GOSFORD HIGH SCHOOL.
Extension 2 Mathematics.
Assessment task December 2005.

Question 1.

(a) Given the complex numbers $A = 3 - 4i$ and $B = 1 + i$, determine the following in the form $x + iy$.

(i) $A - B$ 1

(ii) $\frac{A}{B}$ 2

(iii) B^2 2

(iv) $\sqrt{A}$ 3

(b) Given $C = 1 + \sqrt{3}i$

(i) Write C in mod–arg form. 2

(ii) Hence, using De Moivre's theorem find C^6. 2

(c) Factorise $z^2 + 4z + 5$ over the complex field. 2

(d) Given that $1, \omega, \omega^2$ are the cubed roots of unity show that:
$(1 + 2\omega + 3\omega^2)(1 + 2\omega^2 + 3\omega) = 3$ 2

(e) Solve the equation $z^2 + 2\bar{z} + 6 = 0$ giving the solutions in the form $z = x + iy$ where x and y are real numbers 3

(f) Draw a neat accurate sketch of the following.

(i) $\text{Re}(z) = 4$ 1

(ii) $z\bar{z} - 4(z + \bar{z}) = 10$ 3

(iii) $\arg(z-1) - \arg(z+1) = \frac{\pi}{2}$ 2

Question 2.

(a) By using implicit differentiation show that the equation of the tangent to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (a and b constants) at the point $(a\cos\theta, b\sin\theta)$ is given by $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$. 3

(b) Find the gradient of the curve $2x^3 - x^2y + y^3 = 1$ at the point $(2,-3)$. 3

(c) If α, β, γ are the roots of $x^3 + px + q = 0$, find in terms of p and q :

(i) $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \alpha\gamma$, $\alpha\beta\gamma$ 1

(ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ 2

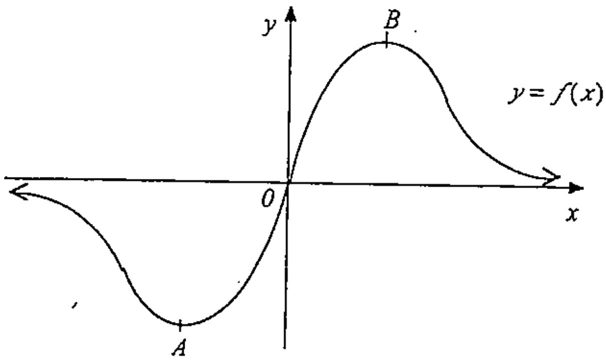
(iii) $\alpha^3 + \beta^3 + \gamma^3$ 2

(d) Given that $-1 + i\sqrt{3}$ is a root of $3x^3 + 5x^2 + 10x - 4 = 0$ find the other roots. 2

Question 3.

(a) the diagram below shows the graph of the function

$f(x) = xe^{-\frac{1}{2}x^2}$.



(i) Find the coordinates of the stationary points A and B. 3

(ii) Find the gradient of the tangent to the curve at the origin O. Hence find the set of values of the real number k such that the equation $f(x) = kx$ has three real roots. 2

(iii) On separate diagrams sketch the following graphs.

(α) $y = |f(x)|$ 2

(β) $y = \{f(x)\}^2$ 2

(γ) $y = \dfrac{1}{f(x)}$ 2

(χ) $y = \sqrt{f(x)}$ 2

(λ) $y^2 = f(x)$ 2

(θ) $y = f'(x)$ 2

(b) Sketch the graph of $y = \dfrac{1}{1+e^{\frac{1}{x}}}$ 3

(c) Solve graphically $2\cos|x| > 1$ for $-\pi \le x \le 2\pi$. 3

Question 4.

(a) (i) Sketch the intersection of the locus described by $|z| \le 3$ and $-\dfrac{\pi}{4} \le \arg(z+3) \le \dfrac{\pi}{4}$. 3

(ii) If the complex number ω lies on the boundary of the region sketch in part (i), find the minimum value of $|\omega|$. 2

(b) OABC is a rectangle on the argand diagram in which side OC is twice the length of OA, where O is the origin.

(i) If A represents the complex number $1+2i$, find the complex numbers represented by B and C given that the argument of the complex number represented by the point C is negative. 2

(ii) If the rectangle is rotated anticlockwise $\dfrac{\pi}{3}$ radians about O, 1

find the complex number represented by the new position of A.

(c) Solve $z^4 + 16 = 0$ over the complex numbers giving your answer in $x + iy$ form. 2

(d) If Z_1, Z_2, Z_3, Z_4 and Z_5 are the roots of $Z^5 = 1$

(i) Determine the values of all these roots in the form $\cos\theta + i\sin\theta$. 2

(ii) Factorise $Z^5 = 1$ in terms of quadratic and linear factors. 2

(iii) Hence show that $\cos\dfrac{2\pi}{5} + \cos\dfrac{4\pi}{5} = -\dfrac{1}{2}$ 2

(e) Given $z = \cos\theta + i\sin\theta$

i) Show $z^n + \dfrac{1}{z^n} = 2\cos n\theta$. 2

ii) Hence by expanding $(z + \frac{1}{z})^4$ find an expression for $\cos^4\theta$ in the form $a\cos 4\theta + b\cos 2\theta$. 3

COMPLEX NUMBERS

1) If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Then: $\bar{z}_1 =$

$|z_1| =$

$\arg(z) =$

$|z_1 z_2| =$

$\arg(z_1 z_2) =$

$\left|\dfrac{z_1}{z_2}\right| =$

$\arg(\dfrac{z_1}{z_2}) =$

2) If $z = r(\cos\theta + i\sin\theta)$

then $z^n =$

3) If ω is a complex cube root of unity then:

a) $\omega^3 =$

b) $1 + \omega + \omega^2 =$

POLYNOMIALS

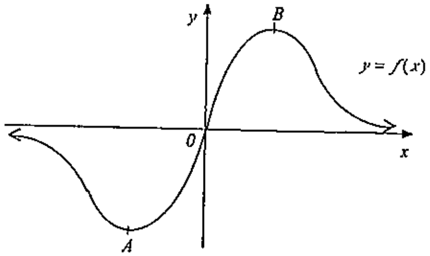
Remainder theorem.
If $P(x)$ is divided by $(x - a)$ then the remainder is ....

Factor theorem
If $(x\text{-}a)$ is a factor of $P(x)$ then ....

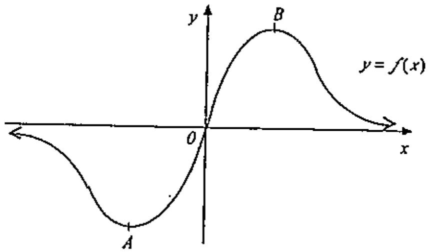
Roots of multiplicity
If $x = a$ is an n fold root of $P(x)$ then $x = a$ is an .... fold root of ....

If a complex number $a+ib$ is a root of a polynomial, whose coefficients are .... then .... is also a root.

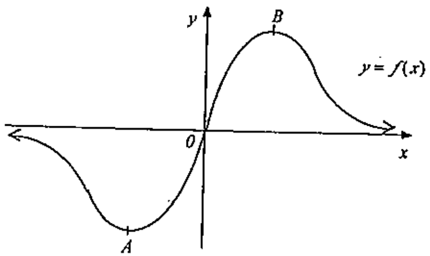
(α) $y = |f(x)|$



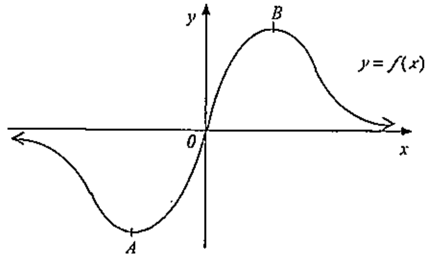
(β) $y = \{f(x)\}^2$



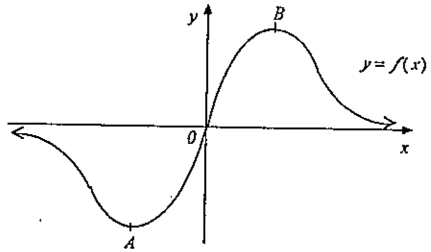
(γ) $y = \dfrac{1}{f(x)}$



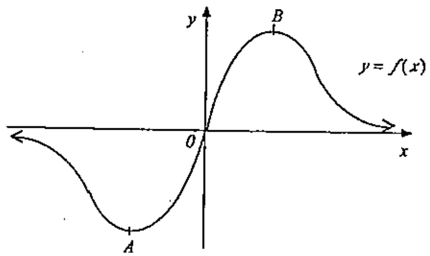
(χ) $y = \sqrt{f(x)}$



(λ) $y^2 = f(x)$



(θ) $y = f'(x)$



1 a) i) $(3-4i)-(1+i)$

$= 2-5i$

ii) $\frac{3-4i}{1+i} \times \frac{1-i}{1-i} = \frac{3-4-(3+4)i}{2}$

$= -\frac{1}{2} + \frac{7}{2}i$

iii) $(1+i)^2 = 1-1+2i = 2i$

iv) $\sqrt{3-4i} = x+iy$

$3-4i = x^2-y^2+2xyi$

$x^2-y^2 = 3$

$xy = -2 \implies y = -\frac{2}{x}$

$x^2-\frac{4}{x^2} = 3 \implies x^4-3x^2-4=0$

$(x^2-4)(x^2+1) = 0$

$x = \pm 2$ or $\pm i$ (not real)

$y = \mp 1$

$\therefore \sqrt{3-4i} = \pm 2 \mp i$

b) i) $1+\sqrt{3}i = 2\operatorname{cis}\frac{\pi}{3}$

ii) $\left(2\operatorname{cis}\frac{\pi}{3}\right)^6 = 64\operatorname{cis}2\pi$

$= 64$

c) $z^2+4z+4+1$

$(z+2)^2+1 = (z+2)^2-i^2$

$(z+2+i)(z+2-i)$

d) $z^3 = 1 \implies z^3-1=0$

$(z-1)(z^2+z+1) = 0$

$z=1$, or w or w^2

$w \neq 1$, $\therefore w^2+w+1=0$

$(1+2w+3w^2) = (1+w+w^2+w+2w^2)$

$= w+2w^2 = w+w^2+w^2$

$= w^2-1$

$(1+2w^2+3w) = 1+w+w^2+w^2+2w$

$= w^2+w+w \implies w-1$

$(w^2-1)(w-1) = (w-1)(w+1)(w-1)$

$= w^3-w^2-w+1$

$= 1-(w^2+w+1)+1+1$

$= 3$

e) $(x+iy)^2+2(x-iy)+6=0$

$x^2-y^2+2x+6=0$

$2xyi-2iy=0$

$2yi(x-1)=0$

$x=1$ or $y=0$

when $x=1$, $y^2=1+2+6$

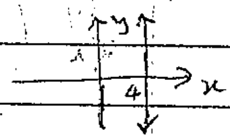
$y=\pm 3$

when $y=0$ $x^2+2x+6=0$

No solutions (real)

$z = 1+3i$ or $1-3i$

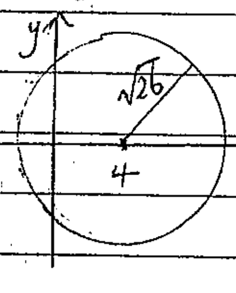
f) i) $x=4$



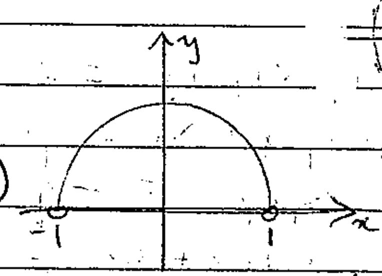
ii) $x^2+y^2-4(2x)=10$

$x^2-8x+16+y^2=26$

$(x-4)^2+y^2=26$



iii)



$x^2+y^2=1$ with $y>0$

2 $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$

$\frac{dy}{dx} = -\frac{2x}{a^2}\times\frac{b^2}{2y}$

$= -\frac{b^2x}{a^2y}$

at $(a\cos\theta, b\sin\theta)$

$$\frac{dy}{dx} = \frac{-b^2 \cdot a\cos\theta}{a^2 \, b\sin\theta}$$

$$= \frac{-b}{a}\frac{\cos\theta}{\sin\theta}$$

$\therefore \; y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$a\sin\theta y - ab\sin^2\theta = -b\cos\theta x + ab\cos^2\theta$

$b\cos\theta x + a\sin\theta y = ab(\sin^2\theta + \cos^2\theta)$

$= ab$

$\div$ b|s by ab

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

b) $2x^3 - x^2y + y^3 = 1$

$6x^2 - x^2\frac{dy}{dx} - y.2x + 3y^2\frac{dy}{dx} = 0$

$\frac{dy}{dx}(3y^2 - x^2) = 2xy - 6x^2$

$$\frac{dy}{dx} = \frac{2xy - 6x^2}{3y^2 - x^2}$$

At $(2, -3)$, $\frac{dy}{dx} = \frac{-12 - 24}{27 - 4}$

$$= \frac{-36}{23}$$

c) i) $\alpha + \beta + \gamma = 0$

$\alpha\beta + \beta\gamma + \alpha\gamma = p$

$\alpha\beta\gamma = -q$

ii) $\frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = -\frac{p}{q}$

iii) α is a root $\therefore$

$\alpha^3 + p\alpha + q = 0 \quad -①$

$\beta^3 + p\beta + q = 0 \quad -②$

$\gamma^3 + p\gamma + q = 0 \quad -③$

$\alpha^3 + \beta^3 + \gamma^3 + p(\alpha + \beta + \gamma) + 3q = 0$

$\alpha^3 + \beta^3 + \gamma^3 = 0 - 3q = -3q$

$-1 + i\sqrt{3}$, $-1 - i\sqrt{3}$ (coeffs are real)

Sum of roots $= -2 + \gamma = -\frac{5}{3}$

$\gamma = 2 - \frac{5}{3} = \frac{1}{3}$

Roots are $-1+i\sqrt{3}$, $-1-i\sqrt{3}$, $\frac{1}{3}$

3) $f'(x) = x \cdot e^{-\frac{1}{2}x^2} \cdot -x + e^{-\frac{1}{2}x^2}$

$= e^{-\frac{1}{2}x^2}(-x^2 + 1)$

$f'(x) = 0$ for stat. pts

$x = \pm 1 \qquad y = \pm e^{-\frac{1}{2}} \doteqdot .6$

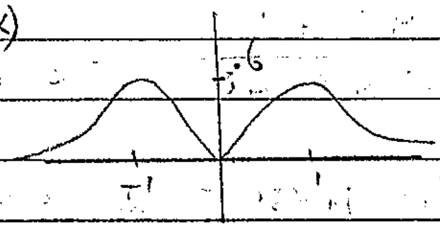
$A\left(-1, -\frac{1}{\sqrt{e}}\right) \quad B\left(1, \frac{1}{\sqrt{e}}\right)$

ii) When $x = 0$, $f'(0) = e^0 = 1$

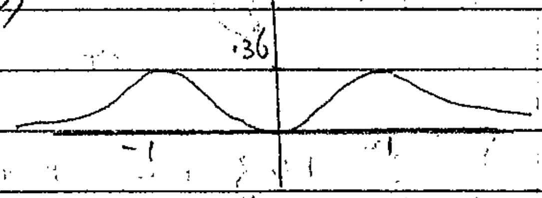
$y = kx$ intersects 3 times

$0 < k < 1$

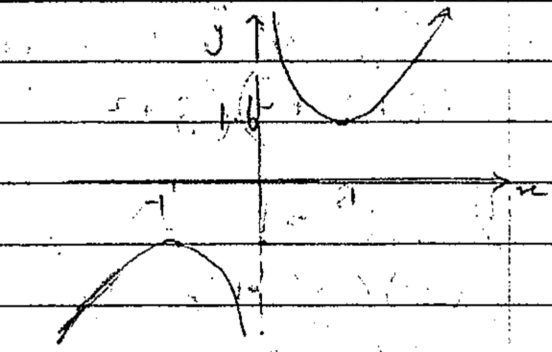
iii) α)



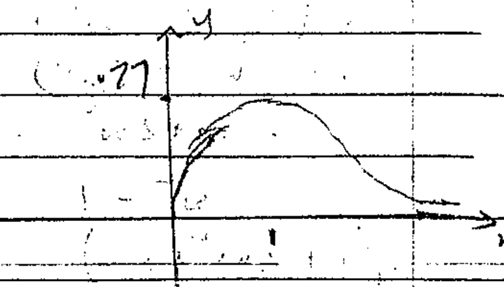
β)



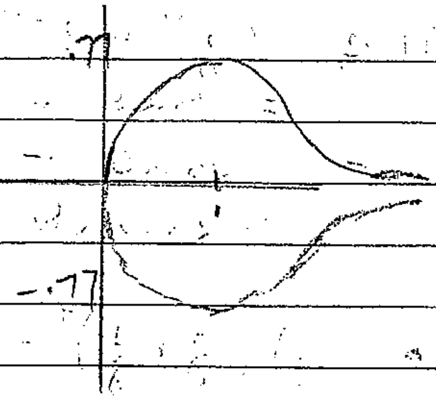
γ)



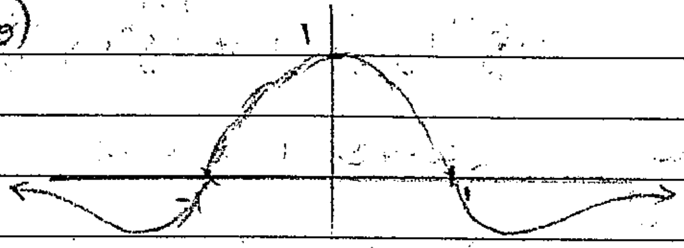
δ)



ii)

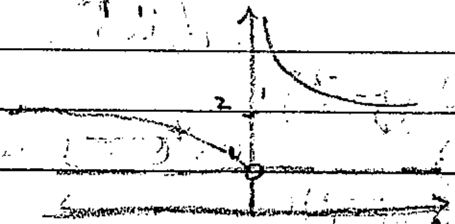


i)

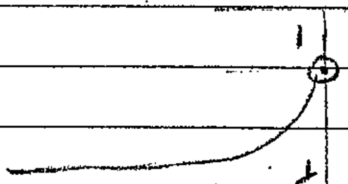


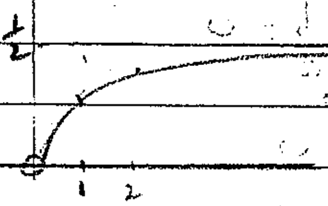
b) $e^{\frac{1}{x}}$

$1+e^{\frac{1}{x}}$



$y = \dfrac{1}{1+e^{\frac{1}{x}}}$



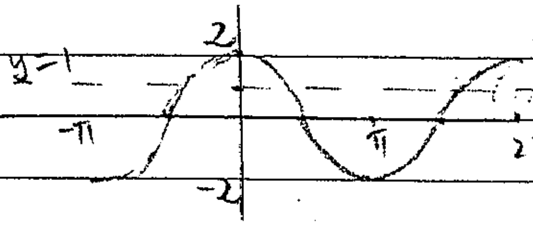


$y' = -(1+e^{\frac{1}{x}})^{-2} \cdot e^{\frac{1}{x}} \cdot -x^{-2}$

$= \dfrac{e^{\frac{1}{x}}}{x^2(1+e^{\frac{1}{x}})}$

Infinite gradient at $x=0$

c)

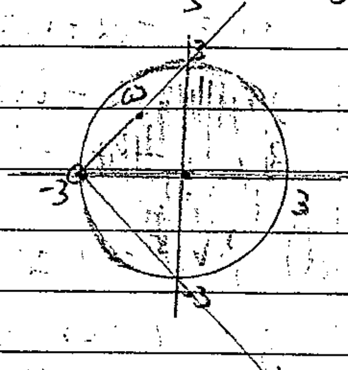


$\cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}$

$\therefore \cos|x| > \frac{1}{2}$

$-\frac{\pi}{3} < x < \frac{\pi}{3}$ or $\frac{5\pi}{3} < x < 2\pi$

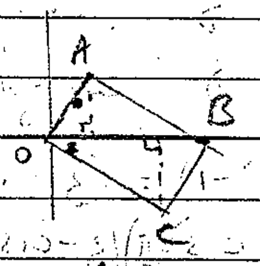
4 a)



ii) Min Val(ω) is perp dist from O to $y = x+3$

$d = \left|\dfrac{0-0+3}{\sqrt{1^2+1^2}}\right| = \dfrac{3}{\sqrt{2}}$

b)



$|OC| = 2|OA|$

Transform A $z \to 2z$ + rotate $-\frac{\pi}{2}$ ie multiply by $-i$

C: $z_3 = (-i)(2)(1+2i)$
$= 4-2i$

B: $z_2 = (1+2i)+(4-2i)$
$= 5$

iii) $|z| = \sqrt{5}$
Δ is equilateral

multiply A by $\text{cis}\frac{\pi}{3}$

$(1+2i)\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$

$= \frac{1}{2} - \sqrt{3} + i\left(\frac{\sqrt{3}}{2}+1\right)$

c) $z^4 = -16 = 16 \text{ cis } \pi$

$z = 2 \text{ cis } \dfrac{\pi+2k\pi}{4}$

$z = 2\text{ cis}\frac{\pi}{4}$, $2\text{ cis}\frac{3\pi}{4}$

$2\text{ cis}-\frac{\pi}{4}$, $2\text{ cis}-\frac{3\pi}{4}$

c) Cont: $z_1 = \frac{2}{\sqrt{2}} + \frac{i2}{\sqrt{2}} = \sqrt{2} + i\sqrt{2}$

$z_2 = -\sqrt{2} + i\sqrt{2}$

$z_3 = \sqrt{2} - i\sqrt{2}$

$z_4 = -\sqrt{2} - i\sqrt{2}$

d) i) $z^5 = 1 \text{ cis } 0$

$z = 1 \text{ cis } \frac{2k\pi}{5}$

$k=0 \quad z = 1$

$k=1 \quad z = \text{cis } \frac{2\pi}{5} = \omega$

$k=2 \quad z = \text{cis } \frac{4\pi}{5} = \omega^2$

$k=3 \quad z = \text{cis } \frac{6\pi}{5} = \text{cis} -\frac{4\pi}{5}$

$k=4 \quad z = \text{cis } \frac{8\pi}{5} = \text{cis} -\frac{2\pi}{5} = \bar{\omega}$

ii) $(z^5 - 1) = (z-1)\left((z-\omega)(z-\bar{\omega})(z-\omega^2)(z-\bar{\omega}^2)\right)$

$= (z-1)\left(z^2 - (\omega+\bar{\omega})z + \omega\bar{\omega}\right)\left(z^2 - (\omega^2+\bar{\omega}^2)z + \omega^2\bar{\omega}^2\right)$

$= (z-1)\left(z^2 - 2\cos\frac{2\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5}z + 1\right)$

iii) Sum of roots $= \frac{-b}{a} = 0$

$1 + \omega + \bar{\omega} + \omega^2 + \bar{\omega}^2 = 0$

$1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = 0$

$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$

e) $z = \cos\theta + i\sin\theta$

$z^{-1} = (\cos\theta + i\sin\theta)^{-1}$

$= \cos -\theta + i\sin -\theta$

$z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos -n\theta + i\sin -n\theta$

(by De Moivre's Theorem)

But $\cos -n\theta = \cos n\theta$

& $\sin -n\theta = -\sin n\theta$

$\therefore z^n + z^{-n}$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

ii) $\left(z + \frac{1}{z}\right)^4 = z^4 + 4z^3z^{-1} + 6z^2z^{-2} + 4zz^{-3} + z^{-4}$

$= (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$

$= 2\cos 4\theta + 8\cos 2\theta + 6$

But $z + \frac{1}{z} = 2\cos\theta$

$\therefore \left(z + \frac{1}{z}\right)^4 = 2^4\cos^4\theta$

$\therefore 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$