GOSFORD HIGH SCHOOL.
Extension 2 Mathematics.
HSC Assessment Task 1 December 2008.

Time Allowed: 2 hours.
Show all necessary working.
Start each section on a new page.

Section 1: Complex Numbers

Question 1.

(a) If $z=\sqrt{3}+i$ find

(i) $|z|$ (1)

(ii) $\arg(z)$ (1)

(iii) z^2 (2)

(iv) $\frac{1}{z}$ (2)

(v) iz (1)

(b) Plot $z, \bar{z}, z^2, \frac{1}{z}, iz$ on an Argand Diagram labelling them as P, Q, R, S, and T. (2)

Question 2.

(a) Express $z=1-\sqrt{3}i$ in mod-arg form. (2)

(b) Hence find z^6 in the form $a+ib$ where a and b are real. (2)

Question 3.

Sketch the region on an Argand Diagram for which

$|z-1| \leq 4$, $\frac{-\pi}{6} \leq \arg(z) \leq \frac{\pi}{6}$, $z+\bar{z} \geq 6$ hold simultaneously. (3)

Question 4.

The quadratic equation $z^2+(1+i)z+k=0$ has a root of $1-2i$. Find, in the form $a+ib$, the value of k and the other root of the equation. (3)

Question 5.

If $\omega = z^2+\bar{z}^2$ find and draw a neat sketch of the locus of ω given that $\omega = 0$ (4)

Question 6.

(a) Show that for any two complex numbers z_1 & z_2 that:

$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ (2)

(b) Sketch the locus of z where $\arg\left(\frac{-z}{i}\right) = \arg\left(\frac{1}{z}\right)$. (3)

Question 7.

(a) Find the five fifth roots of 1 and indicate their position on an Argand Diagram. (4)

(b) If ω is one root of $z^5 = 1$, find the value of $\omega + \omega^2 + \omega^3 + \omega^4$. (2)

Question 8.

(a) Use DeMoivre's Theorem with $n = 2$ to show that:

$\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$. (2)

(b) Hence express $\cos\frac{2\pi}{n}$ & $\sin\frac{2\pi}{n}$ in terms of $\cos\frac{\pi}{n}$ & $\sin\frac{\pi}{n}$. (1)

(c) Using the result of (b) prove that:

$(1 + \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n})^n = -2^n \cos^n\frac{\pi}{n}$. (4)

Question 9.

(a) Show that $\frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}$ and hence that $\tan\frac{5\pi}{12} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$. (2)

Find a similar expression for $\tan\frac{\pi}{12}$ (2)

(b) Express $\sqrt{-6i}$ in the form $a+ib$ where a and b are real. (2)

(c) Solve $z^2+(1+i)z+2i=0$ expressing the roots in the form $x+iy$ where x and y are real. (2)

(d) If these two roots are z_1 & z_2 prove that:

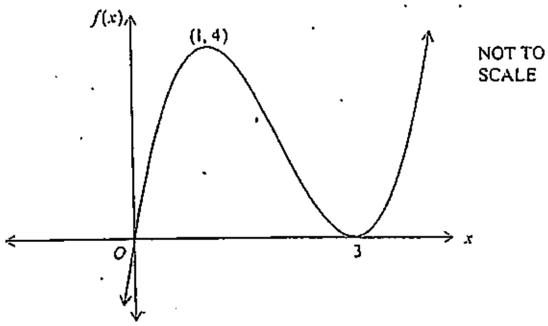
(i) $|z_1| = |z_2| = \sqrt{2}$. (1)

(ii) $\arg(z_1) + \arg(z_2) = \frac{\pi}{2}$. (2)

Section 2: Graphs (Start a new page.)

Question 1.

The function defined by $f(x) = x(x-3)^2$ is drawn below.



Draw separate, one-third page sketches, of each of the following:

(a) $y = |\ f(x)\ |$ (2)

(b) $y = f(|x|)$ (2)

(c) $y = f(-x)$ (2)

(d) $y = \dfrac{1}{f(x)}$ (2)

(e) $y^2 = f(x)$ (2)

Question 2.

Draw a neat sketch of $y = 1 + x^2$ and hence sketch on separate, one-third page diagrams, each of the following:

(a) $y = \dfrac{1}{1+x^2}$ (2)

(b) $y = \dfrac{1+x^2}{x}$ (2)

(c) $y = \left|\dfrac{1+x^2}{x}\right|$ (2)

(d) $y = \sqrt{\dfrac{1+x^2}{x}}$ (2)

Question 3.

(a) Sketch the graphs of $f(x) = x - 2$ and $g(x) = \dfrac{3}{x+2}$ on the same number plane. (2)

(b) Show that $\dfrac{x^2-1}{x+2} = x - 2 + \dfrac{3}{x+2}$ (2)

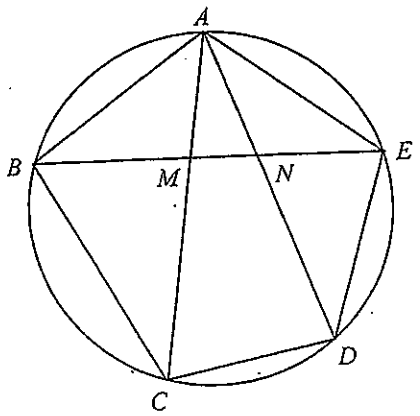
(c) By considering the sum of $f(x)$ & $g(x)$ sketch the graph of $y = \dfrac{x^2-1}{x+2}$ (2)

(d) Hence solve the inequality $\dfrac{x^2-1}{x+2} \le 0$ (2)

Section 3: Circle Geometry (Start a new page.)

Question 1.

ABCDE is a pentagon inscribed in a circle. AB = AE. BE meets AC and AD at M and N respectively.



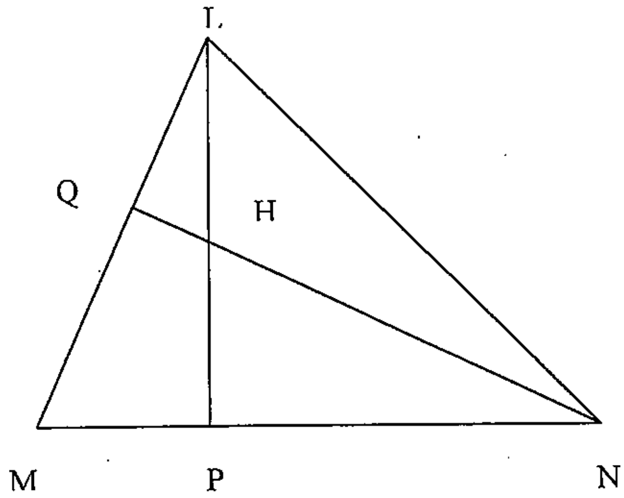
(a) Show that $\angle BEA = \angle ACE$. (2)

(b) Hence show CDMN is a cyclic quadrilateral. (3)

Question 2.

In an acute angled triangle with vertices L, M and N the foot of the perpendicular from L to MN is P and the foot of the perpendicular from N to LM is Q. The lines LP and QN intersect at H.

(P.T.O.)



(a) Prove that $\angle PHM = \angle PQM$. (2)

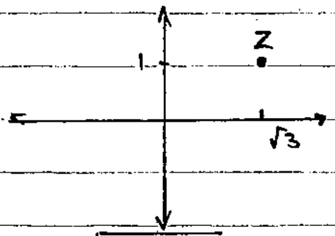
(b) Prove that $\angle PHM = \angle LNM$. (2)

(c) Produce MH to meet LN at R. Prove that MR $\perp$ LN. (3)

SOLUTIONS.

COMPLEX NUMBERS:

Q1 a) $z = \sqrt{3} + i$



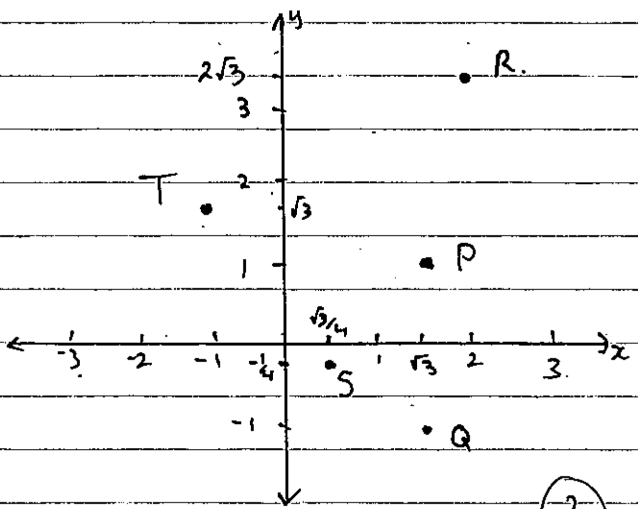
(i) $|z| = \sqrt{\sqrt{3}^2 + 1^2}$
$= 2.$ (1)

(ii) $\arg(z) = \tan^{-1}(1/\sqrt{3})$
$= \pi/6$ (1)

(iii) $z^2 = (\sqrt{3} + i)^2$
$= 3 + 2\sqrt{3}i + i^2$
$= 2 + 2\sqrt{3}i$ (2)

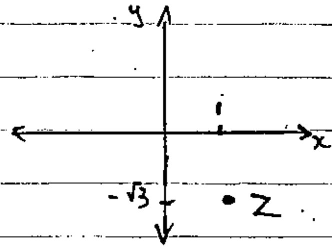
(iv) $\frac{1}{z} = \frac{1}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i}$
$= \frac{\sqrt{3}-i}{4}$ (2)

(v) $iz = i(\sqrt{3} + i)$
$= \sqrt{3}i + i^2$
$= -1 + \sqrt{3}i$ (1)



(2)

Q2 a) $z = 1 - \sqrt{3}i$

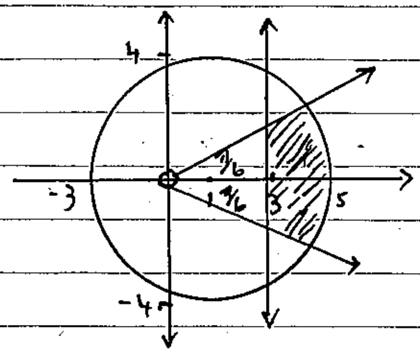


$|z| = 2$, $\arg(z) = -\frac{\pi}{3}$

$\therefore z = 2 \text{ cis} \left(-\frac{\pi}{3}\right)$ (2)

b) $z^6 = \left[2 \text{ cis}\left(-\frac{\pi}{3}\right)\right]^6$
$= 2^6 \text{ cis} -\frac{6\pi}{3}$
$= 64 \text{ cis} -2\pi$
$= 64 \text{ cis } 0$
$= 64 (\cos 0 + i \sin 0)$
$= 64 (1 + 0i)$
$= 64 + 0i$ (2)

Q3. If $z + \bar{z} \geq 6$
$x + iy + x - iy \geq 6$
$2x \geq 6$
$x \geq 3$



(3)

Q4.

$$z^2 + (1+i)z + k = 0$$

When $z = 1-2i$

$$(1-2i)^2 + (1+i)(1-2i) + k = 0$$

$$-3 - 4i + 3 - i + k = 0$$

$$-5i + k = 0$$

$$k = 5i. \quad (2)$$

If the roots are α, β

$$\alpha + \beta = -(1+i)$$

$$(1-2i) + \beta = -1 - i$$

$$\beta = -2 + i. \quad (1)$$

Q5. Let $z = x + iy$

$$w = (x+iy)^2 + (x-iy)^2$$

$$= x^2 - y^2 + 2xyi + x^2 - y^2 - 2xyi$$

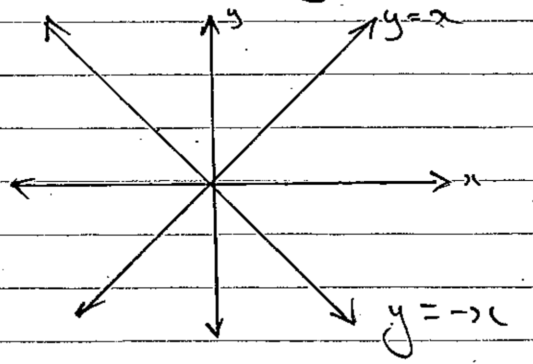
$$= 2x^2 - 2y^2$$

$$\therefore 2x^2 - 2y^2 = 0$$

$$x^2 - y^2 = 0$$

$$(x+y)(x-y) = 0 \quad (3)$$

$\therefore y = x$ or $y = -x$.



(1)

Q6. a) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\frac{[\cos\theta_1 + i\sin\theta_1]}{[\cos\theta_2 + i\sin\theta_2]} \times \frac{[\cos\theta_2 - i\sin\theta_2]}{[\cos\theta_2 - i\sin\theta_2]}$$

$$= \frac{r_1}{r_2}\left[\frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}\right]$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$$\therefore \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$$

$$= \arg z_1 - \arg z_2 \quad (2)$$

b) If $\arg\left(\frac{-z}{i}\right) = \arg\left(\frac{1}{z}\right)$

$$\arg(-z) - \arg(i) = \arg(1) - \arg(z)$$

$$\arg(-1) + \arg(z) - \arg(i) = \arg(1) - \arg(z)$$

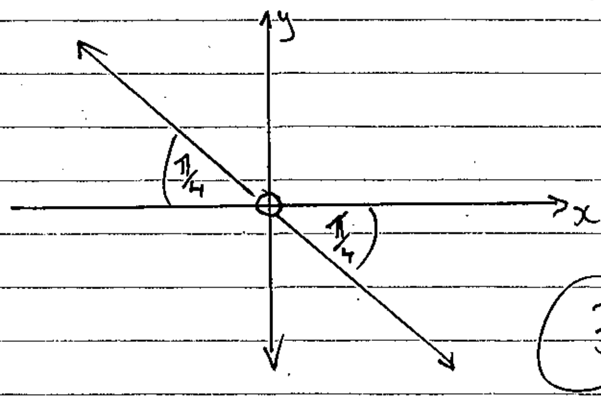
$$\arg(-1) + 2\arg(z) = 0 + \frac{\pi}{2}$$

$$\pm\pi + 2\arg(z) = \frac{\pi}{2}$$

$$2\arg(z) = \mp\pi + \frac{\pi}{2}$$

$$2\arg(z) = -\frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\arg(z) = -\frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$



(3)

Q7. a) $z^5 = 1$

Now $\cos(0+2k\pi) + i\sin(0+2k\pi)$

& $R(\cos\phi + i\sin\phi) = \sqrt[5]{\cos(0+2k\pi) + i\sin(0+2k\pi)}$

$R^5(\cos 5\phi + i\sin 5\phi) = \cos(0+2k\pi) + i\sin(0+2k\pi)$

$\therefore R^5 = 1 \Rightarrow R = 1$

& $5\phi = 0 + 2k\pi, \quad k: 0, 1, 2, 3, 4.$

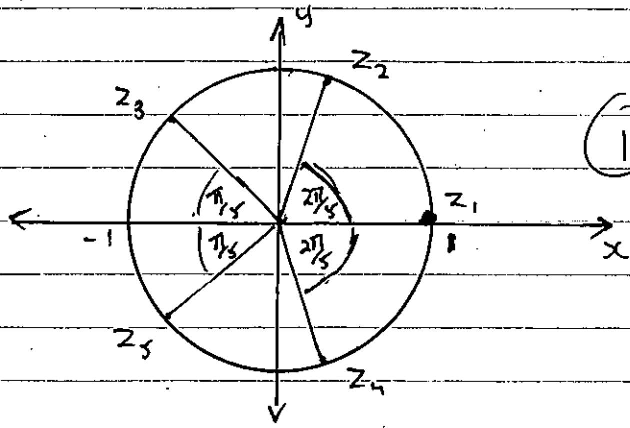
$\therefore$ the five fifth roots are

$\text{cis}\left(\frac{0+2k\pi}{5}\right) \quad k = 0, 1, 2, 3, 4$

ie $1, \text{cis}\frac{2\pi}{5}, \text{cis}\frac{4\pi}{5}, \text{cis}\frac{6\pi}{5}$

& $\text{cis}\frac{8\pi}{5}$ 3

OR $1, \text{cis}\frac{2\pi}{5}, \text{cis}\frac{4\pi}{5}, \text{cis}-\frac{2\pi}{5}, \text{cis}-\frac{4\pi}{5}.$



1

b) Let $\omega = \text{cis}\frac{2\pi}{5}$

$\omega^2 = \text{cis}\frac{4\pi}{5}$

$\omega^3 = \text{cis}\frac{6\pi}{5} = \text{cis}-\frac{4\pi}{5}$

$\omega^4 = \text{cis}\frac{8\pi}{5} = \text{cis}-\frac{2\pi}{5}$

$\therefore 1, \omega, \omega^2, \omega^3, \omega^4$ are the root of $z^5 - 1 = 0$

$\therefore 1 + \omega + \omega^2 + \omega^3 + \omega^4 = -\frac{b}{a}$

$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$ 2

$\therefore \omega + \omega^2 + \omega^3 + \omega^4 = -1$

Q8 a) Using De Moivre's theorem

$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$

But $(\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\sin\theta\cos\theta + i^2\sin^2\theta$

$= \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$

Equating real & imaginary parts

$\cos 2\theta = \cos^2\theta - \sin^2\theta$ 2

& $\sin 2\theta = 2\sin\theta\cos\theta$

b) If $\theta = \pi/n$

$\cos\frac{2\pi}{n} = \cos^2\frac{\pi}{n} - \sin^2\frac{\pi}{n}$ 1

$\sin\frac{2\pi}{n} = 2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$

c) $\left[1 + \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right]^n$

$= \left[1 + \cos^2\frac{\pi}{n} - \sin^2\frac{\pi}{n} + 2i\sin\frac{\pi}{n}\cos\frac{\pi}{n}\right]^n$

$= \left[1 + \cos^2\frac{\pi}{n} - \left(1 - \cos^2\frac{\pi}{n}\right) + 2i\sin\frac{\pi}{n}\cos\frac{\pi}{n}\right]^n$

$= \left[2\cos^2\frac{\pi}{n} + 2i\sin\frac{\pi}{n}\cos\frac{\pi}{n}\right]^n$

$= \left[2\cos\frac{\pi}{n}\left(\cos\frac{\pi}{n} + i\sin\frac{\pi}{n}\right)\right]^n$

$= 2^n\cos^n\frac{\pi}{n}\left(\cos\frac{\pi}{n} + i\sin\frac{\pi}{n}\right)^n$

$= 2^n \cos^n \frac{\pi}{n} \left(\cos n.\frac{\pi}{n} + i \sin n.\frac{\pi}{n} \right)$ by D.M.T

$= 2^n \cos^n \frac{\pi}{n} \left(\cos \pi + i \sin \pi \right)$

$= 2^n \cos^n \frac{\pi}{n} \left(-1 + 0 \right)$

$= -2^n \cos^n \frac{\pi}{n}$.

4

29 a) $\frac{\pi}{6} + \frac{\pi}{4} = \frac{2\pi + 3\pi}{12}$

$= \frac{5\pi}{12}$

$\therefore \tan \frac{5\pi}{12} = \tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$

$= \frac{\tan \frac{\pi}{6} + \tan \frac{\pi}{4}}{1 - \tan \frac{\pi}{6} \tan \frac{\pi}{4}}$

$= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}} \times 1}$

$= \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$

$= \frac{\sqrt{3}+1}{\sqrt{3}-1}$

2

Now $\frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$

$\therefore \tan \frac{\pi}{12} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$= \frac{\tan \frac{\pi}{4} - \tan \frac{\pi}{6}}{1 + \tan \frac{\pi}{4} \tan \frac{\pi}{6}}$

$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$

$= \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}}$

$= \frac{\sqrt{3}-1}{\sqrt{3}+1}$

2

b) Let $\sqrt{-6i} = a + ib$

$\therefore -6i = (a+ib)^2$

$-6i = a^2 - b^2 + 2iab$

$\therefore a^2 - b^2 = 0$ & $2ab = -6$

$a^2 = b^2$

$\pm a = \pm b$

$\therefore -2a^2 = -6$

$a^2 = 3$

$a = \pm\sqrt{3}$

$b = \mp\sqrt{3}$

2

ie. $\sqrt{-6i} = \sqrt{3} - i\sqrt{3}$ OR $-\sqrt{3} + i\sqrt{3}$

c) $z = \frac{-(1+i) \pm \sqrt{(1+i)^2 - 4 \times 1 \times 2i}}{2}$

$= \frac{-(1+i) \pm \sqrt{1 + 2i + i^2 - 8i}}{2}$

$= \frac{-(1+i) \pm \sqrt{-6i}}{2}$

taking $\sqrt{-6i}$ as $\sqrt{3} - i\sqrt{3}$ by convention

$z = \frac{-(1+i) + \sqrt{3} - i\sqrt{3}}{2}$ or $\frac{-(1+i) - (\sqrt{3} - i\sqrt{3})}{2}$

$= \frac{\sqrt{3}-1}{2} + \frac{(-\sqrt{3}-1)}{2}i$ or $\frac{-\sqrt{3}-1}{2} + \frac{(\sqrt{3}-1)}{2}i$

$= \left(\frac{\sqrt{3}-1}{2}\right) - \left(\frac{\sqrt{3}+1}{2}\right)i$ or $-\left(\frac{\sqrt{3}+1}{2}\right) + \left(\frac{\sqrt{3}-1}{2}\right)i$

2

d) (i) Let $z_1 = \left(\frac{\sqrt{3}-1}{2}\right) - \left(\frac{\sqrt{3}+1}{2}\right)i$

& $z_2 = -\left(\frac{\sqrt{3}+1}{2}\right) + \left(\frac{\sqrt{3}-1}{2}\right)i$

$|z_1| = \sqrt{\frac{3 - 2\sqrt{3} + 1 + 3 + 2\sqrt{3} + 1}{4}}$

$= \sqrt{2}$

1

$|z_2| = \sqrt{\frac{3 + 2\sqrt{3} + 1 + 3 - 2\sqrt{3} + 1}{4}}$

$= \sqrt{2}$

ii) $\arg(z_1) = \tan^{-1}\left[\dfrac{-\frac{(\sqrt{3}+1)}{2}}{\frac{(\sqrt{3}-1)}{2}}\right]$

$= -\tan^{-1}\left[\dfrac{\sqrt{3}+1}{\sqrt{3}-1}\right]$

$= -\dfrac{5\pi}{12}$

$\arg(z_2) = \pi - \tan^{-1}\left[\dfrac{\frac{(\sqrt{3}-1)}{2}}{\frac{(\sqrt{3}+1)}{2}}\right]$

$= \pi - \tan^{-1}\left[\dfrac{\sqrt{3}-1}{\sqrt{3}+1}\right]$

$= \pi - \dfrac{\pi}{12}$

$= \dfrac{11\pi}{12}$

2

$\therefore \arg(z_1) + \arg(z_2) = -\dfrac{5\pi}{12} + \dfrac{11\pi}{12}$

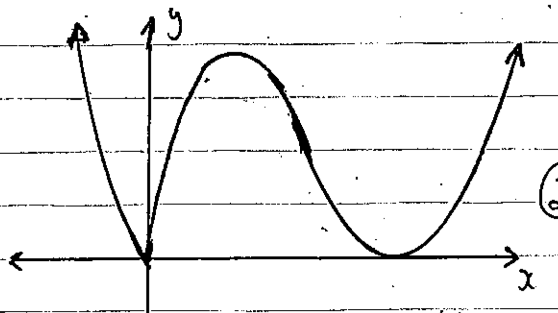
$= \dfrac{\pi}{2}$

SOLUTIONS.

GRAPHS

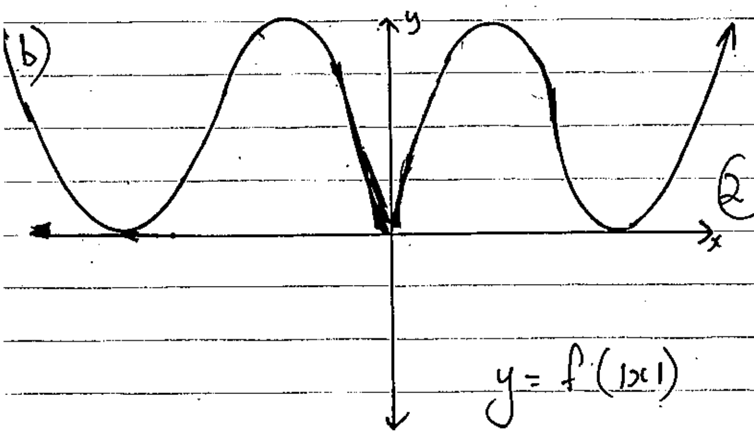
1/

a)



$y = |f(x)|$

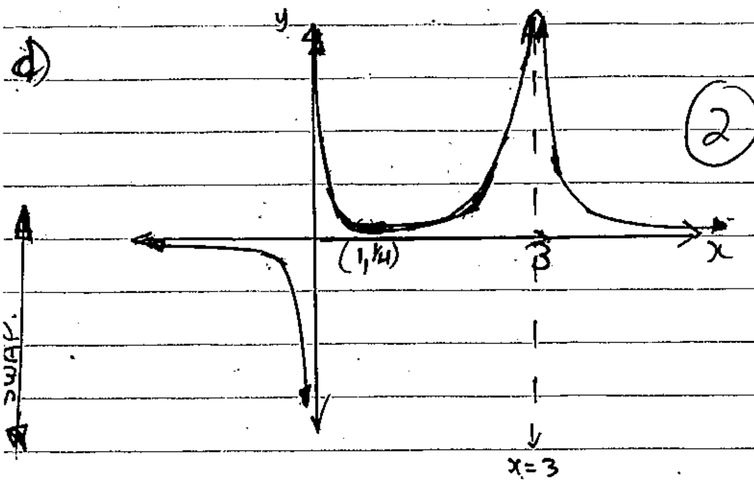
b)



$y = f(|x|)$

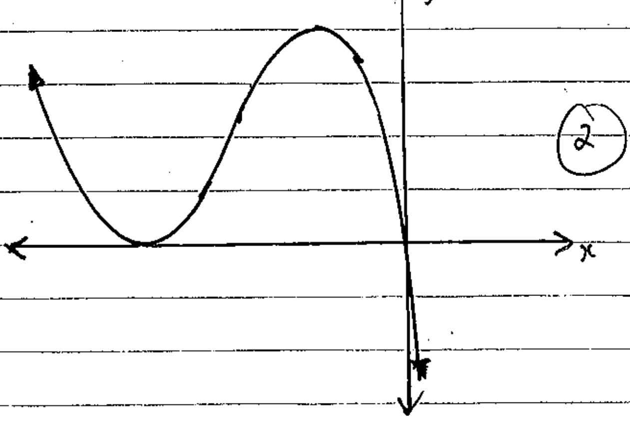
(2)

d)



(2)

d)



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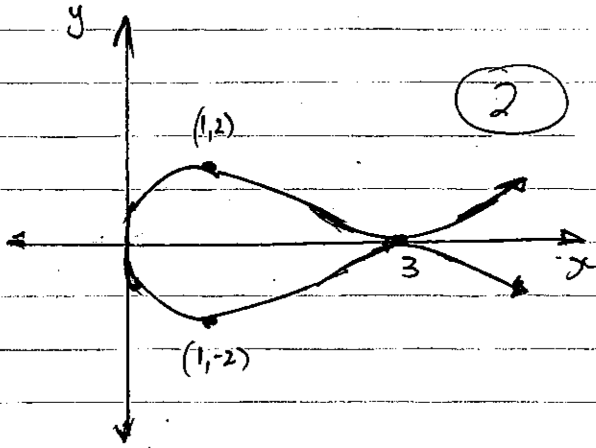
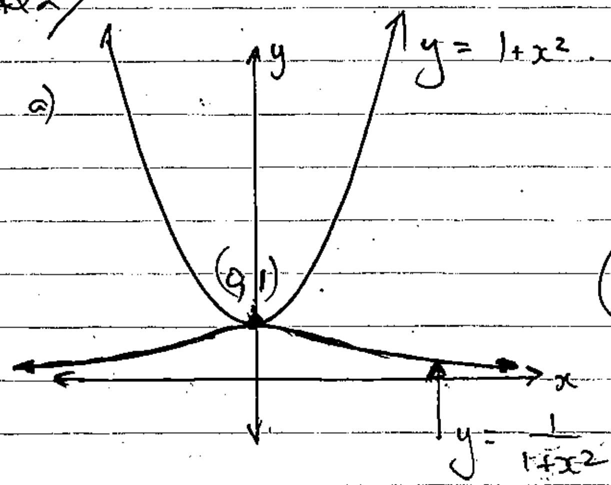
e)



(2)

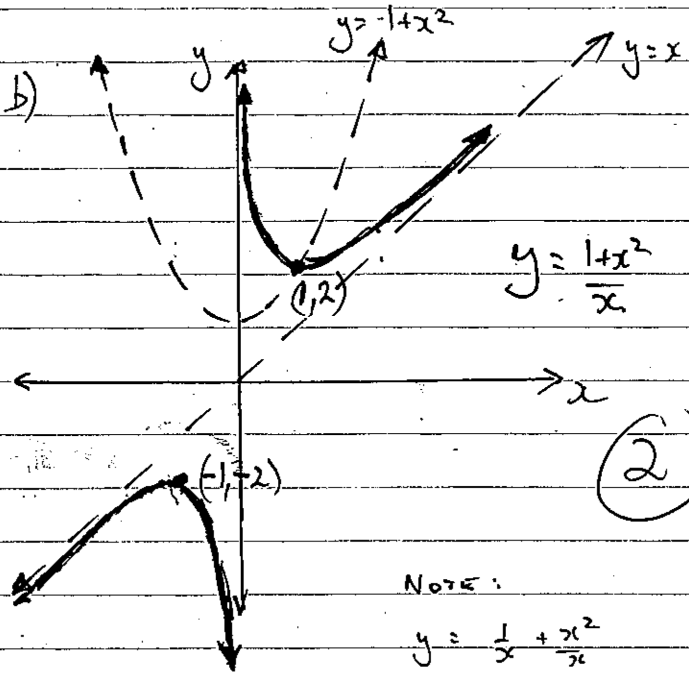
Q2/

a)



(2)

b)



$y = \frac{1+x^2}{x}$

(2)

NOTE:

$y = \frac{1}{x} + \frac{x^2}{x}$

$= \frac{1}{x} + x$

$y' = -1x^{-2} + 1$

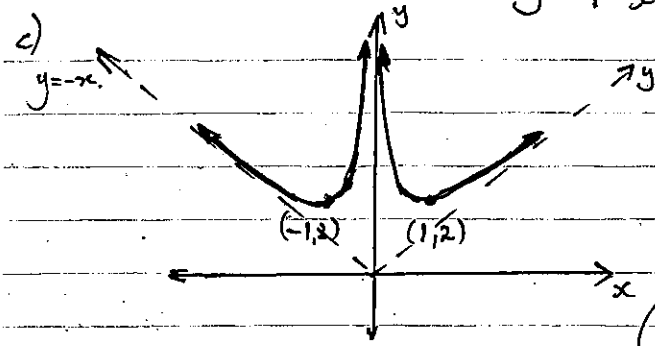
$= \frac{-1}{x^2} + 1$

If $y' = 0$, $\frac{1}{x^2} = 1$, $\therefore x = \pm 1$

$y = \pm 2$,

$y = \left|\frac{1+x^2}{x}\right|$

c)



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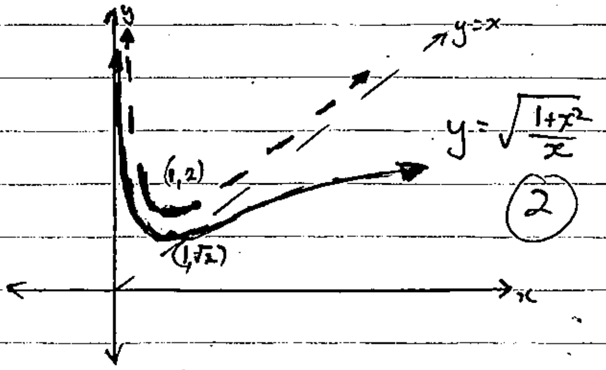
d)



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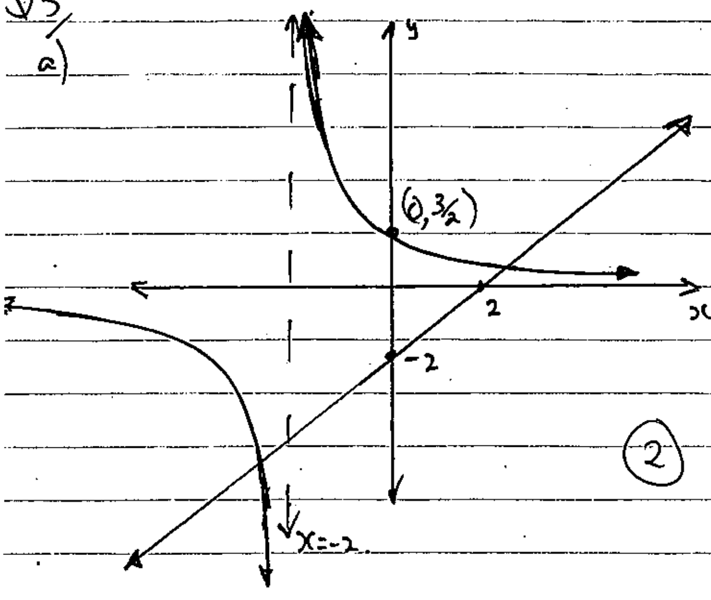
Q3

a)



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b) $RHS = x-2 + \frac{3}{x+2}$

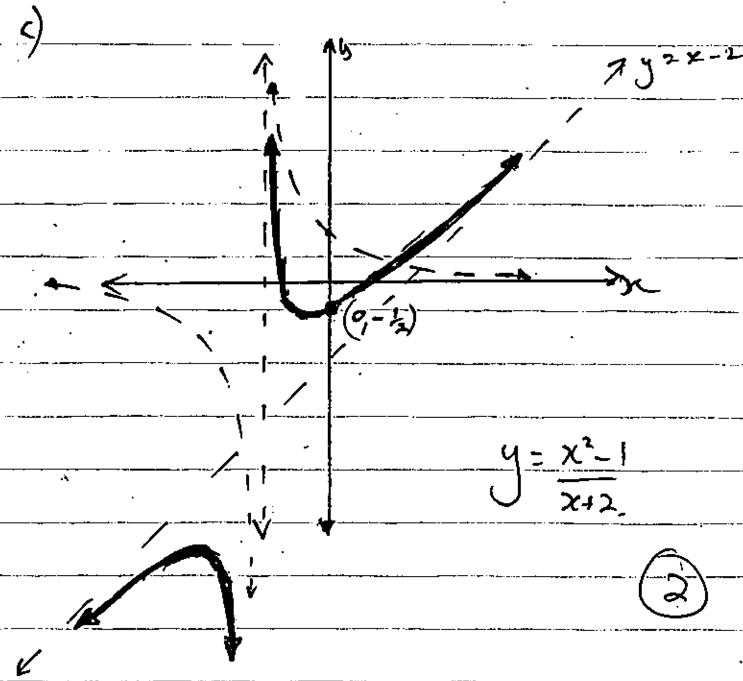
$= \frac{(x-2)(x+2)+3}{x+2}$

$= \frac{x^2-4+3}{x+2}$

$= \frac{x^2-1}{x+2}$

2

c)



2

d) If $\frac{x^2-1}{x+2} = 0$

$x^2-1 = 0$

$x = \pm 1$

2

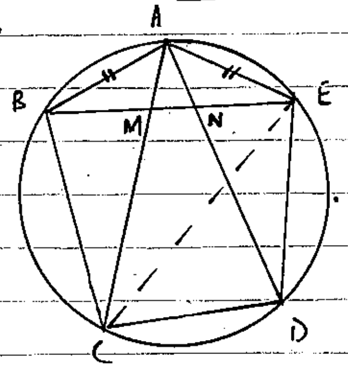
$\therefore \frac{x^2-1}{x+2} \le 0$ when

$x < -2$ or $-1 \le x \le 1$

Solutions.

Circle Geometry

Q1.



a) Join C to E

$\angle BEA = \angle ABE$ (base $\angle$s of a isosc Δ are equal, $AB = AE$)

$\angle ABE = \angle ACE$ ($\angle$s in the same segment standing on AE are equal)

$\therefore \angle BEA = \angle ACE$ (2)

b) $\angle EAD = \angle ECD$ ($\angle$s in the same segment standing on ED are equal)

$\therefore \angle BEA + \angle EAD = \angle ACE + \angle ECD$

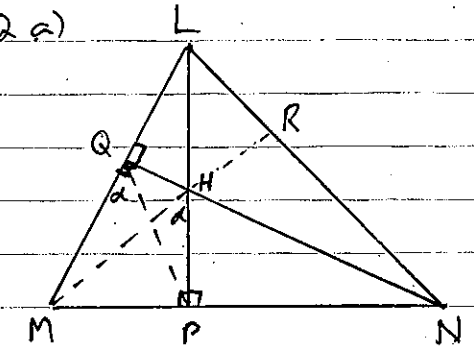
But $\angle BEA + \angle EAD = \angle END$ (ext $\angle$ of a Δ theorem)

Also $\angle ACE + \angle ECD = \angle ACD$.

$\therefore \angle END = \angle ACD$

Hence, CDNM is a cyclic quad as the ext $\angle$ is equal to the opp. int. $\angle$. (3)

Q2 a)



(2)

a) Join M to H & P to Q

Now PMQH is a cyclic quadrilateral as $\angle$s MQH & MPH are opp suppl. $\angle$s.

$\therefore \angle PHM = \angle PQM$ ($\angle$s in the same segment standing on MP are equal). (2)

b) Now LQPN is a cyclic quad as $\angle LQN$ & $\angle LPN$ form a pair of equal $\angle$s in the same segment on arc LN.

Let $\angle PHM = \angle PQM$ be α

$\angle LQP = 180° - \alpha$ (adj. suppl. $\angle$s)

$\therefore \angle LNP = \alpha$ (opp $\angle$s of a cyclic quad are suppl.) (2)

$\therefore \angle PHM = \angle LNM$ (both α)

c) $\angle RHP = 180° - \alpha$ (adj supp $\angle$s)

$\therefore \angle RHP$ & $\angle RNP$ form a pair of opp. suppl. $\angle$s

$\therefore$ RHPN is cyclic.

$\therefore \angle NRH + \angle NPH = 180°$ (opp $\angle$s of a cyclic quad)

ie $\angle NRH + 90 = 180°$ (3)

$\therefore MR \perp LN$.