cis}\,\frac{2\pi}{7}\right)^2 = \text{Cis}\,\frac{4\pi}{7}$

$w^3 = \left(\text{Cis}\,\frac{2\pi}{7}\right)^3 = \text{Cis}\,\frac{6\pi}{7}$

$w^4 = \left(\text{Cis}\,\frac{2\pi}{7}\right)^4 = \text{Cis}\,\frac{8\pi}{7}$

$w^5 = \left(\text{Cis}\,\frac{2\pi}{7}\right)^5 = \text{Cis}\,\frac{10\pi}{7}$

$w^6 = \left(\text{Cis}\,\frac{2\pi}{7}\right)^6 = \text{Cis}\,\frac{12\pi}{7}$

2

$\therefore$ the roots can be written as $1, w, w^2, w^3, w^4, w^5$ and w^6.

(iii) $1 + w + w^2 + w^3 + w^4 + w^5 + w^6$

$$= -\frac{\text{Coefficient of } z^6}{\text{Coefficient of } z^7} = -\frac{0}{1} = 0$$ 2

(iv) $z_1 = w + w^2 + w^4,\quad z_2 = w^3 + w^5 + w^6$

$z_1 + z_2 = w + w^2 + w^4 + w^3 + w^5 + w^6 = -1$

$$z_1 z_2 = (w+w^2+w^4)(w^3+w^5+w^6)$$

$$= w^4+w^6+w^7+w^5+w^7+w^8+w^7+w^9+w^{10}$$

$$= w^4+w^6+1+w^5+1+w+1+w^2+w^3$$

$$= 0+2 = 2$$

The quadratic equation with roots z_1 and z_2 is

$$z^2-(-1)z+2=0 \quad (3)$$

$$z^2+z+2=0$$

(v) $z^2+z+2=0$

$$z = \frac{-1 \pm \sqrt{1-4\times1\times2}}{2}$$

$$= \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm \sqrt{-7}}{2}$$

$$= \frac{-1 \pm i\sqrt{7}}{2}$$

$$= \frac{-1}{2} + \frac{i\sqrt{7}}{2} \text{ or } \frac{-1}{2} - \frac{i\sqrt{7}}{2} \quad \text{—— (1)}$$

Consider $z_1 = w+w^2+w^4$

$$= \operatorname{cis}\frac{2\pi}{7} + \operatorname{cis}\frac{4\pi}{7} + \operatorname{cis}\frac{8\pi}{7}$$

$$= \left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}\right) + i\left(\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}\right) \quad (2)$$

Comparing (1) and (2) we get

$$\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \frac{\sqrt{7}}{2} \text{ or } \frac{-\sqrt{7}}{2} \quad \text{—— (3)}$$

But $\sin\frac{4\pi}{7} = \sin\left(\pi - \frac{4\pi}{7}\right) = \sin\frac{3\pi}{7}$

$\sin\frac{8\pi}{7} = -\sin\left(\frac{8\pi}{7} - \pi\right) = -\sin\frac{\pi}{7}$

(3) becomes

(4)

$$\sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} - \sin\frac{\pi}{7} = \frac{\sqrt{7}}{2} \text{ or } -\frac{\sqrt{7}}{2}$$

$$\sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} - \sin\frac{\pi}{7} > 0$$

$$\therefore \underline{\underline{\sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} - \sin\frac{\pi}{7} = \frac{\sqrt{7}}{2}}}$$

(c) $z^5 + z^4 + z^3 + z^2 + z + 1 = 0$

$z^6 - 1 = (z-1)(z^5 + z^4 + z^3 + z^2 + z + 1)$

$z^5 + z^4 + z^3 + z^2 + z + 1 = \frac{z^6 - 1}{z - 1}$

(3)

$z^6 - 1 = 0, \quad z \neq 1$

$z^6 = 1 = \cos 0 + i\sin 0$

$= \cos 2k\pi + i\sin 2k\pi \quad k \in J$

$z = \left(\cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6}\right) \quad k = 0, 1, 2, 3, 4, 5$

by De Moivre's theorem

$= \cos\frac{k\pi}{3} + i\sin\frac{k\pi}{3} \quad k = 0, 1, 2, 3, 4, 5$

when $k = 0$, $z_1 = 1$

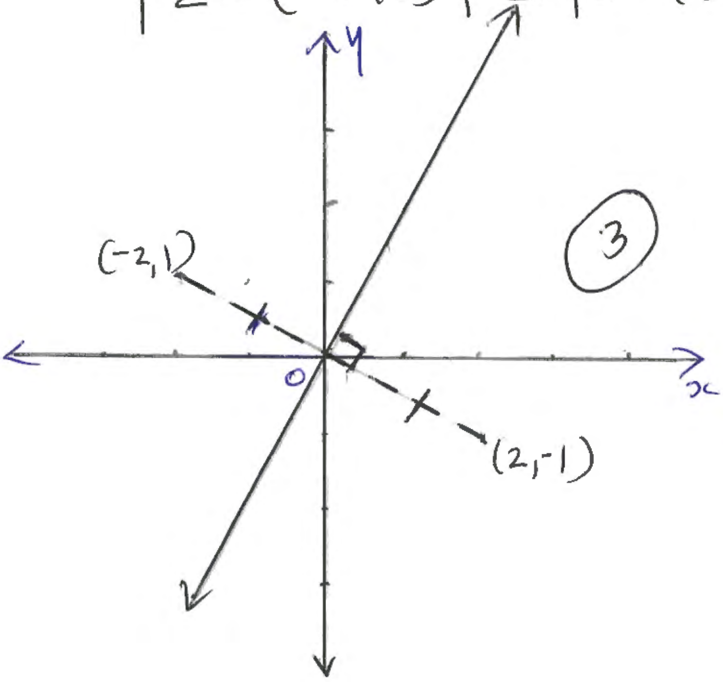
$\therefore$ the roots of $z^5 + z^4 + z^3 + z^2 + z + 1 = 0$ are

given by $\underline{\underline{z = \cos\frac{k\pi}{3} + i\sin\frac{k\pi}{3},}}$ $k = 1, 2, 3, 4, 5$

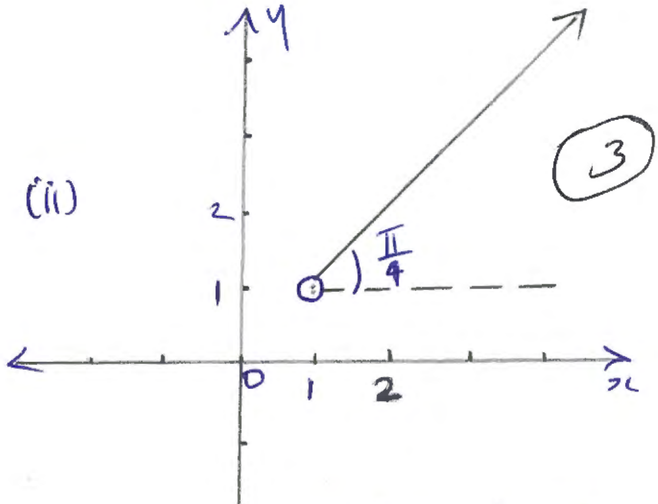
Question 9 (20 marks)

(a) (i) $|z+2-i| = |z-2+i|$

$$|z-(-2+i)| = |z-(2-i)|$$

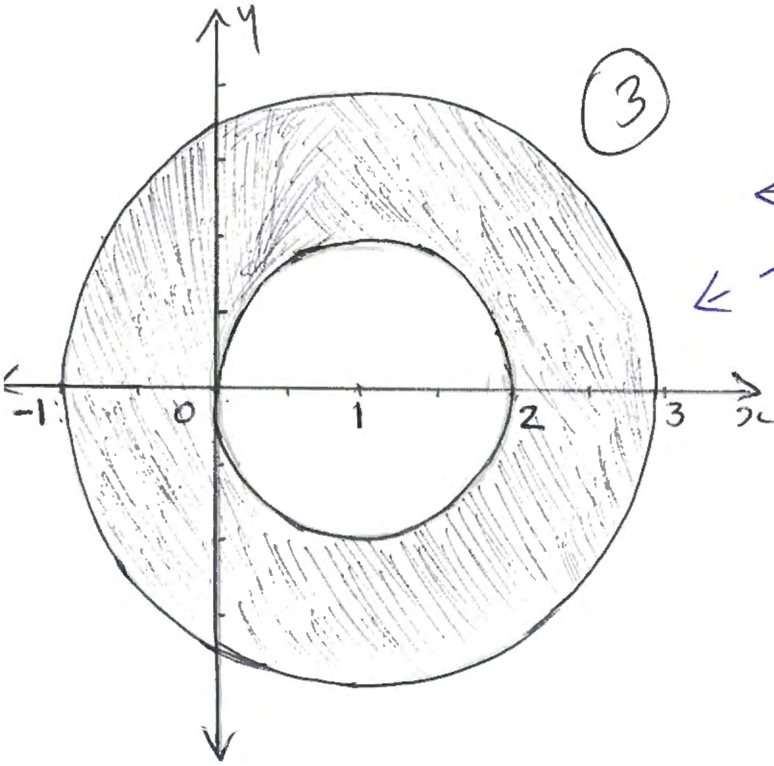


(ii)

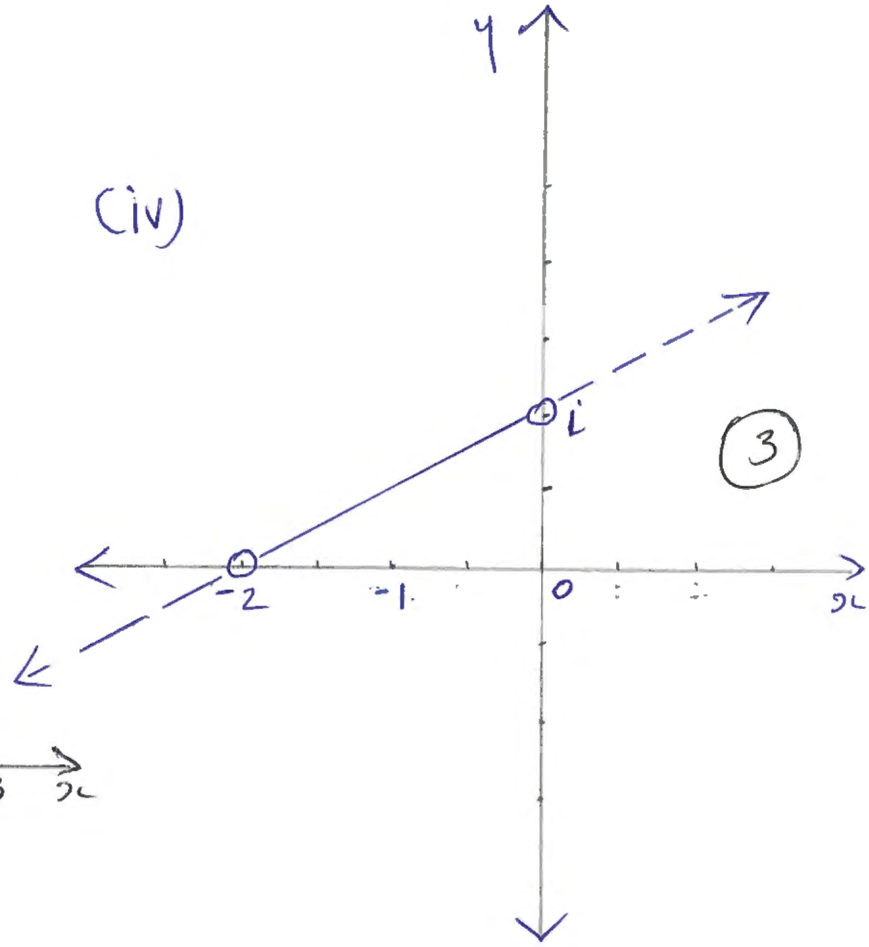


$\arg(z-1-i) = \frac{\pi}{4}$

(iii) $1 \le |z-1| \le 2$



(iv)



$\arg(z+2) - \arg(z-i) = \pi$

(b) $|z-2| = Re(z)$

$|x+iy-2| = x$

$|x-2+iy| = x$

$\sqrt{(x-2)^2+y^2} = x$

$(x-2)^2+y^2 = x^2$ (4)

$x^2-4x+4+y^2 = x^2$

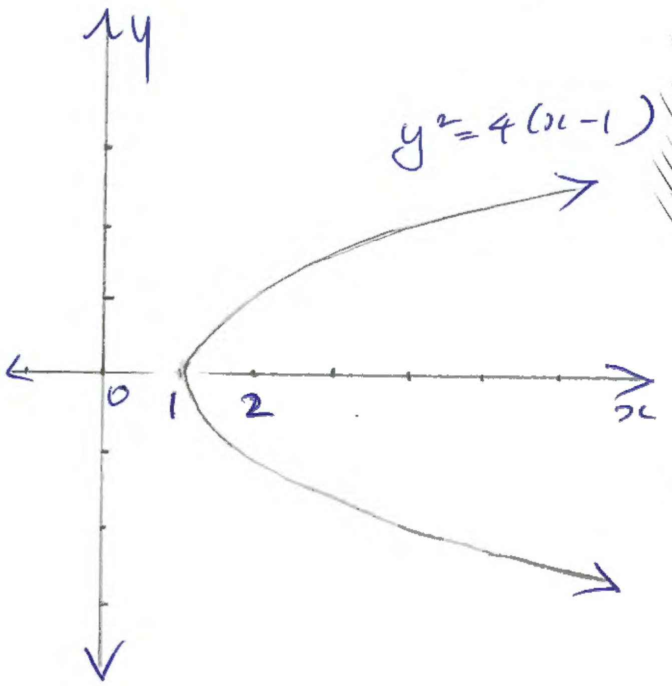
$y^2 = 4x-4$

$= 4(x-1)$

$4a = 4 \quad a = 1$

Vertex $(1,0)$ Focus $(2,0)$

Directrix $x = 0$



(ii) $|z^2-\bar{z}^2| \geq 8$

$|(x+iy)^2-(x-iy)^2| \geq 8$

$|(x+iy+x-iy)(x+iy-x+iy)| \geq 8$

$|(2x)(2iy)| \geq 8$

$|4ixy| \geq 8$

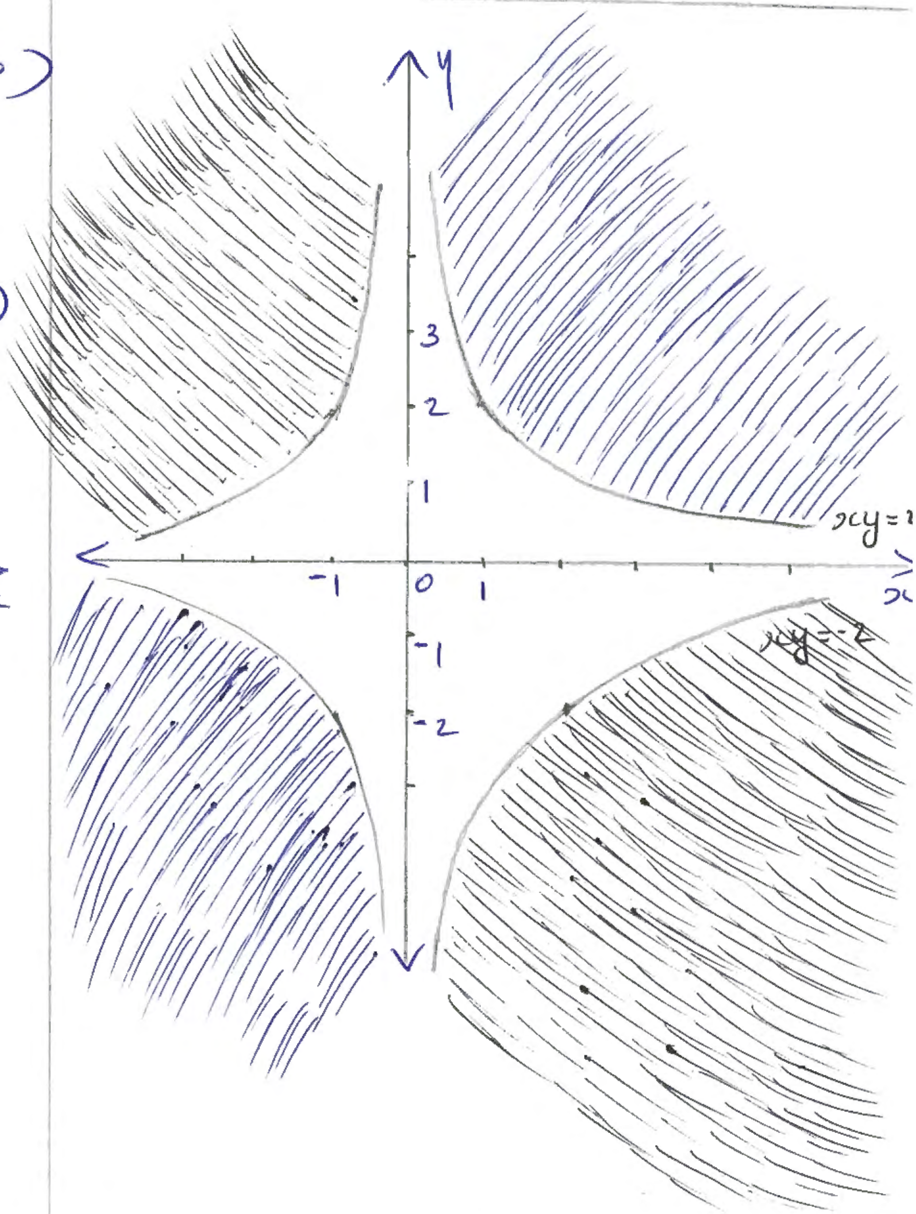
$|4i||xy| \geq 8$

$4|xy| \geq 8$

$|xy| \geq 2$ (4)

$xy \geq 2$ or $-xy \geq 2$

$xy \geq 2$ or $xy \leq -2$



Question 10 (10 marks)

(a) (i) $OA = 2 \times OB$

$\vec{OA} = 2zi$

$\therefore A$ is $2zi$

1

$\vec{OC} = \vec{OA} + \vec{AC}$

$= 2zi + z$

$= (1+2i)z$

(ii) D is the midpoint of O and C ($\because$ diagonals bisect each other)

$D = \frac{(1+2i)z}{2}$

3

$= \frac{1}{2}z + iz$

(b) (i) $\angle AOB = \angle AOX - \angle BOX$

$= \arg(z_1) - \arg(z_2)$

$= \frac{4\pi}{5} - \frac{7\pi}{15} = \frac{\pi}{3}$

3

$|z_1| = |z_2| = 2$

$\angle OAB = \angle OBA = \frac{\pi - \frac{\pi}{3}}{2} = \frac{\pi}{3}$

(angles opposite equal side in an isosceles Δ)

$\therefore \Delta OAB$ is equilateral.

(ii) $z_2 - z_1$ is represented by $\vec{AB}$.

AB is the rotation of z_2 in the clockwise direction through an angle of $\frac{\pi}{3}$

3

$\vec{AB} = OB \times cis\left(\frac{-\pi}{3}\right)$

$z_2 - z_1 = z_2\, cis\left(\frac{-\pi}{3}\right)$

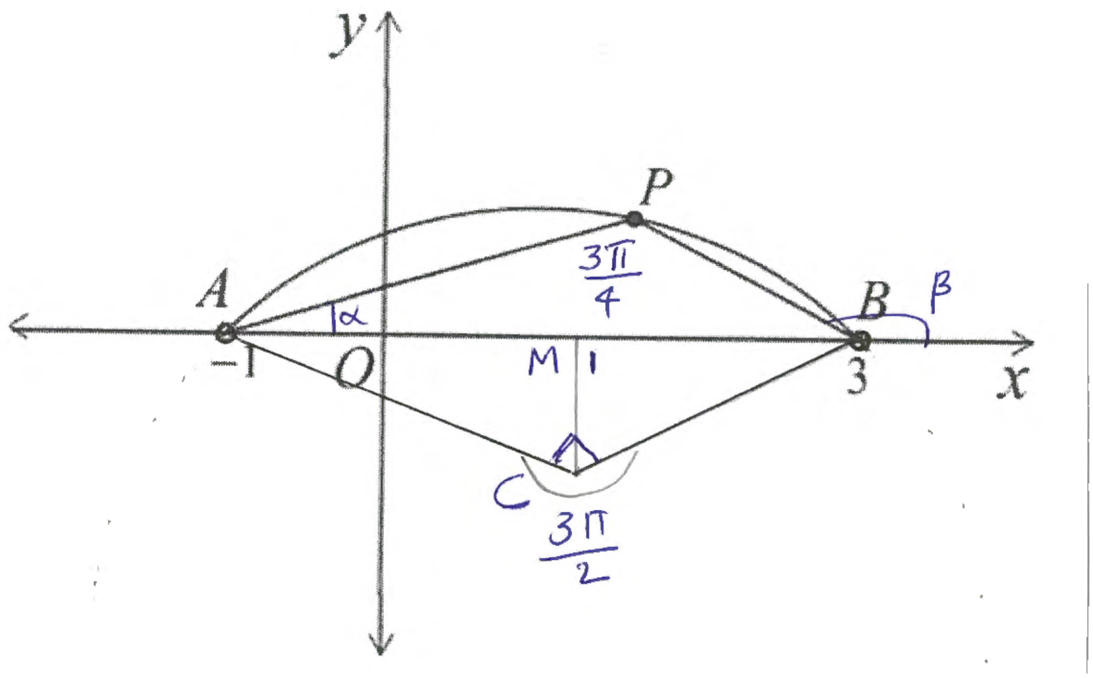
$= 2\left(\cos\frac{7\pi}{15} + i\sin\frac{7\pi}{15}\right)\left(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}\right)$

$= 2\, cis\left(\frac{7\pi}{15} - \frac{\pi}{3}\right)$

$= 2\, cis\left(\frac{2\pi}{15}\right)$

$= 2\left(\cos\frac{2\pi}{15} + i\sin\frac{2\pi}{15}\right)$

Question 11 (11 marks)



(i) $\arg(z-3) = \angle PBX = \beta$

$\arg(z+1) = \angle PAX = \alpha$

Let $\angle APB = \theta$

$\alpha + \theta = \beta$ (exterior angle of ΔAPB)

4

$\theta = \beta - \alpha$

$= \arg(z-3) - \arg(z+1)$

$= \arg\left(\frac{z-3}{z+1}\right) = \frac{3\pi}{4}$

(ii) Let C be the centre of the circle. Draw

$CM \perp AB$

M = Midpoint of $AB = (1,0)$
(The perpendicular from the centre of a circle to a chord bisects the chord)

Reflex $\angle ACB = \frac{3\pi}{2}$ (angle at the centre is twice angle at the circumference)

ΔACB is isosceles ($AC = BC$ = radii)

$\angle CBA = \dfrac{\pi - \frac{\pi}{2}}{2}$ (angles opposite equal side in an isosceles Δ)

$= \dfrac{\pi}{4}$

In ΔCBM

$\tan\frac{\pi}{4} = \dfrac{CM}{BM} = \dfrac{CM}{2}$

$\dfrac{CM}{2} = 1$

$CM = 2$

centre $(1,-2)$

Apply Pythagoras' theorem in ΔCBM

$BC = \sqrt{2^2+2^2}$

$= \sqrt{8}$

radius $= \sqrt{8}$

7

locus is the minor arc of the circle

$(x-1)^2 + (y+2)^2 = 8, \ y > 0$