GIRRAWEEN HIGH SCHOOL

TASK 1 2019 (December 2018)

MATHEMATICS

EXTENSION 2

Complex Numbers

Time allowed – 90 Minutes

DIRECTIONS TO CANDIDATES

- Attempt all questions.
- For Questions 1 -5, shade the circle for the letter corresponding to the correct answer on your answer sheet.
- For Questions 6-11, start each question on a new page. Each question should be clearly labelled.
- All necessary working must be shown for Questions 11– 15.
- Marks may be deducted for careless or badly arranged work.
- Board approved calculators may be used.
- A Mathematics reference sheet is provided.
- All diagrams are NOT TO SCALE.
- The use of the notation $cis\Theta$ is acceptable in this examination.

Question 1

$(-i)^{2019} =$

(A) i **(B)** $-i$ **(C)** 1 **(D)** -1

Question 2

$\frac{\bar{z}}{|z|} =$

(A) $\frac{1}{z}$ **(B)** $\frac{|z|}{z}$ **(C)** $\frac{z}{\bar{z}}$ **(D)** $\frac{1}{z|z|}$

Examination continues on the following page

Question 3

If $w = 2 + i$, $\frac{1}{w^2} =$

(A) $\frac{3+4i}{25}$ (B) $\frac{3-4i}{25}$ (C) $\frac{3+4i}{5}$ (D) $\frac{3-4i}{5}$

Question 4

Multiplying by a complex number rotates the result $\frac{\pi}{6}$ CLOCKWISE about the origin. The complex number I multiplied by could be

(A) $\frac{1-i\sqrt{3}}{2}$ (B) $\frac{\sqrt{3}-i}{2}$ (C) $\frac{1+i\sqrt{3}}{2}$ (D) $\frac{\sqrt{3}+i}{2}$

Question 5

When expressed in modulus/argument form to the nearest degree, $-4 - i =$

(A) $\sqrt{17}\ (cos14^o + isin14^o)$ (B) $\sqrt{17}\ (cos166^o + isin166^o)$

(C) $\sqrt{17}\ (cos{-166^o} + isin{-166^o})$(D) $\sqrt{17}\ (cos{-14^o} + isin{-14^o})$

Question 6 ***(16 marks – show all workings in your answer booklet)*** **Marks**

(a) (i) By realising the denominator, find the value $\frac{\sqrt{3}-i}{1+i}$ in Cartesian form. 2

(ii) By converting both $\sqrt{3} - i$ and $1 + i$ to modulus/ argument form, express the value of $\frac{\sqrt{3}-i}{1+i}$ in modulus/argument form. 5

(iii) Hence use your answers to (i) and (ii) to find the exact value of $cos\frac{5\pi}{12}$ 1

(b) (i) By letting $(x + iy)^2 = 24 - 10i\ x, y\ real$, find $\sqrt{24 - 10i}$ in Cartesian form. 3

(ii) Hence solve $z^2 + (1 + i)z - 6 + 3i = 0$. 2

(c) Find all five fifth roots of $16 - 16i\sqrt{3}$. Leave your answers in modulus/argument form. 3

Examination continues on the following page

Question 7 ***(8 marks – show all workings in your answer booklet)*** **Marks**

(a) Sketch these loci on separate Argand diagrams if $z = x + iy$

(i) $|z + 3 - 4i| = 5$ 2

(ii) $Arg\ (z + 3 - i) = \frac{2\pi}{3}$. 2

(b) Shade the region in the complex plane where 4

$|z - 1 - i| \leq 5$ and $\frac{\pi}{4} < Arg z \leq \frac{\pi}{2}$

Question 8 ***(9 marks – show all workings in your answer booklet)***

If $z = cos\theta + isin\theta$

(i) Use DeMoivre's theorem to show that $z^n - z^{-n} = 2isin\, n\theta$. 2

(ii) Hence express $sin^5\theta$ in the form $Asin5\theta + Bsin3\theta + Csin\theta$. 3

(iii) By substituting $\theta = \frac{\pi}{5}$ and using $sin3\theta = 3sin\theta - 4sin^3\theta$ show 4

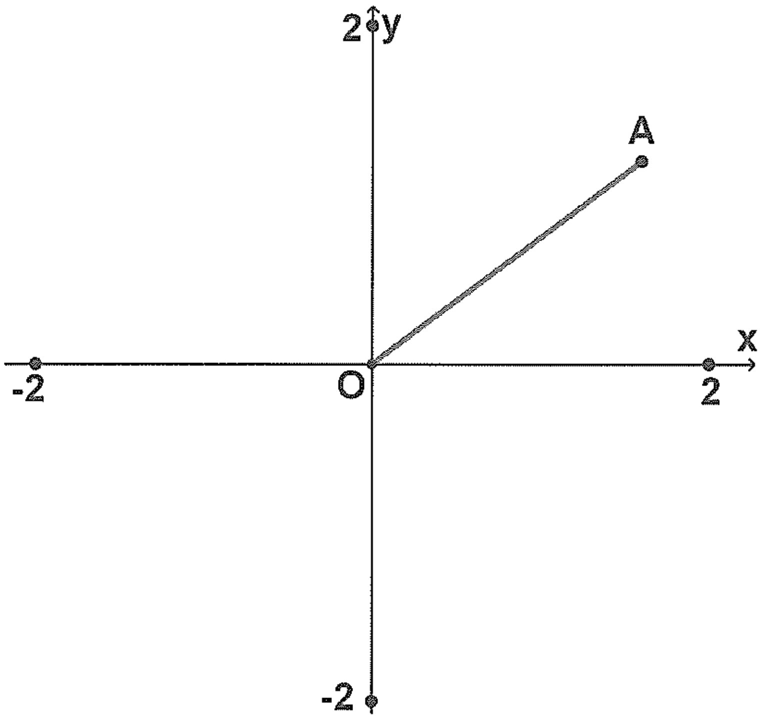
that $\sin\frac{\pi}{5}$ is a root of the equation $16x^4 - 20x^2 + 5 = 0$ and find

the exact value of $\sin\frac{\pi}{5}$. *(You may leave your answer under a square root sign).*

Examination continues on the following page

Question 9 ***(10 Marks – show all workings in your answer booklet)*** **Marks**

(a) In the diagram below, $\overrightarrow{OA} = z$ where $|z| = 2$ and $0 < Arg\ z < \frac{\pi}{2}$.



Copy the diagram in to your answer booklet and draw on it the complex number

(i) $\overrightarrow{OB}$ so that $\overrightarrow{OB} = \bar{z}$ **1**

(ii) $\overrightarrow{OC}$ so that $\overrightarrow{OC} = iz$ **1**

(iii) $\overrightarrow{OD}$ so that $\overrightarrow{OD} = z(\cos(-\frac{\pi}{3}) + isin(-\frac{\pi}{3}))$ **1**

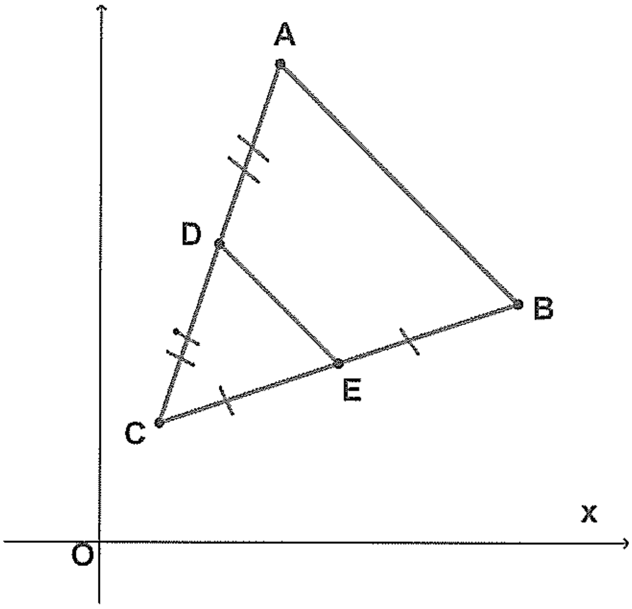
(iv) $\overrightarrow{OE}$ so that $\overrightarrow{OE} = z + 2$ **1**

(v) $\overrightarrow{OF}$ so that $\overrightarrow{OF} = z + i$ **1**

Question 9 continues on the following page

Question 9 (continued) **Marks**

(b) In the Argand diagram below, A, B and C are the vertices of a triangle. D and E are the midpoints of AC and BC respectively.



If the complex number $\overrightarrow{OA} = a$, the complex number $\overrightarrow{OB} = b$ and $\overrightarrow{OC} = c$,

(i) Find expressions for $\overrightarrow{DC}$ and $\overrightarrow{EC}$ in terms of a, b and c. **2**

(ii) HENCE show that $\overrightarrow{AB} = 2 \times \overrightarrow{DE}$. **3**

Question 10 ***(10 Marks – show all workings in your answer booklet)***

(a) Find the locus of z in terms of x and y if $z = x + iy$ and

(i) $\frac{z-2i}{z+4}$ is entirely real. **3**

(ii) $\frac{|z-7-6i|}{|z+5+3i|} = 2$. **3**

(You do not need to draw diagrams for either question in Part (a))

(b) The locus $Arg\left\{\frac{z-7-2i}{z+1-2i}\right\} = \frac{\pi}{3}$ represents part of a circle. **4**

Draw this circle part on an Argand diagram and find its centre, radius and equation.

Examination continues on the following page

Question 11 ***(15 Marks – show all workings in your answer booklet)*** **Marks**

(a)(i) Solve $z^9 - 1 = 0$. **2**

If w is the root of $z^9 - 1 = 0$ with the smallest positive argument

(ii) Show that $w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 + w^8 = -1$ **1**

(You do NOT need to prove that w^2, w^3 etc. are the other non real roots of $z^9 - 1 = 0$)

(iii) Show that $w^8 = \frac{1}{w}$ **1**

(iv) Show that $w + w^2 + w^3 + w^4 + \frac{1}{w} + \frac{1}{w^2} + \frac{1}{w^3} + \frac{1}{w^4} = -1$ **1**

(v) Show that $cos\frac{2\pi}{9} + cos\frac{4\pi}{9} = cos\frac{\pi}{9}$ **2**

(You may assume that $cisn\Theta + cis(-n\Theta) = 2cosn\Theta$).

(b) (i) Resolve $z^9 - 1$ into real linear and quadratic factors. Hence show that **4**

$$z^6 + z^3 + 1 = (z^2 - 2zcos\frac{2\pi}{9} + 1)(z^2 - 2zcos\frac{4\pi}{9} + 1)(z^2 + 2zcos\frac{\pi}{9} + 1)$$

(ii) Hence show that **2**

$$2cos3\Theta + 1 = (2cos\Theta - 2cos\frac{2\pi}{9})(2cos\Theta - 2cos\frac{4\pi}{9})(2cos\Theta + 2cos\frac{\pi}{9})$$

(iii) Hence by an appropriate substitution for Θ, show that **2**

$cos\frac{\pi}{9}\ cos\frac{2\pi}{9} cos\frac{4\pi}{9} = \frac{1}{8}$.

END OF EXAMINATION!!!

Y12 Ext 2 Solutions p.1

Complex Numbers (Task 1)

December 2018 for 2019 HSC

Multiple Choice

Q.(1) A (2) B (3) B (4) B (5) C

(1) $(-i)^{2019} = (-i)^3$
$= i$ (A)

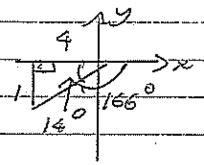
(2) $\frac{\bar{z}}{|z|^2} = \frac{1}{z}$
$\times|z| \quad \times|z|$ (B)

$\therefore \frac{\bar{z}}{|z|} = \frac{|z|}{z}$

(3) $\frac{1}{\omega^2} = \frac{1}{(2+i)^2}$
$= \frac{1}{3+4i} \times \frac{(3-4i)}{(3-4i)}$
$= \frac{3-4i}{25}$ (B)

(4) Arg $= \frac{\pi}{6}$
$= \frac{\sqrt{3}-i}{2}$ (B)

(5)



$\therefore -4-i = \sqrt{17}\,\text{cis}(-166°)$ (C)

(6)(a)(i) $\frac{\sqrt{3}-i}{1+i} \times \frac{(1-i)}{(1-i)}$
$= \frac{(\sqrt{3}-1) - i(1+\sqrt{3})}{2}$

(ii) $\sqrt{3}-i = 2\,\text{cis}\left(-\frac{\pi}{6}\right)$
$1+i = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

$\therefore \frac{\sqrt{3}-i}{1+i} = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{6} - \frac{\pi}{4}\right)$
$= \sqrt{2}\left(\cos -\frac{5\pi}{12} + i\sin -\frac{5\pi}{12}\right)$

(iii) Equating real parts of (i) & (ii)
$\frac{\sqrt{3}-1}{2} = \sqrt{2}\cos\left(-\frac{5\pi}{12}\right)$
$\frac{\sqrt{3}-1}{2\sqrt{2}} = \cos\left(-\frac{5\pi}{12}\right)$
$\therefore$ cos is EVEN, $\cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

(6)(b)(i) $(x+iy)^2 = 24-10i$, x, y real
$\therefore x^2 - y^2 + 2ixy = 24 - 10i$

Equating reals,	Equating imaginaries
$x^2 - y^2 = 24$ (1)	$2xy = -10$
	$y = -\frac{5}{x}$ (2)

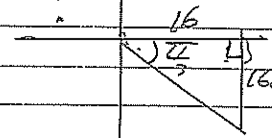
Sub (2) in (1):
$x^2 - \left(-\frac{5}{x}\right)^2 = 24$
$x^4 - 24x^2 - 25 = 0$
$(x^2-25)(x^2+1) = 0$
As x is real, $x = \pm 5$.
As $y = -\frac{5}{x}$, $y = \mp 1$.
$\therefore \pm\sqrt{24-10i} = \pm(5-i)$.

p.2

Q.(6)(b)(ii) $z^2 + (1+i)z - 6 + 3i = 0$

Using $z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
$z = \frac{(-1-i) \pm \sqrt{(1+i)^2 - 4\times1\times(-6+3i)}}{2\times1}$
$= \frac{(-1-i) \pm \sqrt{24-10i}}{2}$
$= \frac{-1-i+(5-i)}{2}$ or $\frac{-1-i-(5-i)}{2}$ (from (i))
$z = 2-i$ or $z = -3$

(c)

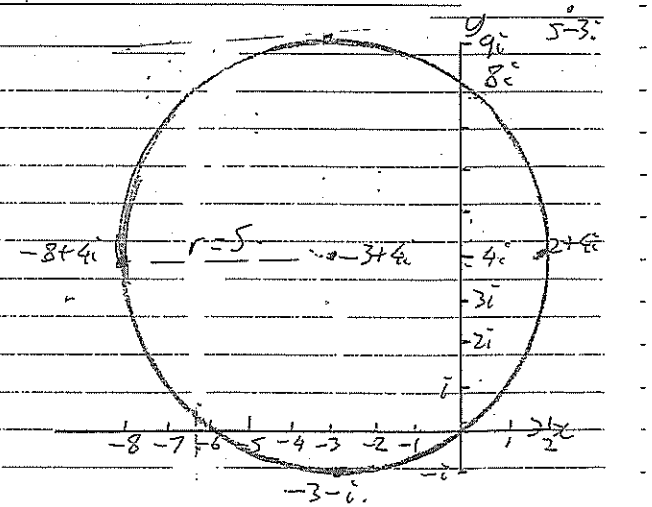


$16 - 16i\sqrt{3} = 32\,\text{cis}\left(\frac{-\pi}{3}\right)$

$\therefore$ The five 5th roots of $16-16i\sqrt{3}$ are
$2\,\text{cis}\left[\frac{\left(-\frac{\pi}{3}\right) + 2k\pi}{5}\right]$, $k = 0, 1, 2, 3, 4$
$= 2\,\text{cis}\,-\frac{\pi}{15},\ 2\,\text{cis}\,\frac{\pi}{3},\ 2\,\text{cis}\,\frac{11\pi}{15},\ 2\,\text{cis}\,\frac{17\pi}{15},\ 2\,\text{cis}\,\frac{23\pi}{15}$.

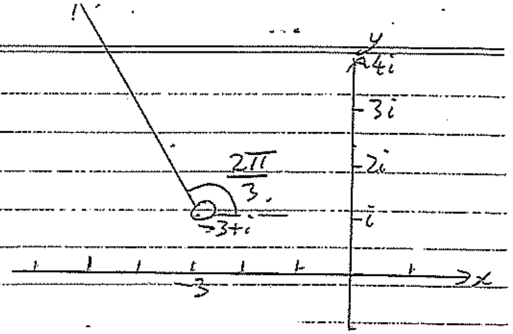
Q.(7)(a)(i)

$|z+3-4i| = 5$.
Circle centre $-3+4i$
$r = 5$.

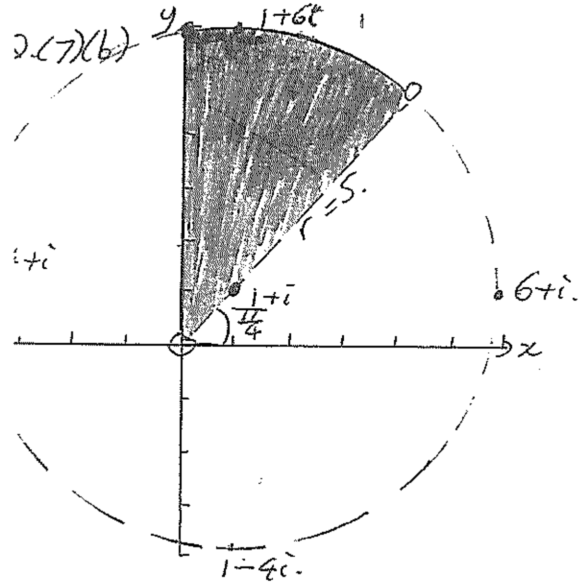


p. 3

Q.(7)(a)(ii) $\operatorname{Arg}(z+3-i)=\frac{2\pi}{3}$



Q.(7)(b)



(8)(i) If $z = \operatorname{cis}\theta$

By De Moivre, $z^n - z^{-n}$

$= \cos n\theta + i\sin n\theta - (\cos(-n\theta) + i\sin(-n\theta))$

$= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$ (as cos even, sin odd)

$= 2i\sin n\theta$.

(ii) Hence if $z = \operatorname{cis}\theta$

$(z - z^{-1})^5 = (2i\sin\theta)^5$

$= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$

$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right) = 32i\sin^5\theta$ $\div 32i$

$2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$

$\frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta = \sin^5\theta$.

p. 4

Q.(8) (iii)

$\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$

Sub. in $\theta = \frac{\pi}{5}$:

$\sin^5\left(\frac{\pi}{5}\right) = \frac{1}{16}\sin\pi - \frac{5}{16}\sin\frac{3\pi}{5} + \frac{5}{8}\sin\frac{\pi}{5}$

As $\sin\pi = 0$ & $\sin\frac{3\pi}{5} = 3\sin\frac{\pi}{5} - 4\sin^3\frac{\pi}{5}$,

$\sin^5\frac{\pi}{5} = -\frac{5}{16}\left(3\sin\frac{\pi}{5} - 4\sin^3\frac{\pi}{5}\right) + \frac{5}{8}\sin\frac{\pi}{5}$

$\times -16$ & $+16\sin^5\frac{\pi}{5}$

$0 = 16\sin^5\frac{\pi}{5} - 20\sin^3\frac{\pi}{5} + 5\sin\frac{\pi}{5}$

As $\sin\frac{\pi}{5} \neq 0$

$0 = 16\sin^4\frac{\pi}{5} - 20\sin^2\frac{\pi}{5} + 5$.

$\therefore \sin\frac{\pi}{5}$ is a root of $16x^4 - 20x^2 + 5 = 0$

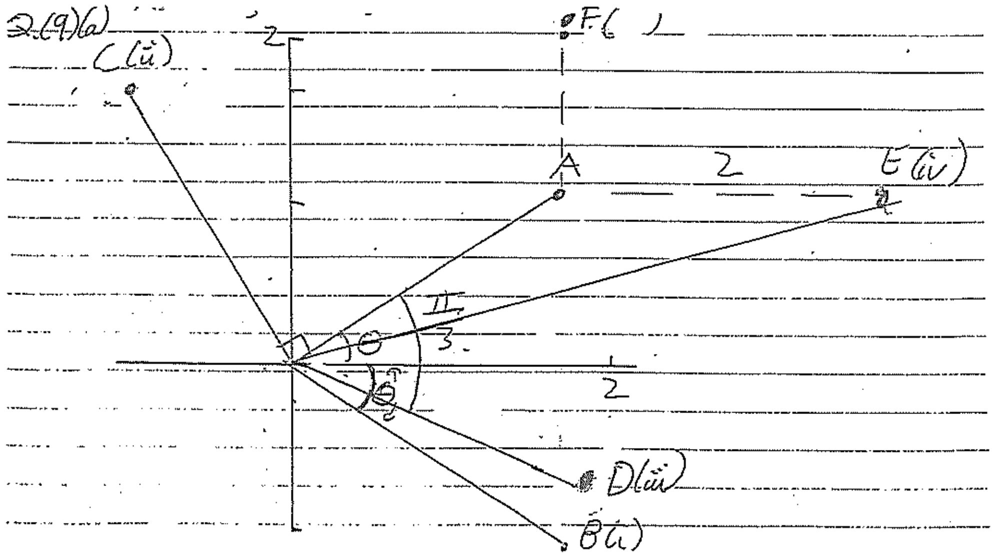
$\therefore \sin^2\frac{\pi}{5} = \frac{20 \pm \sqrt{(-20)^2 - 4\times 16\times 5}}{2\times 16}$

$\sin^2\frac{\pi}{5} = \frac{5 \pm \sqrt{5}}{8}$

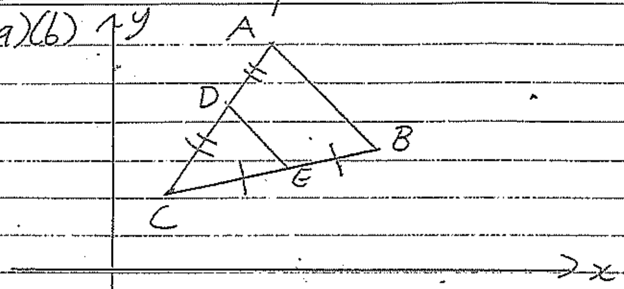
Noting $\frac{\pi}{5} < \frac{\pi}{4}$, $\sin\frac{\pi}{5} < \sin\frac{\pi}{4}$ i.e. $\sin\frac{\pi}{5} < \frac{1}{\sqrt{2}}$ $\therefore \sin^2\frac{\pi}{5} < \frac{1}{2}$.

$\therefore \sin^2\frac{\pi}{5} = \frac{5-\sqrt{5}}{8} \Rightarrow \sin\frac{\pi}{5} = \sqrt{\frac{5-\sqrt{5}}{8}}$

Q.(9)(a)



Q.(9)(b)



(i) $\overrightarrow{DC} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}(c-a)$ $\overrightarrow{EC} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}(c-b)$

(ii) $\overrightarrow{DE} = \overrightarrow{DC} + \overrightarrow{CE}$
$= \frac{1}{2}(c-a) + \frac{1}{2}(b-c)$
$= \frac{1}{2}(b-a).$

$\overrightarrow{AB} = b-a$
$= 2\times\overrightarrow{DE}.$

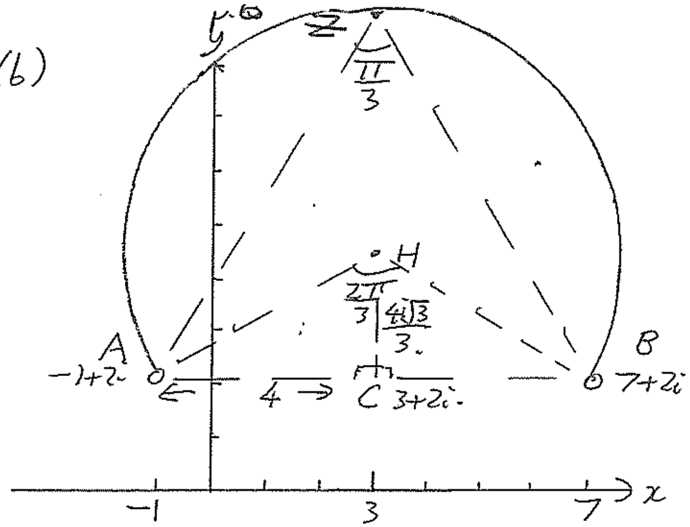
Q.(10)(a)(i) $\frac{z-2i}{z+4}$ is real.

$\frac{z-2i}{z+4} = \frac{x+(y-2)i}{(x+4)+iy} \times \frac{(x+4)-iy}{(x+4)-iy}$
$= \frac{x(x+4) - ixy + i(xy+4y-2x-8) + y(y-2)}{(x+4)^2+y^2}.$

If $\frac{z-2i}{z+4}$ is real then $xy+4y-2x-8-xy = 0$
$\times -\frac{1}{2}$
$x-2y+4 = 0$

(ii) $|z-7-6i| = 2|z+5+3i|$
If $z = x+iy$
$(x-7)^2+(y-6)^2 = 4[(x+5)^2+(y+3)^2]$
$x^2-14x+49+y^2-12y+36 = 4x^2+40x+100+4y^2+24y+36$
$0 = 3x^2+54x+3y^2+36y+51$
$0 = x^2+18x+y^2+12y+17$
$100 = (x+9)^2+(y+6)^2$

Q.(10)(b)



$\text{Arg}\left[\frac{(z-7-2i)}{(z+1-2i)}\right] = \frac{\pi}{3}.$

Letting $\overrightarrow{O\ldots} = z,$
$= A,\ 7\ldots$

$\angle AzB = \frac{\pi}{3}.$

Circle centre $= H.$

$\angle AHB = \frac{2\pi}{3}$ [∠ at centre = 2× ∠ at circumference].

From observation, x at circle centre $= 3$.

Letting midpoint $AB = C = 3+2i$.

$AH = BH$ (circle radii).

$\angle HAC = \frac{\pi}{6}$ [∠s opposite = sides in isosceles △AHC].

$HAC = 4\tan\frac{\pi}{6} = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$ [using trigonometry in △AHC].

$\therefore H = 3+\left(2+\frac{4\sqrt{3}}{3}\right)i.$

$AH = \frac{4}{\cos\frac{\pi}{6}} = \frac{8}{\sqrt{3}}.$

∴ Locus is circle part circle centre $3+\left(2+\frac{4\sqrt{3}}{3}\right)i$, $r = \frac{8}{\sqrt{3}}$ where $y > 2$.

∴ Equation is $(x-3)^2+\left[(y-2-\frac{4\sqrt{3}}{3}\right]^2 = \frac{64}{3}$, $y>2$.

Q(11)(a)(i) $z^9 - 1 = 0$

$z = \text{cis}\,\frac{2k\pi}{9},\ k = 0, 1, 2 \ldots 8.$

(ii) As $1, \omega, \omega^2, \ldots \omega^8$ are roots of $z^9 - 1 = 0$

By $\alpha + \beta + \ldots = -\frac{b}{a}$

$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = 0.$

$\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = -1.$

(iii) $\frac{1}{\omega} = \frac{\omega^8}{\omega^9} = \frac{\omega^8}{1} = \omega^8.$

(iv) Similarly, $\frac{1}{\omega^2} = \omega^7$, $\frac{1}{\omega^3} = \omega^6$ & $\frac{1}{\omega^4} = \omega^5$.

Hence $\omega + \omega^2 + \omega^3 + \omega^4 + \frac{1}{\omega} + \frac{1}{\omega^2} + \frac{1}{\omega^3} + \frac{1}{\omega^4}$

$= \omega + \omega^2 + \omega^3 + \omega^4 + \omega^8 + \omega^7 + \omega^6 + \omega^5$

$= -1$ (by (ii)).

(v) As $\omega + \frac{1}{\omega} + \omega^2 + \frac{1}{\omega^2} + \omega^3 + \frac{1}{\omega^3} + \omega^4 + \frac{1}{\omega^4} = -1$

Using $\omega + \frac{1}{\omega} = 2\cos\frac{2\pi}{9}$, $\omega^2 + \frac{1}{\omega^2} = 2\cos\frac{4\pi}{9}$ etc.

$2\cos\frac{2\pi}{9} + 2\cos\frac{4\pi}{9} + 2\cos\frac{2\pi}{3} + 2\cos\frac{8\pi}{9} = -1.$

Noting $\cos\frac{2\pi}{3} = -\frac{1}{2}$ & $\cos\frac{8\pi}{9} = -\cos\frac{\pi}{9}$

$2\cos\frac{2\pi}{9} + 2\cos\frac{4\pi}{9} - 1 - 2\cos\frac{\pi}{9} = -1.$

$+1 \quad \div 2 \quad + \cos\frac{\pi}{9}$

$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}.$

(b)(i) $z^9 - 1 = (z^3 - 1)(z^6 + z^3 + 1)$ (by $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$)

$= (z-1)(z - \text{cis}\frac{2\pi}{3})(z - \text{cis}\frac{4\pi}{3})(z^6 + z^3 + 1)$

Using $(z - \text{cis}\frac{2\pi}{3})(z - \text{cis}\frac{4\pi}{3}) = (z - \text{cis}\frac{2\pi}{3})(z - \text{cis}(-\frac{2\pi}{3}))$

$= (z^2 - z(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} + \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}) + 1) = z^2 - 2z\cos\frac{2\pi}{3} + 1$

$= z^2 + z + 1$

$z^9 - 1 = (z-1)(z^2+z+1)(z^6+z^3+1)$ (1)

$z^9 - 1$ also $= (z-1)(z - \text{cis}\frac{2\pi}{3})(z - \text{cis}\frac{4\pi}{3})(z - \text{cis}\frac{2\pi}{9})(z - \text{cis}\frac{16\pi}{9})$

$(z - \text{cis}\frac{4\pi}{9})(z - \text{cis}\frac{14\pi}{9})(z - \text{cis}\frac{8\pi}{9})(z - \text{cis}\frac{10\pi}{9})$

$= (z-1)(z^2+z+1)(z - \text{cis}\frac{2\pi}{9})(z - \text{cis}\frac{16\pi}{9})(z - \text{cis}\frac{4\pi}{9})(z - \text{cis}\frac{14\pi}{9})$

$(z - \text{cis}\frac{8\pi}{9})(z - \text{cis}\frac{10\pi}{9})$

(11)(b)(i)(continued):

Noting $(z - \text{cis}\frac{2\pi}{9})(z - \text{cis}\frac{16\pi}{9})$

$= (z - \text{cis}\frac{2\pi}{9})(z - \text{cis}(-\frac{2\pi}{9}))$

$= z^2 - z(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9} + \cos\frac{2\pi}{9} - i\sin\frac{2\pi}{9}) + 1$ (as cos even, sin odd)

$= z^2 - 2z\cos\frac{2\pi}{9} + 1$

(Note: You ARE allowed to assume in Q.11 that $\text{cis}\,n\theta + \text{cis}(-n\theta) = 2\cos n\theta$.

Similarly, $(z - \text{cis}\frac{4\pi}{9})(z - \text{cis}\frac{14\pi}{9}) = z^2 - 2z\cos\frac{4\pi}{9} + 1$

& $(z - \text{cis}\frac{8\pi}{9})(z - \text{cis}\frac{10\pi}{9}) = z^2 - 2z\cos\frac{8\pi}{9} + 1$

$= z^2 + 2z\cos\frac{\pi}{9} + 1.$

Hence $z^9 - 1$ also $=$

$(z-1)(z^2+z+1)(z^2 - 2z\cos\frac{2\pi}{9} + 1)(z^2 - 2z\cos\frac{4\pi}{9} + 1)(z^2 + 2z\cos\frac{\pi}{9} + 1)$ (2)

Equating (1) & (2) & dividing BS by $(z-1)(z^2+z+1)$

$z^6 + z^3 + 1 = (z^2 - 2z\cos\frac{2\pi}{9} + 1)(z^2 - 2z\cos\frac{4\pi}{9} + 1)(z^2 + 2z\cos\frac{\pi}{9} + 1)$

(ii) Hence, dividing (i) by z^3

$z^3 + 1 + \frac{1}{z^3} = (z + \frac{1}{z} - 2\cos\frac{2\pi}{9})(z + \frac{1}{z} - 2\cos\frac{4\pi}{9})(z + \frac{1}{z} + 2\cos\frac{\pi}{9})$

Letting $z = \text{cis}\,\theta$, $z^n + \frac{1}{z^n} = 2\cos n\theta$,

$2\cos 3\theta + 1 = (2\cos\theta - 2\cos\frac{2\pi}{9})(2\cos\theta - 2\cos\frac{4\pi}{9})(2\cos\theta + 2\cos\frac{\pi}{9})$

(iii) Hence letting $\theta = \frac{\pi}{2}$, so $\cos 3\theta = \cos\theta = 0$

$1 = -2\cos\frac{2\pi}{9} \times -2\cos\frac{4\pi}{9} \times 2\cos\frac{\pi}{9}$

$\div 8$

$\frac{1}{8} = \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$

(or $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$).

End of solutions!!