Mathematics
4 Unit Assessment 15/12/1998

Question 1.

(a)

If $A = 3 + 4i$ and $B = 5 - 12i$ write the following in the form $a + ib$.

(i) $A + B$. **(2 marks)**

(ii) AB

(b)

If $z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$ evaluate z^6. **(2 marks)**

(c)

Find the square root of $5 - 12i$. **(3 marks)**

(d)

Draw neat labelled sketches (not on graph paper) to indicate each of the subsets of the Argand diagram described below :

(i) $\text{Im}(z) = 4$

(ii) $|z - 4| \leq |z + 4i|$

(iii) $|z - 2 - 3i| = |z - i|$

(iv) $|z| \leq 1$

(iii) $|z - 3 + 4i| = 5$ **(16 marks)**

(iv) $\arg(z + i) = \frac{\pi}{4}$

Question 2.

(a)

Given that $z = \frac{1 - 7i}{3 + 4i}$, find

(i) $|z|$ **(5 marks)**

(ii) z in the form $a + ib$

(iii) $\bar{z}$

(b)

Find algebraically and describe in geometrical terms , the locus (in the Argand plane) represented by **(6 marks)**

(i) $2|z| = z + \bar{z} + 4$

(ii) $z\bar{z} = z + \bar{z}$

(c)

Find the modulus and argument of

(i) $1 - i$

(ii) $-\sqrt{3} - i$ **(8 marks)**

(iii) $(1 - i)(-\sqrt{3} - i)$

(iv) $\frac{1 - i}{-\sqrt{3} - i}$

(d)

Let z be the complex number so that $z = 1 + \sqrt{3}i$

(*i*) Express z in the mod / arg form **(5 marks)**

(*ii*) Write z^5 in the form $a + ib$.

Question 3.

(a)

Show that if $z = \cos\theta + i\sin\theta$ then **(4 marks)**

$z^n + z^{-n} = 2\cos n\theta$

(b)

Use De Moivre's Theorem to show that

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

Hence show that the solutions of the equation $8x^3 - 6x + \sqrt{3} = 0$. **(6 marks)**

are $x = \sin\frac{\pi}{9}, x = \sin\frac{2\pi}{9}$ and $x = \sin\frac{13\pi}{9}$

Show that $\sin\frac{\pi}{9} + \sin\frac{2\pi}{9} - \sin\frac{4\pi}{9} = 0$

(c)

Show that **(3 marks)**

$\frac{1}{2}(1 + i\sqrt{3})(\cos\theta + i\sin\theta) = \cos(\frac{\pi}{3} + \theta) + i\sin(\frac{\pi}{3} + \theta)$

(d)

What is the maximum value of $|z|$ for $|z - 1 - i| \le 2$ **(3 marks)**

Question 4.

(a)

(i) Find the fifth roots of unity.

(ii) Show these roots on an Argand diagram

(iii) Show that the complex roots occur in conjugate pairs. **(6 marks)**

(iv) If ω is one of the roots of $z^5 = 1$ state the value of

$\omega + \omega^2 + \omega^3 + \omega^4$

(b)

Show by means of an Argand diagram, that if z and w are complex numbers then **(3 marks)**

$|z + w| \le |z| + |w|$

(c)

On separate diagrams draw a neat sketch of the locus specified by each of the following . Include a discription to clarify your sketck where necessary.

(*i*) $0 \le Arg(z+1) \le \frac{\pi}{4}$

(*ii*) $\left|\frac{z+1}{z-1}\right| = 1$ **(6 marks)**

(*iii*) $Arg\left(\frac{z+1}{z-1}\right) = \frac{\pi}{4}$

(d)

Simplify **(3 marks)**

$$\frac{(\cos 5\theta - i\sin 5\theta)^2(\cos 3\theta + i\sin 3\theta)^8}{(\cos 2\theta + i\sin 2\theta)^5}.$$

4 Unit Assessment
15/12/98.

Q1 a)

$A = 3+4i, \; B = 5-12i$

$A+B = (3+4i)+(5-12i)$

$= \underline{8-8i}$

$AB = (3+4i)(5-12i)$

$= 15-36i+20i-48i^2$

$= 15+48-16i$

$= \underline{63-16i}$

b) $z^6 = \left[2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]^6$

$= 2^6\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^6$

$= 2^6\left(\cos\left(\frac{\pi}{6}\times 6\right)+i\sin\left(\frac{\pi}{6}\times 6\right)\right)$

$= 64(\cos\pi + i\sin\pi)$

$= 64(-1+i\times 0)$

$= 64\times -1$

$= \underline{\underline{-64}}$

c) $x+iy = \sqrt{5-12i}$

$(x+iy)^2 = 5-12i$

$x^2+2ixy+i^2y^2 = 5-12i$

$x^2-y^2+2ixy = 5-12i$

$x^2-y^2 = 5$ and $2xy = -12$

$x^2-y^2 = 5$ and $y = \frac{-6}{x}$

$\therefore x^2 - \left(\frac{-6}{x}\right)^2 = 5$

$x^2 - \frac{36}{x^2} = 5$

$x^4 - 36 = 5x^2$

$x^4 - 5x^2 - 36 = 0$

$(x^2-9)(x^2+4) = 0$

$x^2 = 9$ OR $x^2 = -4$

$x = \pm 3$ No solns as x is real

If $x = 3$, $y = \frac{-6}{3}$

$= -2$

If $x = -3$, $y = \frac{-6}{-3}$

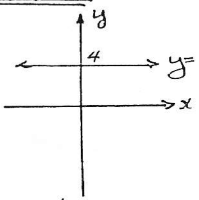
$= 2$

$\therefore \sqrt{5-12i} = 3-2i$ OR $-3+2i$

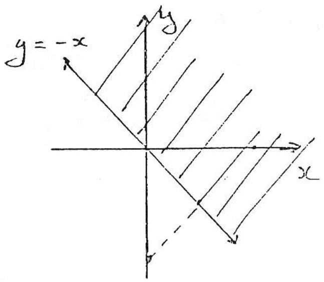
$= \underline{\underline{\pm(3-2i)}}$

d)(i) $\operatorname{Im}(z) = 4$

$y = 4$

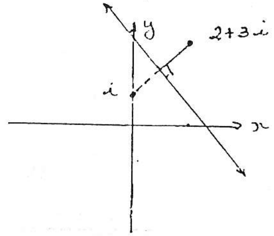


(ii) $|z-4| \le |z+4i|$

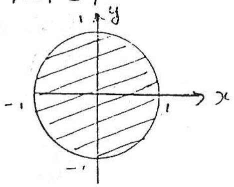


(iii) $|z-2-3i| = |z-i|$

$|z-(2+3i)| = |z-i|$

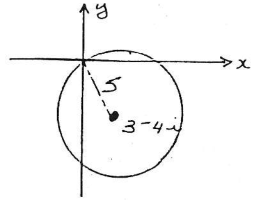


(iv) $|z| \le 1$

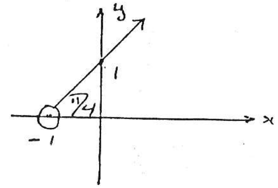


v) $|z-3+4i| = 5$

$|z-(3-4i)| = 5$



(vi) $\arg(z+1) = \frac{\pi}{4}$



Question 2.

a) (i) $z = \frac{1-7i}{3+4i} \times \frac{3-4i}{3-4i}$

$= \frac{3-4i-21i+28i^2}{3^2-16i^2}$

$= \frac{-25-25i}{25}$

$= \underline{\underline{-1-i}}$

(ii) $|\bar{z}| = \sqrt{(-1)^2+(-1)^2}$

$= \sqrt{1+1}$

$= \underline{\sqrt{2}}$

(iii) $\underline{\bar{z} = -1+i}$

b) (i) $2|z| = z+\bar{z}+4$

$2\sqrt{x^2+y^2} = x+iy+x-iy+4$

$2\sqrt{x^2+y^2} = 2x+4$

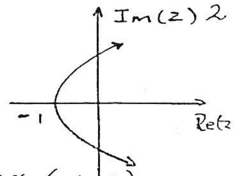
$\sqrt{x^2+y^2} = x+2$

$x^2+y^2 = (x+2)^2$

$x^2+y^2 = x^2+4x+4$

$y^2 = 4x+4$

$y^2 = 4(x+1)$



Parabola vertex $(-1, 0)$

axis the x axis

(ii) $z\bar{z} = z+\bar{z}$

$(x+iy)(x-iy) = x+iy+x-iy$

$x^2-i^2y^2 = 2x$

$x^2+y^2 = 2x$

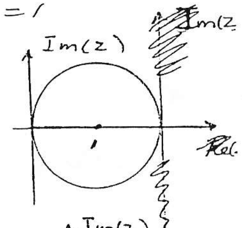
$x^2-2x+1+y^2 = 1$

$(x-1)^2+y^2 = 1$

Circle

Centre $(+1, 0)$

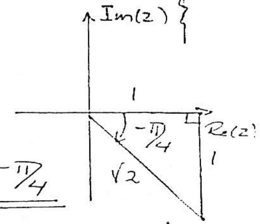
Radius 1



c) (i) $1-i$

$|1-i| = \underline{\sqrt{2}}$

$\arg(1-i) = \underline{\underline{-\frac{\pi}{4}}}$

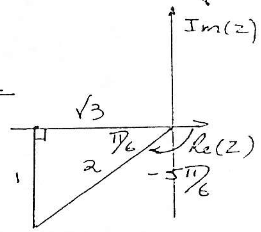


(ii) $-\sqrt{3}-i$

$|-\sqrt{3}-i| = \underline{\underline{2}}$

$\arg(-\sqrt{3}-i)$

$= \underline{\underline{-\frac{5\pi}{6}}}$



(iii) $|(1-i)(-\sqrt{3}-i)|$

$= |1-i||-\sqrt{3}-i|$

$= \sqrt{2}\times 2$

$= \underline{\underline{2\sqrt{2}}}$

$\arg\{(1-i)(-\sqrt{3}-i)\}$

$= \arg(1-i)+\arg(-\sqrt{3}-i)$

$= -\frac{\pi}{4} + \left(-\frac{5\pi}{6}\right)$

$= \frac{-3\pi-10\pi}{12}$

$= -\frac{13\pi}{12}$

$= \frac{11\pi}{12}$

(iv) $\left|\frac{1-i}{-\sqrt{3}-i}\right| = \frac{|1-i|}{|-\sqrt{3}-i|}$

$=$

$= \frac{\sqrt{2}}{2}.$

$\arg\left\{\frac{1-i}{-\sqrt{3}-i}\right\} = \arg(1-i) - \arg(-\sqrt{3}-i)$

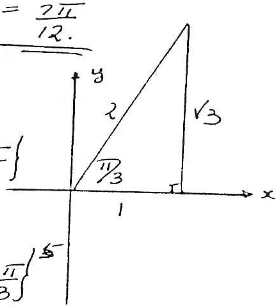
$= -\pi/4 - -\frac{5\pi}{6}$

$= \frac{-3\pi + 10\pi}{12}$

$= \frac{7\pi}{12}.$

d) (i) $z = 1 + \sqrt{3}i$

$= 2\{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\}$



(ii) z^6

$= 2^5\{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\}^5$

$= 2^5\{\cos(5\times\frac{\pi}{3}) + i\sin(5\times\frac{\pi}{3})\}$

$= 2^5(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3})$

$= 2^5 \times (\cos\pi/3 - i\sin\frac{\pi}{3})$

$= \cancel{64}\ 32\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= 16(1 - i\sqrt{3})$

Question 3:

a) $z^n + z^{-n}$

$= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

b) $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta\, i^2\sin^2\theta + i^3\sin^3\theta$

$\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

Equating imaginary parts

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

$= 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta$

$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

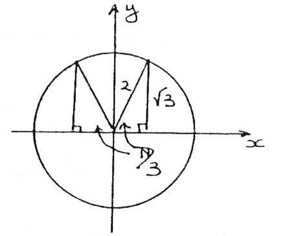
$= 3\sin\theta - 4\sin^3\theta$

let $\sin\theta = x$ and $\sin 3\theta = \frac{\sqrt{3}}{2}$

$\therefore \frac{\sqrt{3}}{2} = 3x - 4x^3$

$\sqrt{3} = 6x - 8x^3$

$8x^3 - 6x + \sqrt{3} = 0.$



$\therefore 3\theta = \pi/3, 2\pi/3, 7\pi/3, 8\pi/3, 13\pi/3, 14\pi/3$

$\theta = \pi/9, 2\pi/9, 7\pi/9, 8\pi/9, 13\pi/9, 14\pi/9$

$\therefore x = \sin\pi/9$

$x = \sin 2\pi/9$

$x = \sin 7\pi/9$

$= \sin 2\pi/9$

$x = \sin 8\pi/9$

$= \sin\pi/9$

$x = \sin 13\pi/9$

$= -\sin 4\pi/9$

$x = \sin 14\pi/9$

$= \sin(-4\pi/9)$

$= -\sin 4\pi/9$

$\therefore$ Roots are

$x = \sin\pi/9$, $x = \sin 2\pi/9$

$x = -\sin 4\pi/9.$

$\sum\alpha = -\frac{b}{a}$

$= \frac{0}{8}$

$= 0$

$\therefore \sin\pi/9 + \sin 2\pi/9 + (-\sin 4\pi/9) = 0$

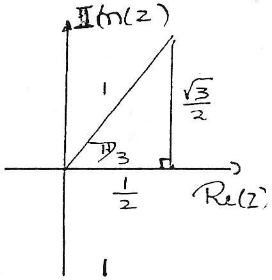
$\sin\pi/9 + \sin 2\pi/9 - \sin 4\pi/9 = 0.$

c) $\frac{1}{2}(1+i\sqrt{3})(\cos\theta + i\sin\theta)$

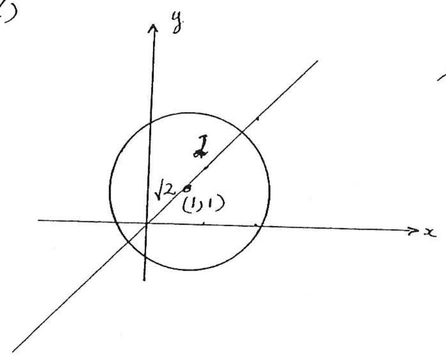
$= 1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \times 1(\cos\theta + i\sin\theta)$

$= 1 \times 1 \left\{\cos\left(\frac{\pi}{3}+\theta\right) + i\sin\left(\frac{\pi}{3}+\theta\right)\right\}$

$= 1\left\{\cos\left(\pi/3+\theta\right) + i\sin\left(\pi/3+\theta\right)\right\}$.



d)



Max. Value is $\sqrt{2}+2$.

Question 4.

(i) $z^5 = 1$

let $z = r(\cos\theta + i\sin\theta)$

$\{r(\cos\theta + i\sin\theta)\}^5 = 1(\cos 0 + i\sin 0)$

$r^n(\cos\theta + i\sin\theta)^5 = 1(\cos(0+2n\pi) + i\sin(0+2n\pi))$

$r^n(\cos 5\theta + i\sin 5\theta) = 1(\cos 2n\pi + i\sin 2n\pi)$

$r^n = 1$ $\qquad 5\theta = 2n\pi$

$r = 1$ $\qquad \theta = \frac{2n\pi}{5}$

let $n = 0, 1, 2, 3, 4$

$\theta = 0, 2\pi/5, 4\pi/5, 6\pi/5, 8\pi/5$

$= 0, 2\pi/5, 4\pi/5, -4\pi/5, -2\pi/5$

$z_1 = \cos 0 + i\sin 0$

$= 1$

$z_2 = \cos 2\pi/5 + i\sin 2\pi/5$.

$= \omega$

$z_3 = \cos 4\pi/5 + i\sin 4\pi/5$.

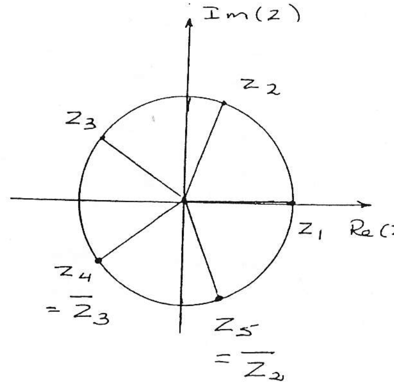
$= \omega^2$

$z_4 = \cos 6\pi/5 + i\sin 6\pi/5$

$= \omega^3$

$z_5 = \cos 8\pi/5 + i\sin 8\pi/5$

$= \omega^4$



$z_4 = \cos 6\pi/5 + i\sin 6\pi/5$

$= \cos(-4\pi/5) + i\sin(-4\pi/5)$

$= \cos 4\pi/5 - i\sin 4\pi/5$

$= \bar{z}_3$

$z_5 = \cos 8\pi/5 + i\sin 8\pi/5$

$= \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right)$

$= \cos\frac{2\pi}{5} - i\sin 2\pi/5$

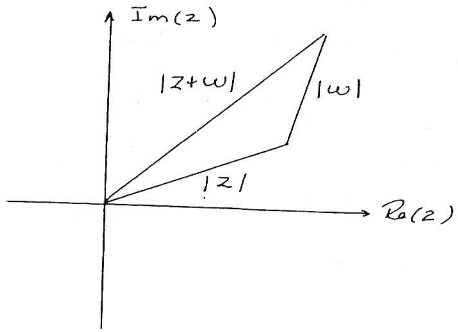
$= \bar{z}_2$

(iv) $z_1 + z_2 + z_3 + z_4 + z_5 = -\frac{b}{a}$

$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

$\therefore \omega + \omega^2 + \omega^3 + \omega^4 = -1$.

b)



TRIANGULAR INEQUALITY

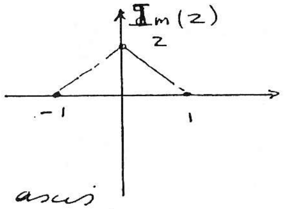
$|z+w| \leq |z| + |w|$

c)

(i) $\left|\frac{z+1}{z-1}\right| = 1$

$\frac{|z+1|}{|z-1|} = 1$

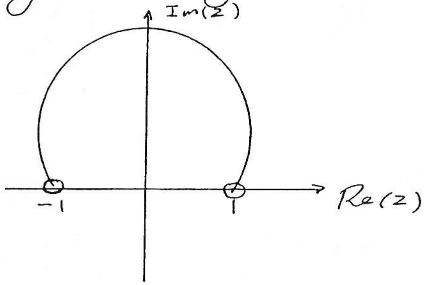
$|z+1| = |z-1|$

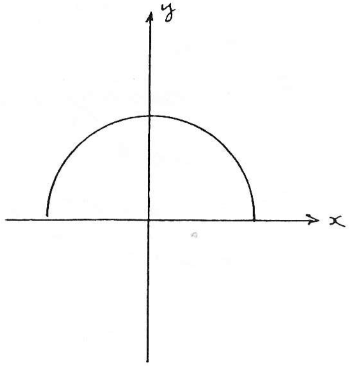


Locus is the y axis.

(ii) $\operatorname{Arg}\left(\frac{z+1}{z-1}\right) = \frac{\pi}{4}$

$\operatorname{Arg}(z+1) - \operatorname{Arg}(z-1) = \frac{\pi}{4}$





d) $\frac{(\cos 5\theta - i\sin 5\theta)^2(\cos 3\theta + i\sin 3\theta)^8}{(\cos 2\theta + i\sin 2\theta)^5}$

$= \frac{(\cos(-5\theta) + i\sin(-5\theta))^2\{(\cos\theta + i\sin\theta)^3\}^8}{\{(\cos\theta + i\sin\theta)^2\}^5}$

$= \frac{\{(\cos\theta + i\sin\theta)^{-5}\}^2(\cos\theta + i\sin\theta)^{24}}{(\cos\theta + i\sin\theta)^{10}}$

$= \frac{(\cos\theta + i\sin\theta)^{-10}(\cos\theta + i\sin\theta)^{24}}{(\cos\theta + i\sin\theta)^{10}}$

$= (\cos\theta + i\sin\theta)^4$

$= \underline{\cos 4\theta + i\sin 4\theta}$.