FINAL MARK

GIRRAWEEN HIGH SCHOOL
MATHEMATICS EXTENSION 2
HSC ASSESSMENT TASK 1, 2017 (HSC 2018)
ANSWERS COVER SHEET

Name: ______________________________

QUESTION	MARK	E2	E3	E4	E5	E6	E7	E8	E9
Multiple choice	/5		✓						✓
Q6	/14		✓						✓
Q7	/21		✓						✓
Q8	/21		✓						✓
Q9	/21		✓						✓
Q10	/8		✓						✓
Q11	/20		✓						✓
TOTAL									
	/110		**/110**						**/110**

HSC Outcomes **Mathematics Extension 2**

E1 appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems.

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings.

E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections.

E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials.

E5 uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces, resisted motion and circular motion

E6 combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions.

E7 uses the techniques of slicing and cylindrical shells to determine volumes.

E8 applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems.

E9 communicates abstract ideas and relationships using appropriate notation and logical argument.

GIRRAWEEN HIGH SCHOOL

TASK 1

2017 (HSC 2018)

MATHEMATICS

EXTENSION 2

Time allowed – 90 minutes

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used.
- Start each question on a separate page. Each paper must show your name.

Multiple Choice (5 marks) Write the letter corresponding to the correct answer in your answer booklet.

1. What is the value of i^{2018} ?

(A) 1 **(B)** -1 **(C)** 0 **(D)** -2

2. If $x+iy=\dfrac{a+ib}{c+id}$ then $x^2+y^2=?$

(A) $\dfrac{a^2+b^2}{c^2+d^2}$ **(B)** $\dfrac{a^2-b^2}{c^2+d^2}$ **(C)** $\dfrac{a^2+b^2}{c^2-d^2}$ **(D)** $\dfrac{a^2-b^2}{c^2-d^2}$

3. Multiplying a non-zero complex number by $\dfrac{1+i}{1-i}$ results in a rotation about the origin on an Argand diagram. What is the rotation ?

(A) clockwise by $\dfrac{\pi}{2}$ **(B)** clockwise by $\dfrac{\pi}{4}$

(C) anticlockwise by $\dfrac{\pi}{2}$ **(D)** anticlockwise by $\dfrac{\pi}{4}$

4. If $z=2+i$ and $w=1-i$, then $z\overline{w}$, in the form $x+iy$ is:

(A) $3-i$ **(B)** $1-3i$ **(C)** $1+3i$ **(D)** $1+i$

5. Let $z=5+6i$. Find $\text{Im}(3i-z)$

(A) $3i$ **(B)** $-3i$ **(C)** 3 **(D)** -3

Question 6 (14 marks) Marks

(a) If $z=\dfrac{-1+i}{\sqrt{3}+i}$

(i) Express z in the form $x+iy$, where x and y are real. 2

(ii) By expressing $-1+i$ and $\sqrt{3}+i$ in the modulus argument form, show that

$$z=\frac{1}{\sqrt{2}}cis\left(\frac{7\pi}{12}\right)$$ 3

(iii) Hence find the exact value of $\tan\dfrac{7\pi}{12}$ 3

(b) (i) Find all pairs of real numbers x and y such that $(x+iy)^2=-5-12i$. 3

(ii) Hence solve: $z^2-4z+(9+12i)=0$ 3

Question 7 (21 marks)

(a) (i) Express $\sin 5\theta$ and $\cos 5\theta$ in terms of $\cos\theta$ and $\sin\theta$. 4

(ii) Hence express $\tan 5\theta$ in terms of $\tan\theta$. 3

(b) (i) If $z=\cos\theta+i\sin\theta$, show that $z^n+\dfrac{1}{z^n}=2\cos n\theta$ and $z^n-\dfrac{1}{z^n}=2i\sin n\theta$. 4

(ii) Use De Moivre's theorem to obtain an expression for $\cos^7\theta$ in the form

$a\cos 7\theta+b\cos 5\theta+c\cos 3\theta+d\cos\theta$. What are the values of a,b,c and d. 4

(c) $1,\omega,\omega^2$ are the roots of the equation $z^3=1$.

(i) Show that $1+\omega+\omega^2=0$ 2

(ii) Find the cubic equation whose roots are $1,1+\omega,1+\omega^2$. 4

Question 8 (21 marks)

(a) **(i)** Express the roots of the equation $z^5 + 1 = 0$ in modulus-argument form. 3

(ii) Show the roots of $z^5 + 1 = 0$ in an Argand diagram . 3

(iii) Show that $z^4 - z^3 + z^2 - z + 1 = \left(z^2 - 2\cos\frac{\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{3\pi}{5} + 1\right)$ 4

(iv) Hence find the value of $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$ and $\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$. 4

(v) Form a quadratic equation with roots $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$. 3

(vi) Hence find the exact value of $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ in simplest surd form. 4

Question 9 (21 marks)

(a) Sketch the following loci on separate Argand diagram.

(i) $|z + 2 - 3i| = |z + 2 + i|$ 3

(ii) $\arg(z - (2 + 3i)) = -\frac{\pi}{3}$ 3

(iii) $2 < |z - 1| \leq 3$ 3

(b) Find the locus of z and sketch on an Argand diagram.

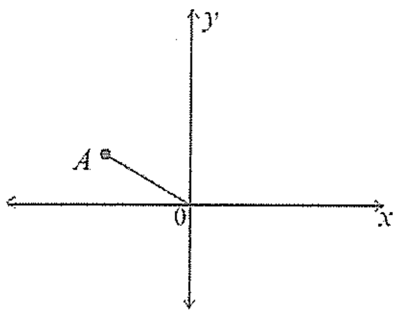
(i) $|z - 4i| = \text{Im}(z)$ 4

(ii) $\left|z^2 - \left(\overline{z}\right)^2\right| \geq 12$ 4

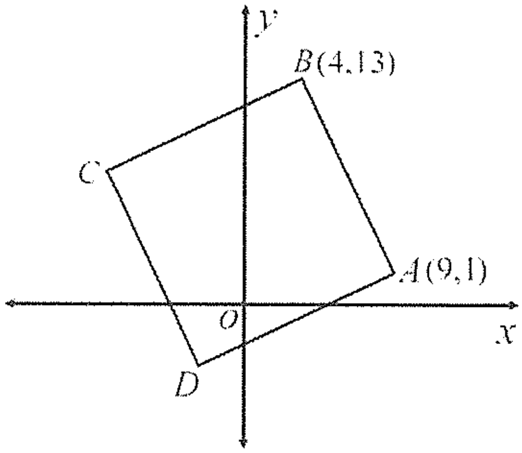
(c) Find the locus of z , given $\frac{z-4}{z-2i}$ is a purely imaginary number. 4

Question 10 (8 marks)

(a) ΔOAB is an isosceles triangle with $OA = OB$ and $\angle OBA = 75°$. If O is the origin and A represent the complex number $-\sqrt{3}+i$, find two possible complex numbers represented by the point B, in the form $a+ib$. 4



(b)



The diagram above shows a square $ABCD$ in the complex plane. The vertices A and B represent the complex numbers $9+i$ and $4+13i$ respectively. Find the complex numbers represented by

(i) The vector AB. 2

(ii) The vertex D 2

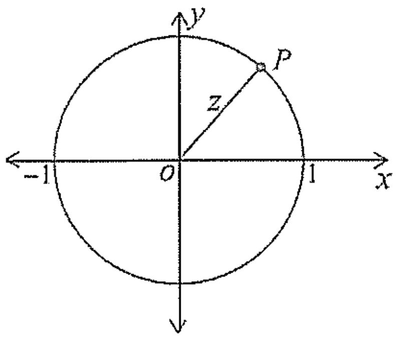
Question 11 (20 marks)

(a) Given the equation $\arg\left(\dfrac{z-2}{z+4}\right)=\dfrac{3\pi}{4}$.

(i) Draw $\overrightarrow{z-2}$, $\overrightarrow{z+4}$, $\arg(z-2)$ and $\arg(z+4)$ in an Argand diagram and mark the angle representing $\arg\left(\dfrac{z-2}{z+4}\right)$ giving reasons. **4**

(ii) Find the equation of the locus of $\arg\left(\dfrac{z-2}{z+4}\right)=\dfrac{3\pi}{4}$, describe the locus and sketch in the Argand diagram drawn in **(i)**. All reasons must be given. **7**

(b) P represents the complex number z such that $|z|=1$. Copy the diagram onto your answer sheet and prove that



(i) $\arg(z+1)=\dfrac{1}{2}\arg z$ **3**

(ii) $\arg(z-1)=\arg(z+1)+\dfrac{\pi}{2}$ **3**

(iii) $\left|\dfrac{z-1}{z+1}\right|=\tan\left(\dfrac{1}{2}\arg z\right)$ **3**

END OF EXAMINATION

Extension 2 Task 1, 2017 (HSC 2018) - solutions

Multiple choice (5 marks)

1B 2A 3C 4C 5D

1. $i^{2018} = i^{(504\times 4)+2}$

$= i^2 = -1$

2. $x+iy = \dfrac{a+ib}{c+id}$

$x-iy = \dfrac{a-ib}{c-id}$

$(x+iy)(x-iy)$

$= \dfrac{a+ib}{c+id} \times \dfrac{a-ib}{c-id}$

$x^2+y^2 = \dfrac{a^2+b^2}{c^2+d^2}$

3. $\dfrac{1+i}{1-i} = \dfrac{1+i}{1-i} \times \dfrac{1+i}{1+i}$

$= \dfrac{(1+i)^2}{2}$

$= \dfrac{1+2i-1}{2} = i$

4. $z = 2+i$, $w = 1-i$

$\overline{w} = 1+i$

$z\overline{w} = (2+i)(1+i)$

$= 2+2i+i+i^2$

$= 2+3i-1$

$= 1+3i$

5. $z = 5+6i$

$3i - z = 3i-5-6i$

$= -5-3i$

$\text{Im}(3i-z) = -3$

Question 6 (14 marks)

(a)(i) $z = \frac{-1+i}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i}$

$= \frac{-\sqrt{3}+i+i\sqrt{3}-i^2}{3-i^2}$

$= \frac{-\sqrt{3}+i+i\sqrt{3}+1}{4}$

$= \frac{1-\sqrt{3}}{4} + \frac{i(1+\sqrt{3})}{4}$ (2)

(ii) $-1+i$

$|z| = \sqrt{1^2+1^2} = \sqrt{2}$

$\tan\alpha = 1, \alpha = \frac{\pi}{4}$

$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$-1+i = \sqrt{2}\ \text{cis}\ \frac{3\pi}{4}$

$z = \sqrt{3}+i$

$|z| = \sqrt{3+1} = 2$

$\tan\alpha = \frac{1}{\sqrt{3}} \quad \alpha = \frac{\pi}{6}$

$\theta = \alpha = \frac{\pi}{6}$

$\sqrt{3}+i = 2\ \text{cis}\ \frac{\pi}{6}$

$\frac{-1+i}{\sqrt{3}+i} = \frac{\sqrt{2}}{2}\ \text{cis}\left(\frac{3\pi}{4} - \frac{\pi}{6}\right)$

$= \frac{1}{\sqrt{2}}\ \text{cis}\left(\frac{7\pi}{12}\right)$

(3)

(iii)

$\frac{1-\sqrt{3}}{4} + \frac{i(1+\sqrt{3})}{4} = \frac{1}{\sqrt{2}}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$

Equating real and imaginary parts

$\frac{1}{\sqrt{2}}\cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{4}$

$\frac{1}{\sqrt{2}}\sin\frac{7\pi}{12} = \frac{1+\sqrt{3}}{4}$

$\sin\frac{7\pi}{12} = \frac{\sqrt{2}(1+\sqrt{3})}{4}$

$\cos\frac{7\pi}{12} = \frac{\sqrt{2}(1-\sqrt{3})}{4}$ (3)

$\tan\frac{7\pi}{12} = \frac{\sqrt{2}(1+\sqrt{3})}{4} \times \frac{4}{\sqrt{2}(1-\sqrt{3})}$

$= \frac{1+\sqrt{3}}{1-\sqrt{3}}$

(b)(i) $(x+iy)^2 = -5-12i$

$x^2 + 2ixy - y^2 = -5-12i$

$x^2 - y^2 = -5$ —— (1)

$2xy = -12$

$xy = -6$ —— (2)

(2) $\Longrightarrow y = \frac{-6}{x}$

substitute in (1)

$$\frac{x^2-36}{x^2} = -5$$

$x^4-36 = -5x^2$

$x^4+5x^2-36=0$

Let $m=x^2$

$m^2+5m-36=0$

$(m+9)(m-4)=0$

$m=-9$ or $m=4$

$x^2=-9$ or $x^2=4$

$x=\pm 2$ ($\because x$ is real)

$x=2,\ y=\frac{-6}{2}=-3$

$x=-2,\ y=\frac{-6}{-2}=3$

$\therefore$ the roots are

$2-3i,\ -2+3i$ (3)

$=\pm(2-3i)$

(ii) $z^2-4z+(9+12i)=0$

$$z=\frac{4\pm\sqrt{16-4(9+12i)}}{2}$$

$$=\frac{4\pm\sqrt{-20-48i}}{2}$$

$$=\frac{4\pm\sqrt{4(-5-12i)}}{2}$$

$$=\frac{4\pm 2\sqrt{-5-12i}}{2}$$

$$=\frac{2(2\pm\sqrt{-5-12i})}{2}$$ page 3

$=2\pm\sqrt{-5-12i}$

$=2\pm 2-3i$

$=2+2-3i$ or $2-(2-3i)$

$=4-3i$ or $2-2+3i$

$=4-3i$ or $3i$ (3)

Question 7 (21 marks)

(a)(i) $(\cos\theta+i\sin\theta)^5=\cos5\theta+i\sin5\theta$ (1)

by De Moivre's Theorem

Using binomial theorem,

$(\cos\theta+i\sin\theta)^5$

$=\cos^5\theta+{}^5C_1\cos^4\theta\times i\sin\theta$

$+{}^5C_2\cos^3\theta\,(i\sin\theta)^2$

$+{}^5C_3\cos^2\theta\times i^3\sin^3\theta$

$+{}^5C_4\cos\theta\, i^4\sin^4\theta+i^5\sin^5\theta$

$=\cos^5\theta+5i\cos^4\theta\sin\theta$

$-10\cos^3\theta\sin^2\theta-10i\cos^2\theta\sin^3\theta$

$+5\cos\theta\sin^4\theta+i\sin^5\theta$ (2)

Equating real and imaginary parts of (1) and (2)

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ (4)

(ii) $\tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta}$

$= \dfrac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$

Divide numerator and denominator by $\cos^5\theta$

$$\frac{5\dfrac{\sin\theta}{\cos\theta} - 10\dfrac{\sin^3\theta}{\cos^3\theta} + \dfrac{\sin^5\theta}{\cos^5\theta}}{1 - 10\dfrac{\sin^2\theta}{\cos^2\theta} + 5\dfrac{\sin^4\theta}{\cos^4\theta}}$$ (3)

$$= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

(b)(i) $z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta$ by De Moivre's theorem — (1)

$\frac{1}{z^n} = z^{-n} = (\cos\theta + i\sin\theta)^{-n}$

$= \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$ — (2) $\left(\because \begin{array}{l} \sin(-\theta) = -\sin\theta \\ \cos(-\theta) = \cos\theta \end{array}\right)$

Adding (1) and (2)

$z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

① − ②

$$z^n - \frac{1}{z^n} = \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$$

$$= 2i\sin n\theta$$

4

(ii) $(2\cos\theta)^7 = \left(z + \frac{1}{z}\right)^7$

$$= z^7 + {}^7C_1 z^6 \times \frac{1}{z} + {}^7C_2 z^5 \times \frac{1}{z^2} + {}^7C_3 z^4 \times \frac{1}{z^3} + {}^7C_4 z^3 \times \frac{1}{z^4}$$

$$+ {}^7C_5 z^2 \times \frac{1}{z^5} + {}^7C_6 z \times \frac{1}{z^6} + {}^7C_7 \frac{1}{z^7}$$

$$= z^7 + 7z^5 + 21z^3 + 35z + 35 \times \frac{1}{z} + 21 \times \frac{1}{z^3} + 7 \times \frac{1}{z^5} + \frac{1}{z^7}$$

$$= \left(z^7 + \frac{1}{z^7}\right) + 7\left(z^5 + \frac{1}{z^5}\right) + 21\left(z^3 + \frac{1}{z^3}\right) + 35\left(z + \frac{1}{z}\right)$$

$$= 2\cos 7\theta + 7 \times 2\cos 5\theta + 21 \times 2\cos 3\theta + 35 \times 2\cos\theta$$

$$= 2\cos 7\theta + 14\cos 5\theta + 42\cos 3\theta + 70\cos\theta$$

$$2^7\cos^7\theta = 2\cos 7\theta + 14\cos 5\theta + 42\cos 3\theta + 70\cos\theta$$

$$2^6\cos^7\theta = \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta$$

$$\cos^7\theta = \frac{1}{64}\cos 7\theta + \frac{7}{64}\cos 5\theta + \frac{21}{64}\cos 3\theta + \frac{35}{64}\cos\theta$$

$a = \frac{1}{64}$, $b = \frac{7}{64}$, $c = \frac{21}{64}$, $d = \frac{35}{64}$ 4

(c) (i) $z^3 - 1 = 0$

$$1 + \omega + \omega^2 = -\frac{\text{coefficient of } z^2}{\text{coefficient of } z^3} = -\frac{0}{1} = 0$$

2

(ii) sum of roots $1 + 1 + \omega + 1 + \omega^2 = 2$

Sum of roots taken two at a time

$1+\omega+1+\omega^2+(1+\omega)(1+\omega^2)$

$= 1+\omega+1+\omega^2+1+\omega^2+\omega+\omega^3.$

$= \cancel{1}+\cancel{\omega}+1+\cancel{\omega^2}+\cancel{1}+\cancel{\omega^2}+\cancel{\omega}+1 = 2$

Product of the roots

$(1+\omega)(1+\omega^2) = 1+\omega^2+\omega+\omega^3$ (4)

$= 1+\omega^2+\omega+1 = 1$

$\therefore$ the equation is $z^3-2z^2+2z-1=0$

or

$$\underline{\underline{x^3-2x^2+2x-1=0}}$$

Question 8 (24 marks)

(a) (i) $z^5+1=0$

$z^5=-1=\cos\pi+i\sin\pi$

$=\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)\quad k=0,1,2\ldots$

The five fifth roots of -1 are given by

$$z=\left[\cos(2k+1)\pi+i\sin(2k+1)\pi\right]^{\frac{1}{5}}$$

$$=\cos\frac{(2k+1)\pi}{5}+i\sin\frac{(2k+1)\pi}{5}\quad k=0,1,2,3,4$$

by De Moivre's theorem

When $k=0$

$z_1=\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$

$k=1$

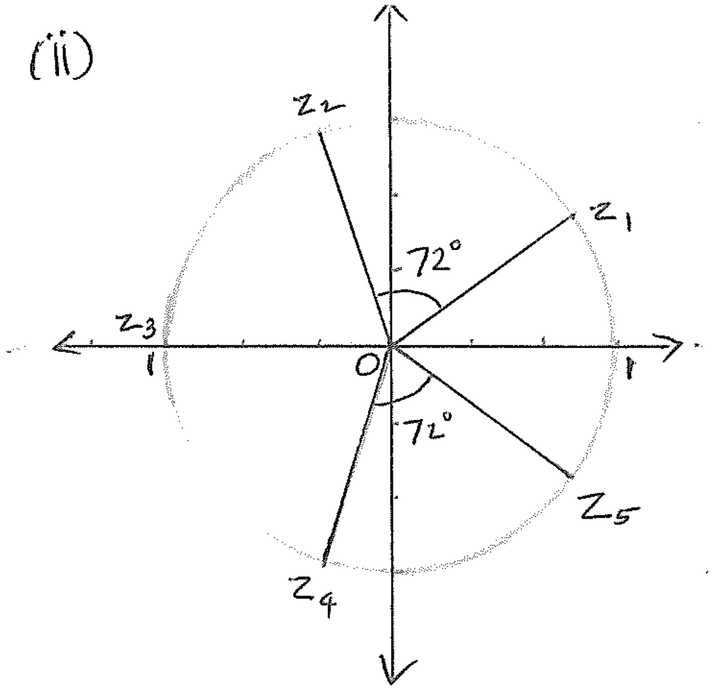
$z_2=\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}$

$k=2,\ z_3=\operatorname{cis}\pi=-1$

$k=3,\ z_4=\operatorname{cis}\frac{7\pi}{5}$ (3)

$k=4,\ z_5=\operatorname{cis}\frac{9\pi}{5}$

(ii)



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(iii) From the above figure

$$z_5 = \overline{z_1} \quad \text{and} \quad z_4 = \overline{z_2}$$

$$z_1 + z_5 = z_1 + \overline{z_1} = 2\cos\frac{\pi}{5}$$

$$z_2 + z_4 = z_2 + \overline{z_2} = 2\cos\frac{3\pi}{5}$$

$$z_1 z_5 = z_1\overline{z_1} = |z_1|^2 = 1$$

$$z_2 z_4 = z_2\overline{z_2} = |z_2|^2 = 1$$

$$z^5 + 1 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)$$

$$= (z+1)(z - z_1)(z - z_5)(z - z_2)(z - z_4)$$

$$= (z+1)\left(z^2 - z(z_1 + z_5) + z_1 z_5\right)\left(z^2 - z(z_2 + z_4) + z_2 z_4\right)$$

$$= (z+1)\left(z^2 - 2\cos\frac{\pi}{5}\, z + 1\right)\left(z^2 - 2\cos\frac{3\pi}{5}\, z + 1\right) \quad \text{——} ①$$

$$z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1) \quad \text{——} ②$$

From ① and ② we get

$$Z^4 - Z^3 + Z^2 - Z + 1$$

$$= \left(z^2 - 2\cos\frac{\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{3\pi}{5}z + 1\right)$$

4

(iv) $z^4 - z^3 + z^2 - z + 1 = z^4 - 2\cos\frac{3\pi}{5}z^3 + z^2 - 2\cos\frac{\pi}{5}z^3$

$$+ 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}z^2 - 2\cos\frac{\pi}{5}z + z^2 - 2\cos\frac{3\pi}{5}z + 1$$

Equating coefficients of z^2

$$1 = 1 + 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5} + 1$$

$$1 = 2 + 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$$

$$-1 = 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$$

$$\cos\frac{\pi}{5}\cos\frac{3\pi}{5} = -\frac{1}{4}$$

Equating Coefficients of z^3

$$-1 = -2\cos\frac{3\pi}{5} - 2\cos\frac{\pi}{5}$$

$$-1 = -2\left(\cos\frac{3\pi}{5} + \cos\frac{\pi}{5}\right)$$

4

$$\cos\frac{3\pi}{5} + \cos\frac{\pi}{5} = \frac{1}{2}$$

(v) The quadratic equation with roots $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ is given by $x^2 - \frac{1}{2}x - \frac{1}{4} = 0$

3

$$4x^2 - 2x - 1 = 0$$

(vi) $4x^2 - 2x - 1 = 0$

$$x = \frac{2 \pm \sqrt{4 - 4 \times 4 \times -1}}{8}$$

$$= \frac{2 \pm 2\sqrt{5}}{8} = \frac{1 \pm \sqrt{5}}{4}$$

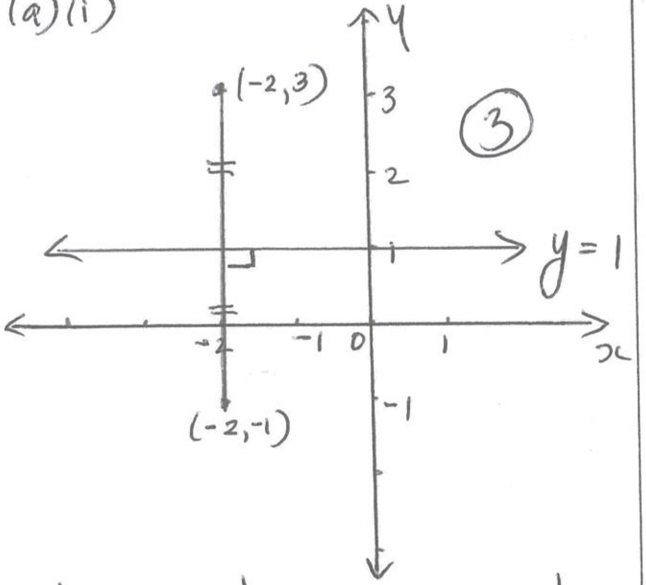
$\cos \frac{\pi}{5}$ is positive.

$\cos \frac{3\pi}{5}$ is negative

$\therefore \cos \frac{\pi}{5} = \underline{\underline{\frac{1+\sqrt{5}}{4}}}$ and $\cos \frac{3\pi}{5} = \underline{\underline{\frac{1-\sqrt{5}}{4}}}$

Question 9 (21 marks)

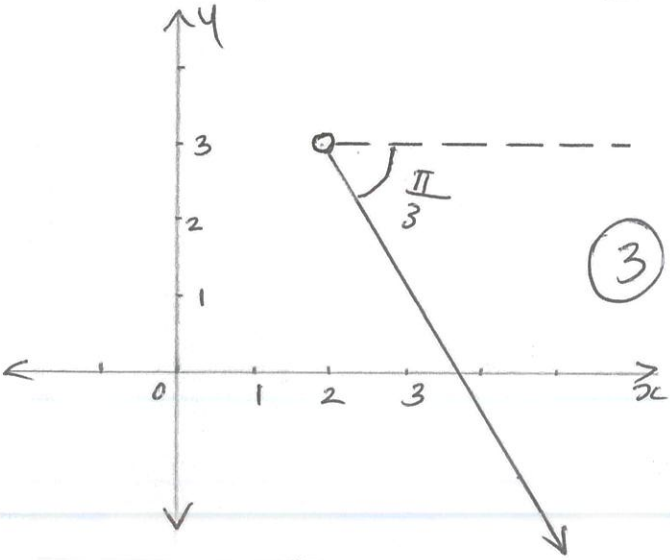
(a)(i)



3

$$|z+2-3i| = |z+2+i|$$

(ii) $\arg\left(z-(2+3i)\right) = -\frac{\pi}{3}$



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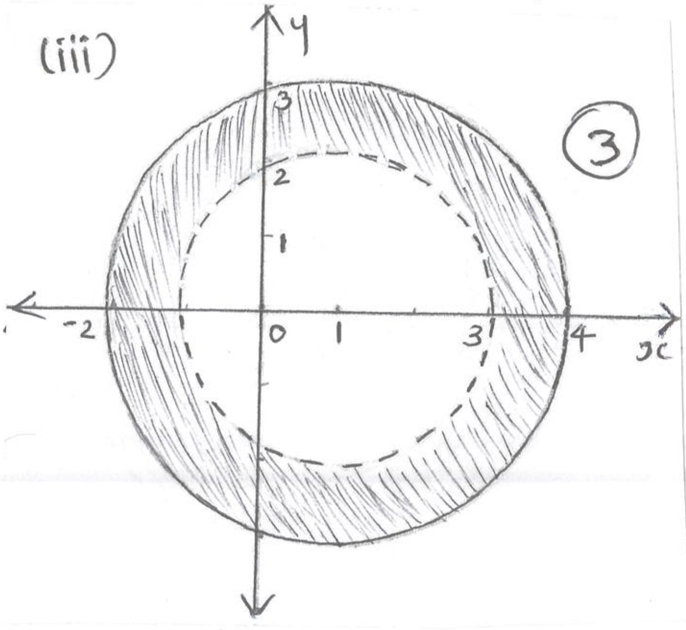
(iii)



3

(b) $|z-4i| = \text{Im}(z)$

$$|x+iy-4i| = y$$

$$|x+i(y-4)| = y$$

$$\sqrt{x^2+(y-4)^2} = y$$

$$x^2+(y-4)^2 = y^2$$

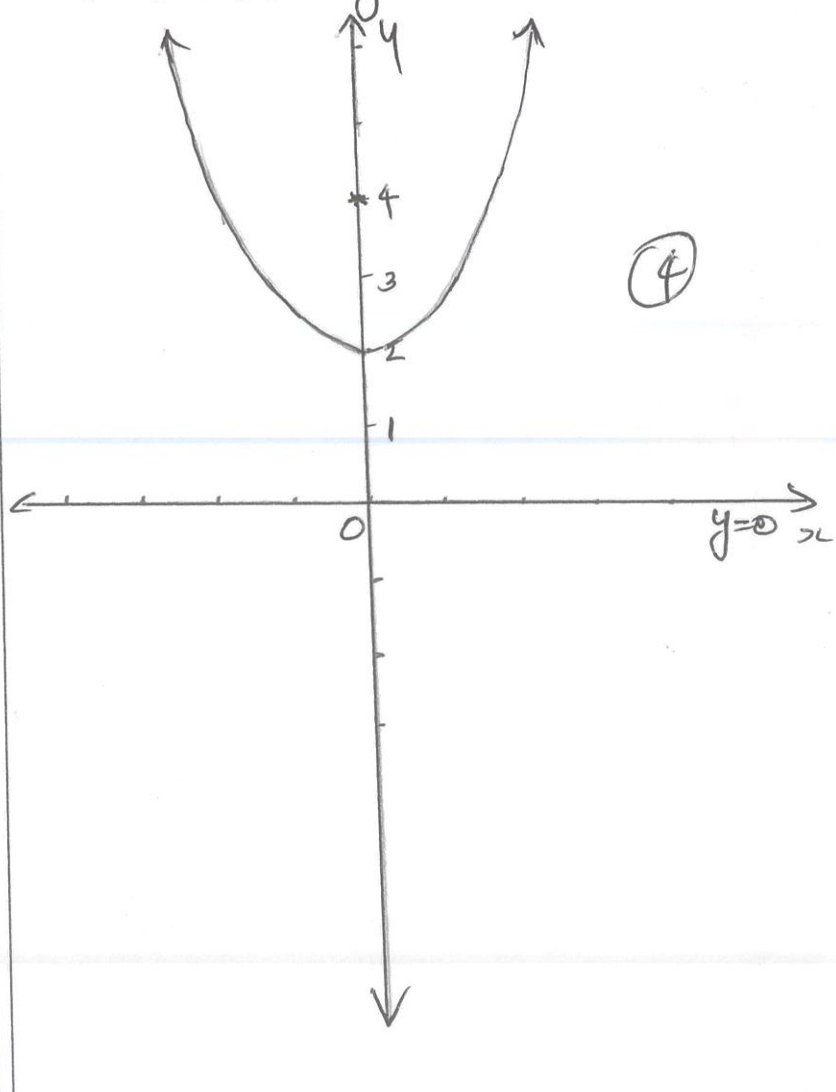
$$x^2+y^2-8y+16 = y^2$$

$$x^2 = 8y-16$$

$$= 8(y-2)$$

Vertex $(0,2)$ Focus $(0,4)$

Directrix $y=0$



4

(ii) $\left|z^2-(\bar{z})^2\right| \geq 12$

$\left|(x+iy)^2-(x-iy)^2\right| \geq 12$

$\left|(x+iy+x-iy)(x+iy-x+iy)\right| \geq 12$

$\left|(2x)(2iy)\right| \geq 12$

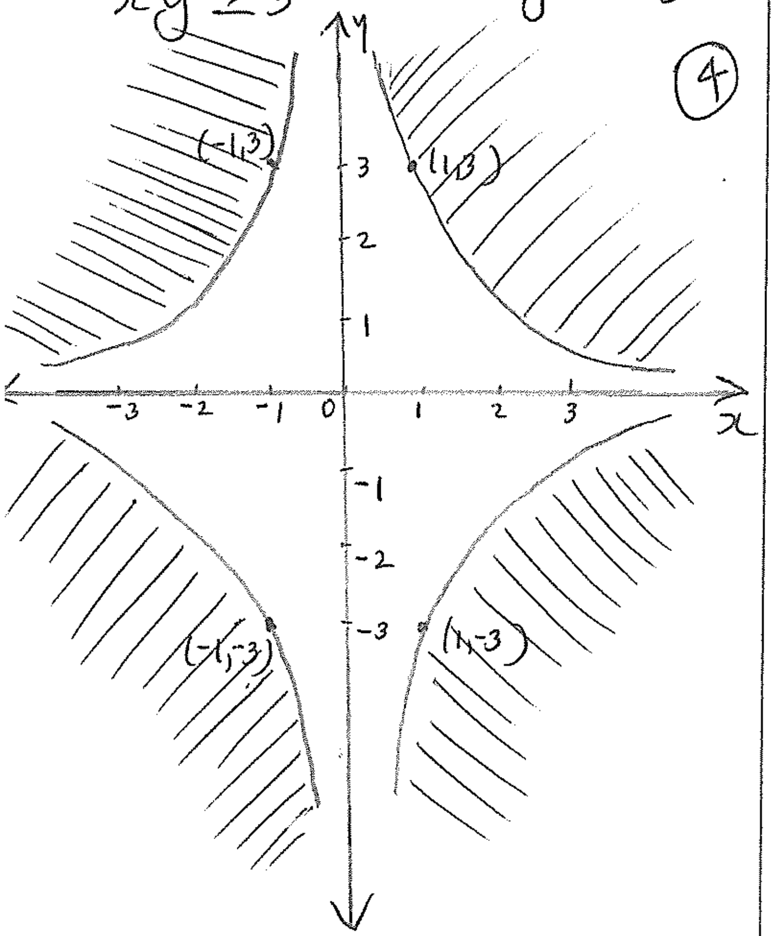
$\left|4ixy\right| \geq 12$

$4|xy| \geq 12$

$|xy| \geq 3$

$xy \geq 3$ or $-xy \geq 3$

$xy \geq 3$ or $xy \leq -3$



4

(iii) Let $z=x+iy$

$$\frac{z-4}{z-2i} = \frac{x+iy-4}{x+iy-2i}$$

$$= \frac{x-4+iy}{x+i(y-2)} \times \frac{x-i(y-2)}{x-i(y-2)}$$

$$= \frac{x(x-4)-i(x-4)(y-2)+ixy-i^2y(y-2)}{x^2-i^2(y-2)^2}$$

$$= \frac{x(x-4)+y(y-2)+ixy-i(x-4)(y-2)}{x^2+(y-2)^2}$$

$$Re\left(\frac{z-4}{z-2i}\right) = \frac{x(x-4)+y(y-2)}{x^2+(y-2)^2} = 0$$

$x(x-4)+y(y-2)=0$

$x^2-4x+y^2-2y=0$

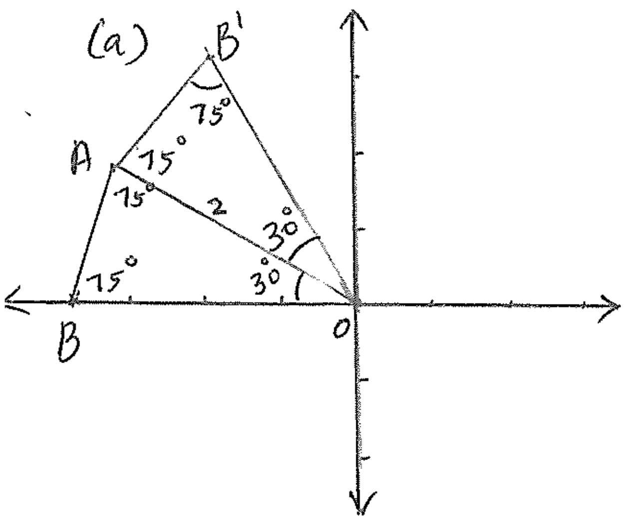
$x^2-4x+4+y^2-2y+1=0$

$$\underline{\underline{(x-2)^2+(y-1)^2=5}}$$

4

Question 10 (8 marks)

(a)



$Z = \sqrt{3} + i$

$|z| = \sqrt{3+1} = 2$

$\tan\alpha = \frac{1}{\sqrt{3}} \quad \alpha = \frac{\pi}{6}$

$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\arg z = \frac{5\pi}{6}$

$\angle BOA = 30°$ and

$\angle AOB' = 30$

$\overrightarrow{OB} = -2$ (4)

$\overrightarrow{OB'} = 2\,\text{cis}\,\frac{2\pi}{3}$

$= 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= 2\left(-\frac{1}{2} + i \times \frac{\sqrt{3}}{2}\right)$

$= -1 + i\sqrt{3}$

$B = -2 + 0i$

$B' = -1 + i\sqrt{3}$

(b)

(i) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 4 + 13i - 9 - i$

$= -5 + 12i$ (2)

(ii) $\overrightarrow{AD} = i(-5 + 12i)$

$= -5i + 12i^2 = -12 - 5i$

$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$

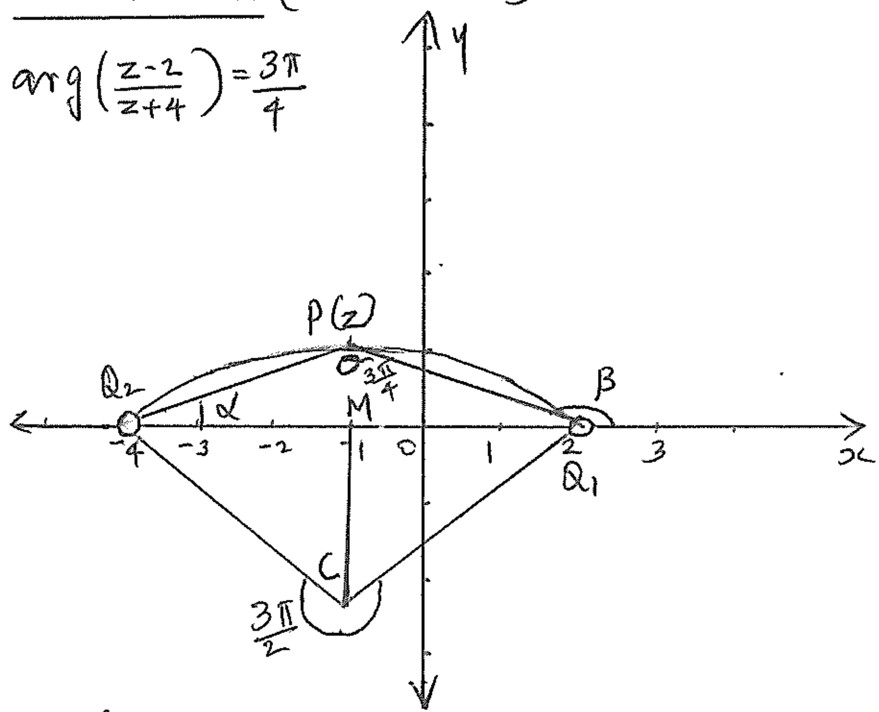
$= (9 + i) + (-12 - 5i)$

$= -3 - 4i$ (2)

$D = (-3 - 4i)$

Question 11 (20 marks)

$\arg\left(\frac{z-2}{z+4}\right) = \frac{3\pi}{4}$



$\arg(z-2) = \angle PQ_1X = \beta$

$\arg(z+4) = \angle PQ_2X = \alpha$

$\alpha + \theta = \beta$ (exterior angle of ΔQ_2PQ_1)

$\theta = \beta - \alpha$

$= \arg(z-2) - \arg(z+4)$

$= \arg\left(\frac{z-2}{z+4}\right) = \frac{3\pi}{4}$ (4)

(ii) Let C be the centre of the circle. Draw $CM \perp Q_1Q_2$

M is the mid point of Q_1Q_2 since the perpendicular from the centre of a circle to a chord bisects the chord. $M = (-1, 0)$

Reflex $\angle Q_2CQ_1 = \frac{3\pi}{2}$ (angle at the centre is twice angle at the circumference standing on the same arc)

ΔQ_2CQ_1 is isosceles ($Q_2C = Q_1C$ = radii)

$\angle Q_2CQ_1 = 2\pi - \frac{3\pi}{2} = \frac{\pi}{2}$ (angles at a point)

$\angle CQ_2Q_1 = \frac{\pi - \frac{\pi}{2}}{2}$ (angles opposite equal sides in an isosceles Δ, angle sum of Δ)

$= \frac{\pi}{4}$

(7)

In ΔQ_2CM

$\tan\frac{\pi}{4} = \frac{CM}{Q_2M} = \frac{CM}{3}$

$\frac{CM}{3} = 1$; $CM = 3$

Centre $(-1, -3)$

Apply Pythagoras' theorem in ΔQ_2CM

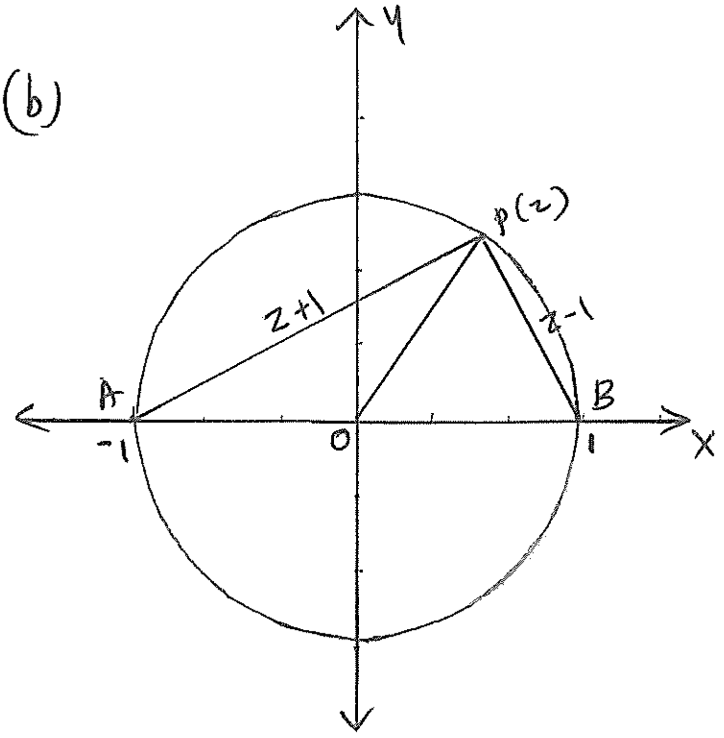
$Q_2C = \sqrt{9+9} = \sqrt{18}$
$= 3\sqrt{2}$

locus is the minor arc of the circle

$(x+1)^2 + (y+\ )^2 = 18, y > 0$

excluding the points $(-4, 0)$ and $(2, 0)$

(b)



(i) $\angle PAO + \angle APO = \angle POB$ (exterior $\angle$ of ΔAOP)

$\arg(z+1) + \arg(z+1) = \arg z$ ($\because$ ΔAOP is isosceles $OA = OP = 1$, angles opposite equal sides in isosceles Δ)

$2\arg(z+1) = \arg z$

$\arg(z+1) = \frac{1}{2}\arg z$

3

(ii) $\angle APB = 90°$ (angle in a semicircle)

$\angle PAB + \angle APB = \angle PBX$

$\arg(z+1) + \frac{\pi}{2} = \arg(z-1)$

3

$\underline{\underline{\arg(z-1) = \arg(z+1) + \frac{\pi}{2}}}$

(iii) $\tan \angle PAB = \dfrac{PB}{PA} = \dfrac{|z-1|}{|z+1|} = \left|\dfrac{z-1}{z+1}\right|$

$\tan(\arg(z+1)) = \left|\dfrac{z-1}{z+1}\right| = \tan\left(\frac{1}{2}\arg z\right)$

3

From (i)