GOSFORD HIGH SCHOOL.
Extension 2 Mathematics.
Assessment task December 2004.

Part A.

Question 1.
If $z = 3 + 4i$ and $\omega = 1 + i$ find in the form $a + ib$

a) $z + \omega$ 1

b) $z\omega$ 2

c) $\dfrac{z}{\omega}$ 2

d) $\bar{z}$ 1

e) $|z|$ 2

Question 2.

$$\text{Simplify } \frac{(\cos\theta + i\sin\theta)^9 . (\cos 3\theta + i\sin 3\theta)^{-5}}{(\cos 2\theta - i\sin 2\theta)^4}$$ 2

Question 3.

If $z = 2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$ evaluate z^6 2

Question 4.
Factorise $z^2 + 4z + 5$ over the complex field. 2

Question 5.
   a) Find the square root of $-8 - 8i\sqrt{3}$ 3
   b) Hence solve the quadratic equation $x^2 - i2\sqrt{2}x + i2\sqrt{3} = 0$. 2

Question 6.
Given that $1, \omega, \omega^2$ are the cube roots of unity show that $\dfrac{1}{1+\omega} + \dfrac{1}{1+\omega^2} = 1$ 2

Question 7.
Write down the conjugate of $a+ib$ and hence show that if $z = x + iy$ 2

then $\dfrac{z + \bar{z}}{z\bar{z}}$ is real.

Question 8.

Find the modulus and argument of the quotient when $\sqrt{3}-i$ is divided by $-1-i$. 3

Question 9.

Find all the fourth roots of 16 and show that if β_2 and β_4 are the two imaginary roots then $\beta_2^3+2\beta_2^2+4\beta_2+8=0$. 4

Question 10.

Find all values of z such that $z^5+1=0$. 2

Question 11.

Find $\cos 4\theta$ in terms of:

a) $\sin\theta$ and $\cos\theta$. 3

b) $\cos\theta$ alone. 2

c) Hence solve $8\cos^4 x-8\cos^2 x+1=0$ for $0\le x\le\pi$. 3

Question 12.

Draw a neat sketch of the following.

a) $\mathrm{Im}(z)=4$ 1

b) $|z|\le 4$ 2

c) $|z-3+4i|=5$ 2

d) $\arg(z+i)=\dfrac{\pi}{4}$ 2

e) $z\bar{z}-4(z+\bar{z})=10$ 3

Question 13.

The quadratic equation $z^2+(1+i)z+k=0$ has a root of $1-2i$. Find, in the form $a+ib$, the value of k and the other root of the equation. 3

Question 14.

Given $z=\cos\theta+i\sin\theta$

a) Show $z^n+\dfrac{1}{z^n}=2\cos n\theta$. 2

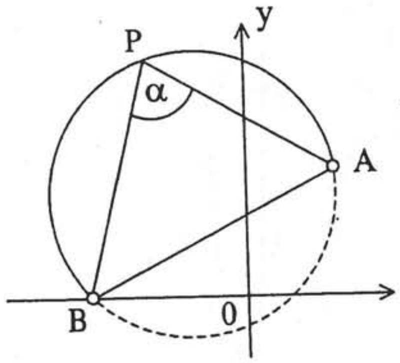
b) Hence using (a) and expanding $(z+\dfrac{1}{z})^3$ show that

$\cos^3\theta=\dfrac{1}{4}\cos 3\theta+\dfrac{3}{4}\cos\theta$. 3

Question 15.

What is the maximum value of $|z|$ for $|z-1-i|\le 2$? 3

Question 16.



NOT TO SCALE

The points P, A and B represent the complex numbers Z, $\sqrt{3}+4i$ and $-3\sqrt{3}$ respectively. The locus of Z which is moving in the complex plane such that $Arg(Z-\sqrt{3}-4i)-Arg(Z+3\sqrt{3})=\dfrac{\pi}{3}$, is part of a circle. The angle between the lines from PA and PB is α as shown in the diagram.

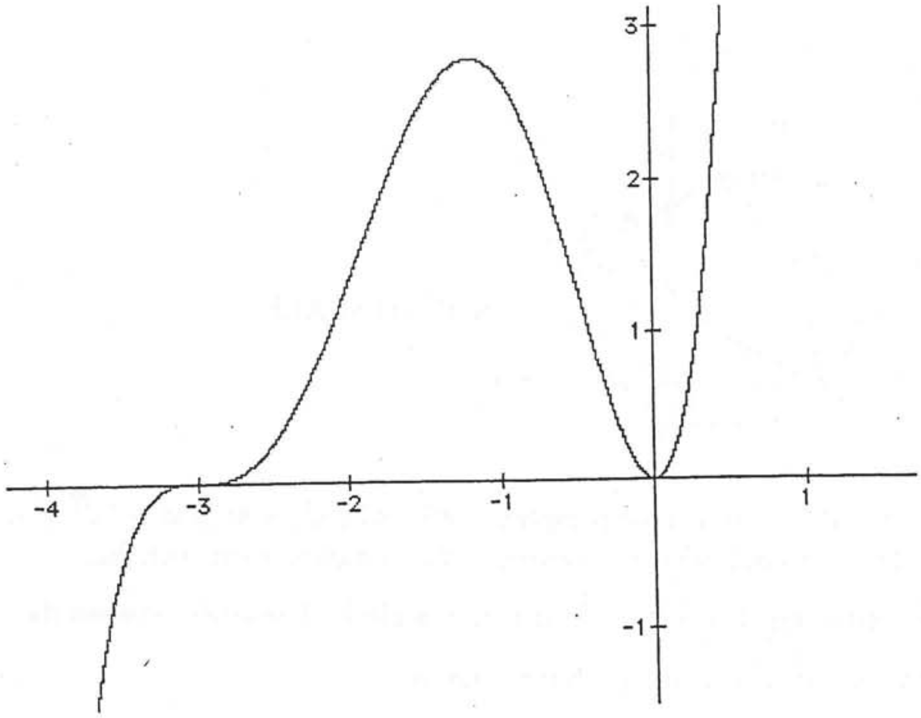
a) Show that $\alpha = \dfrac{\pi}{3}$. 2

b) Find the centre and radius of the circle. 3

Part B continued over the page

Part B.

Question 1



(a) Consider the graph of $y = f(x)$ as shown above.
On the answer sheet provided, use the graphs of $y = f(x)$ to clearly sketch separately the graphs of:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y^2 = f(x)$ **2**

(iii) $y = f'(x)$ **1**

(b) Suggest a possible polynomial equation for the graph of $y = f(x)$ shown in part (a) **1**

Question 2.

(a) If $f(x) = (x-1)(x-3)$ then sketch

(i) $y = \sqrt{f(x)}$ **2**

(ii) $y = f(|x|)$ **2**

(iii) $|y| = f(x)$ **2**

(b) (i) Find the stationary points and the asymptotes of the function **2**

$$y = \frac{(x+1)^4}{x^4+1}$$

(ii) Sketch this function labelling all essential features. **1**

(iii) Use the graph to find the set of values of k for which $(x+1)^4 = k\left(x^4+1\right)$ has two distinct real roots. **2**

Question 3.

A curve is given parametrically in terms of the real number t by the equations $x = \dfrac{3t}{1+t^3}$ and $y = \dfrac{3t^2}{1+t^3}$.

(i) Express t in terms of x and y. Hence show that the curve has Cartesian equation $x^3 + y^3 = 3xy$. Deduce that the curve is symmetrical about the line $y = x$. **3**

(ii) Show that $\dfrac{dy}{dx} = \dfrac{y - x^2}{y^2 - x}$. Hence show that the curve has a horizontal tangent when $x = \sqrt[3]{2}$. **3**

Write down the coordinates of a point on the curve where the tangent is vertical.

PART A

Ques 1

$z = 3+4i, \quad \omega = 1+i$

a) $z+\omega$

$= 3+4i+1+i$

$= 4+5i$

b) $z\omega$

$= (3+4i)(1+i)$

$= 3+3i+4i-4$

$= -1+7i$

c) $\dfrac{z}{\omega}$

$= \dfrac{3+4i}{1+i} \times \dfrac{1-i}{1-i}$

$= \dfrac{(3+4i)(1-i)}{1^2-(i)^2}$

$= \dfrac{7+i}{2}$

$= \dfrac{7}{2} + \dfrac{i}{2}$

d) $\bar{z} = 3-4i$

e) $|z| = \sqrt{3^2+4^2}$

$= 5$

Ques 2

$$\frac{(\cos\theta+i\sin\theta)^9(\cos3\theta+i\sin3\theta)^{-5}}{(\cos2\theta-i\sin2\theta)^4}$$

$$= \frac{(\cos\theta+i\sin\theta)^9\left((\cos\theta+i\sin\theta)^3\right)^{-5}}{\left((\cos\theta+i\sin\theta)^{-2}\right)^4}$$

$$\frac{(\cos\theta+i\sin\theta)^9(\cos\theta+i\sin\theta)^{-15}}{(\cos\theta+i\sin\theta)^{-8}}$$

$= (\cos\theta+i\sin\theta)^2$

$= \cos2\theta + i\sin2\theta$

Ques 3

$z = 2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$z^6 = 2^6\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^6$

$= 64(\cos\pi+i\sin\pi)$

$= 64(-1+0)$

$= -64$

Ques 4

z^2+4z+5

$= z^2+4z+4+1$

$= (z+2)^2-i^2$

$= (z+2+i)(z+2-i)$

Ques 5

a) let $a+ib = \sqrt{-8-8i\sqrt{3}}$

$\therefore a^2-b^2+2iab = -8-8i\sqrt{3}$

equating real and imaginary parts

$a^2-b^2 = -8 \quad \text{----}(1)$

$2ab = -8\sqrt{3}$

$\therefore ab = -4\sqrt{3} \quad \text{--}(2)$

$(2) \Rightarrow b = \dfrac{-4\sqrt{3}}{a}$

Sub into (1)

$$a^2 - \left(\frac{-4\sqrt{3}}{a}\right)^2 = -8$$

$$a^2 - \frac{48}{a^2} = -8$$

$$a^4 - 48 = -8a^2$$

$$a^4 + 8a^2 - 48 = 0$$

$$(a^2-4)(a^2+12) = 0$$

$a^2 = 4$ or $a^2 = -12$ → no solution

$\therefore a = \pm 2.$

Sub into (2)

$\therefore b = -2\sqrt{3},\ 2\sqrt{3}.$

$\therefore \sqrt{-8-8i\sqrt{3}} = 2-3\sqrt{2}i,\ -2+2\sqrt{3}i$

b) $x^2 - i2\sqrt{2}x + i2\sqrt{3} = 0$

$$x = \frac{i2\sqrt{2} \pm \sqrt{-8-8i\sqrt{3}}}{2.}$$

$$= \frac{2\sqrt{2}i \pm (2-2\sqrt{3}i)}{2.}$$

$$= i\sqrt{2} \pm (1-\sqrt{3}i)$$

$$= 1+i(\sqrt{2}-\sqrt{3}),\ -1+i(\sqrt{3}-\sqrt{2})$$

Q6) $\frac{1}{1+\omega} + \frac{1}{1+\omega^2} = 1$

L.HS. (using $1+\omega+\omega^2 = 0$)

$$= \frac{1}{-\omega^2} + \frac{1}{-\omega}$$

$$= \frac{1+\omega}{-\omega^2}$$

$$= \frac{-\omega^2}{-\omega^2}$$

$$= 1$$

$= R.HS$

Ques 7

Conjugate $a+ib = a-ib$

$$\frac{z+\bar{z}}{z\bar{z}}$$

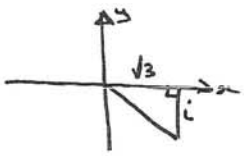
$$= \frac{x+iy+x-iy}{(x+iy)(x-iy)}$$

$$= \frac{2x}{x^2+y^2}$$

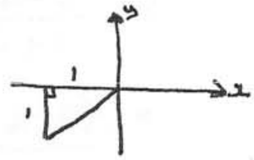
which is real.

Ques 8

$\sqrt{3}-i$



$-1-i$



$|\sqrt{3}-i| = 2.$ $\quad |-1-i| = \sqrt{2}$

$\arg(\sqrt{3}-i) = \frac{-\pi}{6}$ $\quad \arg(-1-i) = -\frac{3\pi}{4}$

$$\therefore \left|\frac{\sqrt{3}-i}{-1-i}\right| = \frac{2}{\sqrt{2}} = \sqrt{2}$$

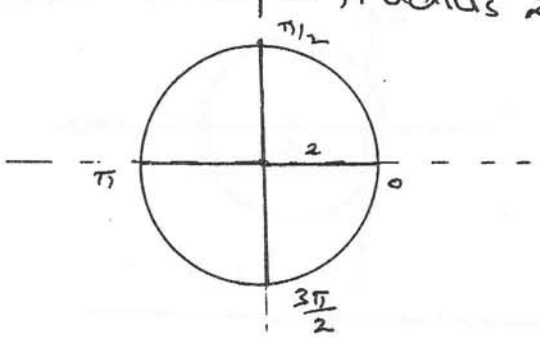
$$\arg\left(\frac{\sqrt{3}-i}{-1-i}\right) = \arg(\sqrt{3}-i) - \arg(-1-i)$$

$$= -\frac{\pi}{6} - \left(\frac{-3\pi}{4}\right)$$

$$= \frac{7\pi}{12}.$$

Ques 9

roots are equally spaced around a circle, radius 2.



$z_1 = 2(\cos 0° + i\sin 0°) = 2$

$z_2 = 2(\cos \frac{\pi}{2} + i\sin \frac{\pi}{2}) = 2i$

$z_3 = 2(\cos \pi + i\sin \pi) = -2$

$z_4 = 2(\cos \frac{3\pi}{2} + i\sin \frac{3\pi}{2}) = -2i$

Let $\beta_2 = 2i, \quad \beta_4 = -2i$

$\beta_2^3 + 2\beta_2^2 + 4\beta_2 + 8 = 0$

L.H.S

$= (2i)^3 + 2(2i)^2 + 4(2i) + 8$

$= -8i + (-8) + 8i + 8$

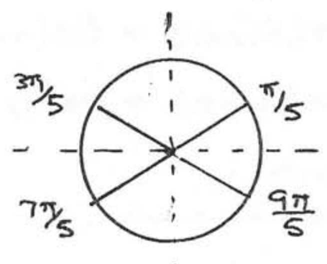
$= 0$

$=$ R.H.S

Q10) $z^5 + 1 = 0$

$z^5 = -1$

roots are equally spaced around the unit circle starting at $\frac{\pi}{5}$



$z_1 = \cos \pi/5 + i\sin \frac{\pi}{5}$

$z_2 = \cos \frac{3\pi}{5} + i\sin \frac{3\pi}{5}$

$z_3 = \cos \pi + i\sin \pi = -1$

$z_4 = \cos \frac{7\pi}{5} + i\sin \frac{7\pi}{5}$

$z_5 = \cos \frac{9\pi}{5} + i\sin \frac{9\pi}{5}$

Ques 11 a)

$(\cos\theta + i\sin\theta)^4 = (\cos\theta + i\sin\theta)^4$

$\therefore \cos 4\theta + i\sin 4\theta$

$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$

$= \cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

$\therefore$ equating real parts

$\cos 4\theta = \cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta$

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b) $\cos 4\theta = \cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta$

$= \cos^4\theta + (1-\cos^2\theta)^2 - 6\cos^2\theta(1-\cos^2\theta)$

$= \cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta$

$= 8\cos^4\theta - 8\cos^2\theta + 1$

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c) $8\cos^4 x - 8\cos^2 x + 1 = 0$

ie using (b)

$\cos 4x = 0$

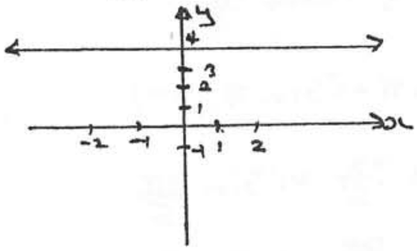
$4x = \pi/2, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \ldots$

$x = \pi/8, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$

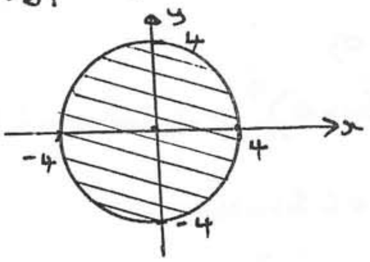
for $0 \le x \le \pi$

Ques 12

a) $\operatorname{Im}(z) = 4$

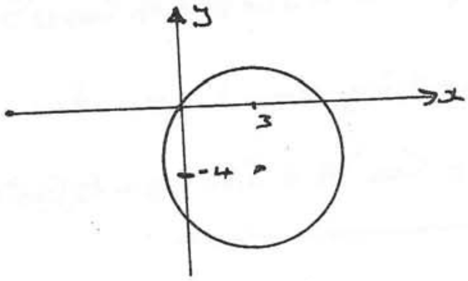


b) $|z| \leq 4$

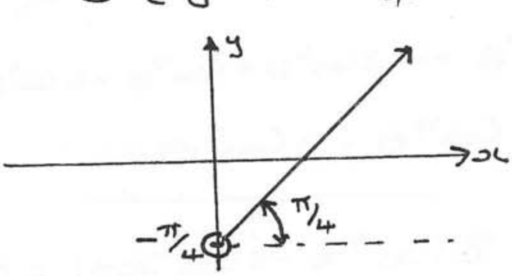


c) $|z - \ + 4i| = 5$

$|z - (3-4i)| = 5$



d) $\arg(z + i) = \frac{\pi}{4}$



e) $z\bar{z} - 4(z + \bar{z}) = 10$

Let $z = x + iy$

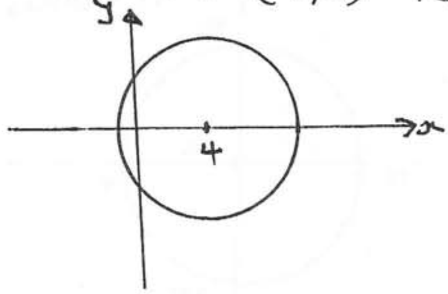
$(x+iy)(x-iy) - 4(x+iy+x-iy) = 10$

$x^2 + y^2 - 8x = 10$

$x^2 - 8x + 16 + y^2 = 10 + 16$

$(x-4)^2 + y^2 = 26$

Circle centre $(4,0)$ radius $\sqrt{26}$



Ques 13

$z^2 + (1+i)z + k = 0$

$(1-2i)$ is a root

$(1-2i)^2 + (1+i)(1-2i) + k = 0$

$1 - 4i - 4 + 1 - 2i + i + 2 + k = 0$

$k = 5i$

Let the other root be β

$\therefore \quad \beta + (1-2i) = -\frac{b}{a}$

$\beta + 1 - 2i = -1 - i$

$\beta = -2 + i$

Q14) $z = \cos\theta + i\sin\theta$

$z^n + \frac{1}{z^n} = 2\cos n\theta$

L.HS

$= z^n + z^{-n}$

$= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta.$

b) $\left(z+\frac{1}{z}\right)^3$

$= z^3 + 3z^2 \cdot \frac{1}{z} + 3z \cdot \frac{1}{z^2} + \frac{1}{z^3}$

$= z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}$

$= z^3 + \frac{1}{z^3} + 3\left(z + \frac{1}{z}\right)$

$= 2\cos 3\theta + 6\cos\theta$

also $\left(z+\frac{1}{z}\right)^3$

$= (2\cos\theta)^3$

$= 8\cos^3\theta$

$\therefore\ 8\cos^3\theta = 2\cos 3\theta + 6\cos\theta$

$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta.$

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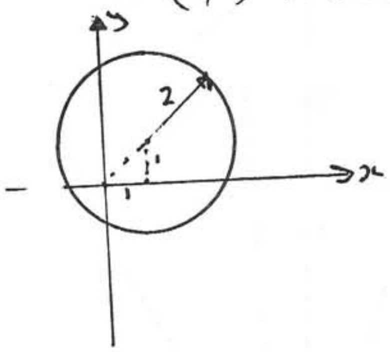
15) $|z-1-i| \leq 2.$

$|z-(1+i)| \leq 2.$

circle centre $(1,1)$ radius 2.



from the diagram

max value $|z| = \sqrt{2}+2.$

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(5)

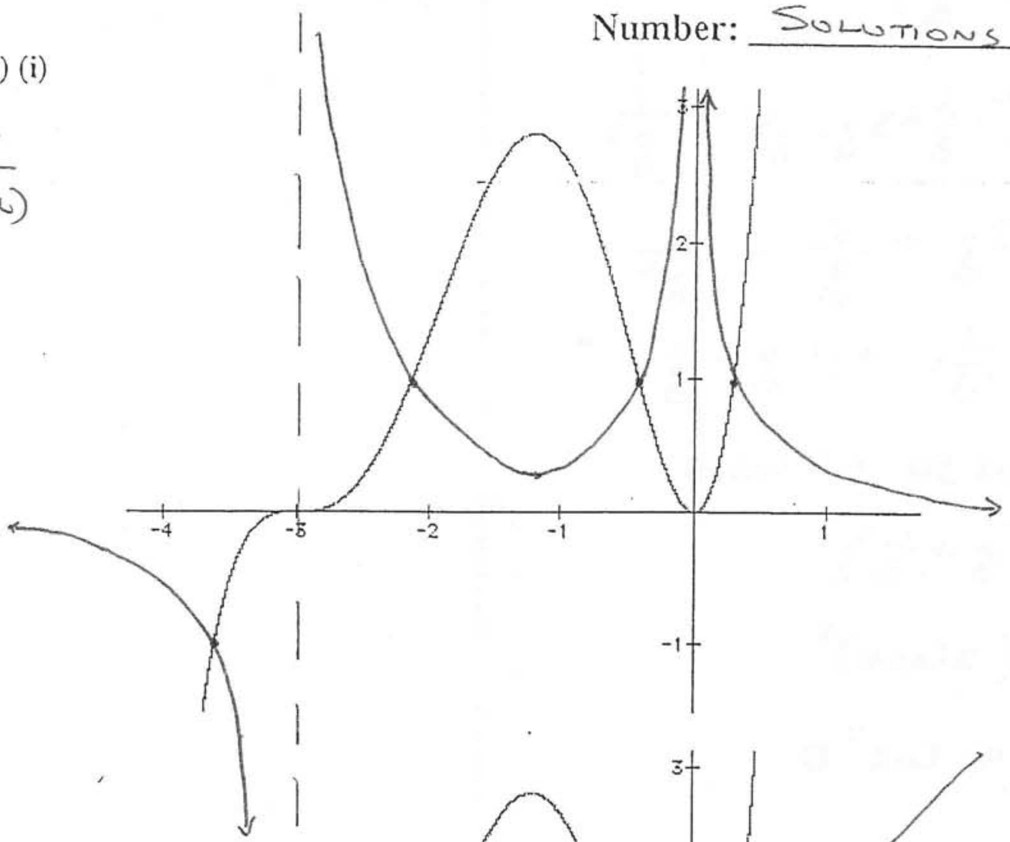
16)

PART B

Number: SOLUTIONS

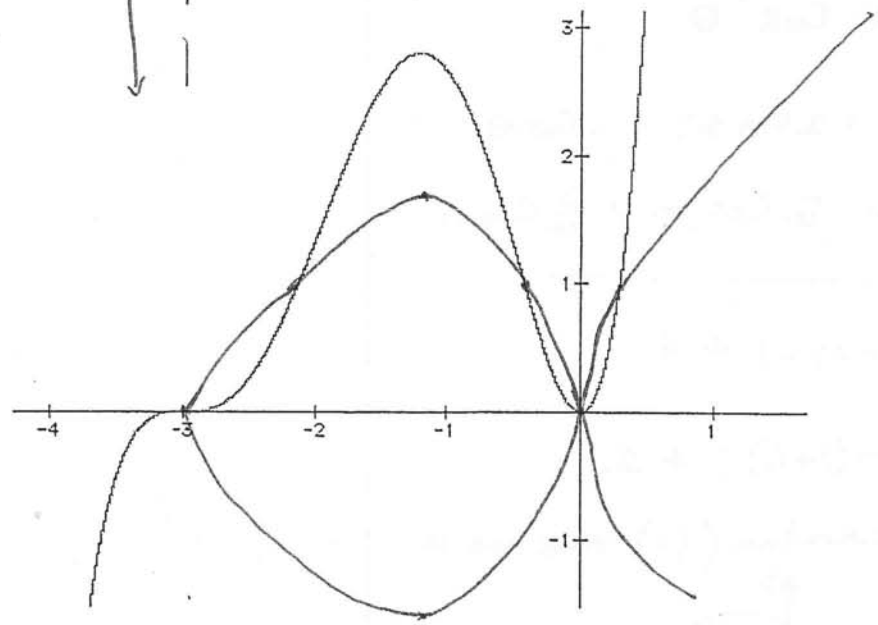
Question 1. (a) (i)

$y = \frac{1}{f(x)}$



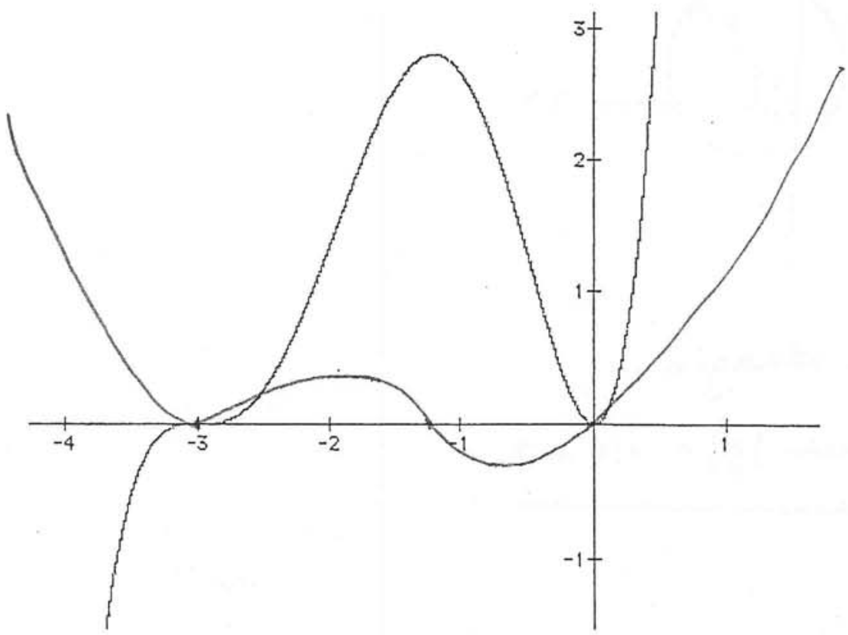
(ii)

$y^2 = f(x)$



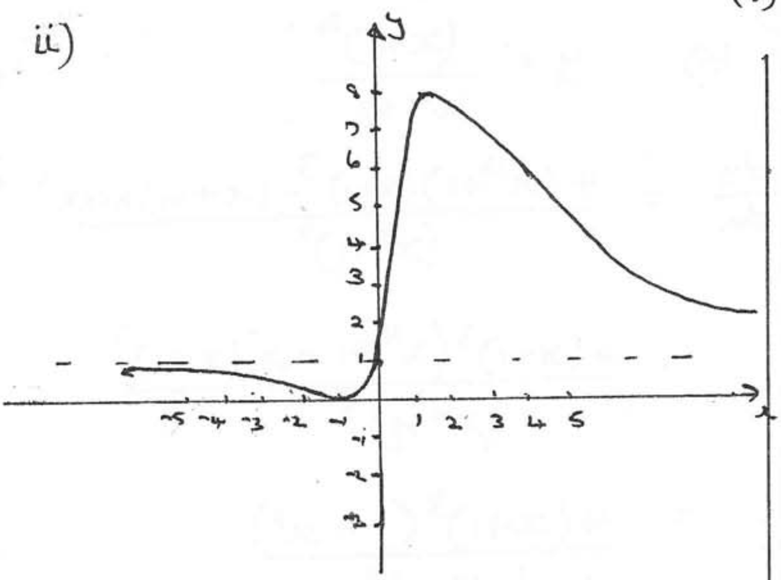
(iii)

$y = f'(x)$



(b) $y = x^2(x+3)^3$

ii)



iii) $0 < k < 1 \quad , \quad 1 < k < 8$

Question 3

$x = \dfrac{3t}{1+t^3} \qquad y = \dfrac{3t^2}{1+t^3}$

$(1+t^3)x = 3t \quad \text{---(1)}$

$(1+t^3)y = 3t^2 \quad \text{---(2)}$

(2) $\div$ (1) $\dfrac{y}{x} = t$

Sub into (1)

$\left(1 + \dfrac{y^3}{x^3}\right)x = \dfrac{3y}{x}$

$x + \dfrac{y^3}{x^2} = \dfrac{3y}{x}$

$x^3 + y^3 = 3xy$

the curve is Symmetrical about $y = x$ because when you interchange x + y you have the Same equation.

ii) $x^3 + y^3 = 3xy$

$3x^2 + 3y^2 \dfrac{dy}{dx} = 3\left(x \cdot \dfrac{dy}{dx} + y\right)$

$3x^2 + 3y^2 \dfrac{dy}{dx} = 3x\dfrac{dy}{dx} + 3y$

$\dfrac{dy}{dx}(3y^2 - 3x) = 3y - 3x^2$

$\dfrac{dy}{dx} = \dfrac{3y - 3x^2}{3y^2 - 3x}$

$= \dfrac{y - x^2}{y^2 - x}.$

horizontal tangent when $\dfrac{dy}{dx} = 0$

$\dfrac{y - x^2}{y^2 - x} = 0$

$y - x^2 = 0$

$y = x^2$

$\therefore x^3 + (x^2)^3 = 3x(x^2)$

$x^3 + x^6 = 3x^3$

$x^6 - 2x^3 = 0$

$x^3(x^3 - 2) = 0$

$x = 0, \quad x = \sqrt[3]{2}.$

Vertical tangent $\dfrac{dx}{dy} = 0$

$\dfrac{y^2 - x}{y - x^2} = 0$

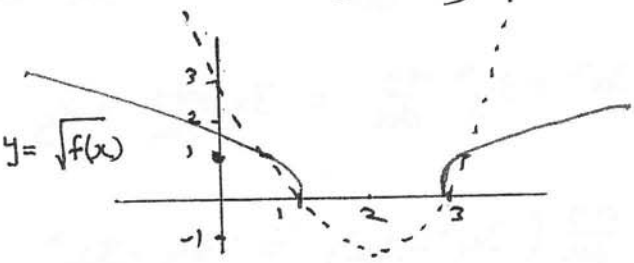
ie $y^2 = x$

PART B

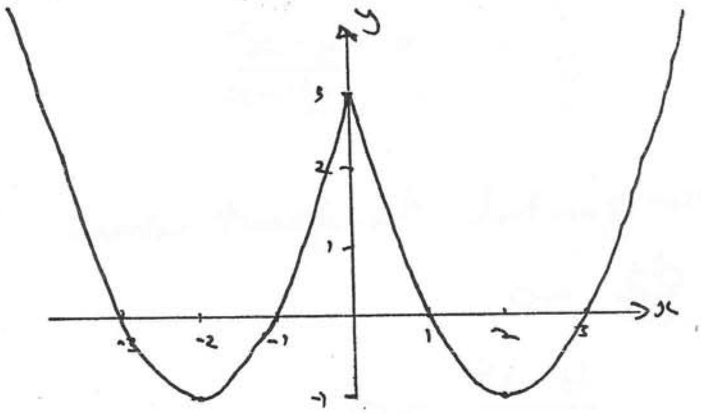
Question 2.

a) $f(x) = (x-1)(x-3)$

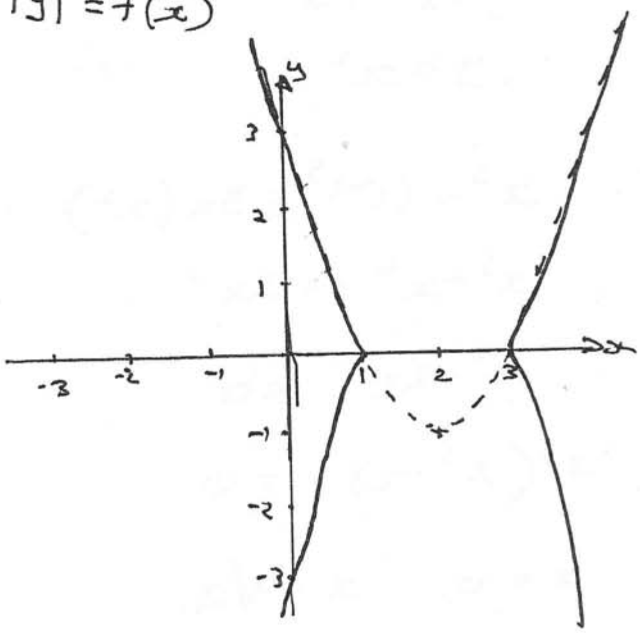
i) $y = \sqrt{f(x)}$



ii) $y = f(|x|)$



iii) $|y| = f(x)$



(6)

b) $y = \dfrac{(x+1)^4}{x^4+1}$

$$\frac{dy}{dx} = \frac{4(x^4+1)(x+1)^3 - (x+4) \times 4x^3}{(x^4+1)^2}$$

$$= \frac{4(x+1)^3\left(x^4+1 - x^3(x+1)\right)}{(x^4+1)^2}$$

$$= \frac{4(x+1)^3(1-x^3)}{(x^4+1)^2}$$

S.P. $\dfrac{dy}{dx} = 0$

$4(x+1)^3(1-x^3) = 0$

$\therefore x = -1, 1$

$f'(-2) < 0$ $\qquad$ $f'(2) < 0$

$f'(0) > 0$ $\qquad$ $f'(0) > 0$

$\therefore (-1, 0)$ min $\qquad$ $\therefore (1, 8)$ max

asymptotes

$$\frac{(x+1)^4}{x^4+1}$$

$$= \frac{x^4+4x^3+6x^2+4x+1}{x^4+1}$$

$$= \frac{1 + \frac{4}{x} + \frac{6}{x^2} + \frac{4}{x^3} + \frac{1}{x^4}}{1 + \frac{1}{x^4}}$$

as $x \to \infty$ $\quad y \to 1$.

$\therefore$ asymptote $y = 1$.

$\therefore \; y^6 + y^3 = 3y^3$

$y^6 - 2y^3 = 0$

$y^3(y^3 - 2) = 0$

$y = 0, \; \sqrt[3]{2}.$

now $\; y^2 = x$

$\therefore \; x = (\sqrt[3]{2})^2$

$= \sqrt[3]{4}$

$\therefore$ vertical tangent $(\sqrt[3]{4}, \sqrt[3]{2})$