1) If $u = 3-4i$ and $v = 4-3i$, find each of the following in the form $a+ib$ (4)

(a) $u+iv$ (b) vu

(c) v^2-u^2 (d) $\frac{u}{v}$

2) Find the square roots of $-8-15i$ (3)

3) Show that $(x+1-i)$ is a factor of x^3-2-2i (2)

4) If $z = x+iy$, find x and y when $\frac{2z}{1+i}-\frac{2z}{i}=\frac{5}{2+i}$ (3)

5) (a) Given $z = x+iy$, express $\frac{z-i}{z+1}$ in the form $a+ib$ (3)

(b) If $\frac{z-i}{z+1}$ is real, find the equation of the locus of the point P representing z on an Argand diagram. (1)

6) If $z_1 = r_1 cis\theta_1$ and $z_2 = r_2 cis\theta_2$, prove that $\overline{z_1 z_2} = \overline{z_1}\cdot\overline{z_2}$ (2)

7) Find the modulus and principal argument of each of the complex numbers

(a) -3 (b) $-2i$ (c) $-\sqrt{3}+i$ (3)

8) Express $2\text{cis}\left[-\frac{5\pi}{6}\right]$ in the form $x+iy$ (2)

9) Find the four fourth roots of $8(\sqrt{3}+i)$ (4)

10) Use De Moivre's Theorem to show that

(a) $(1-i\sqrt{3})^9 = -512$ (3)

(b) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ (3)

11) Draw neat diagrams to represent the following (6)

(a) $|z-1+i|=\sqrt{2}$

(b) $\left|\dfrac{z-1}{z+1}\right|\le 1$

(c) $\text{Arg}(z+i)=-\dfrac{3\pi}{4}$

12) (a) Solve $z^3=1$, giving the complex roots in Mod-Arg form. (1)

(b) If ω is one of the complex cube roots of unity, show that

(i) ω^2 is the other complex root. (1)

(ii) $\omega+\omega^2=-1$ (1)

(iii) $(1+2\omega+3\omega^2)(1+2\omega^2+3\omega)=3$ (2)

13) On an Argand diagram P and Q represent the complex numbers z_1 and z_2 respectively. OPQ is an equilateral triangle in which $|z_1|=|z_2|=1$.

(a) Write an expression in terms of z_1 and z_2 for the complex number represented by the vector PQ. (1)

(b) Find $|z_1+z_2|$ (3)

(c) Find a possible value of k if $\dfrac{z_1+z_2}{z_1-z_2}=k$ (2)

14) If $z=\cos\theta+i\sin\theta$, prove that

(a) $\dfrac{2}{z+1}=\dfrac{1+\cos\theta-i\sin\theta}{1+\cos\theta}$ and hence or otherwise prove that (2)

(b) $\dfrac{2}{z+1}=1-i\tan\dfrac{\theta}{2}$ (2)

13) If $15° = 45° - 30°$,

(a) show that $\cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$,

(b) given $\sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$ and $z = \dfrac{1+\sqrt{3}i}{1+i}$, find

(i) $|z|$ (ii) $\operatorname{Arg}(z)$

13) Given $z = \dfrac{1-\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - i\sin 2\theta}$, find

(a) Re(z) (b) Arg(z)

(c) $|z|$ (d) $\bar{z}$

15) Find x in the domain $0 < x < \dfrac{\pi}{2}$, if $\dfrac{\sqrt{2}(\cos x - i\sin x)}{2+i} = \dfrac{1-3i}{5}$

EXT 2 ASSESS TASK #1 SOLUTIONS 2009

PART I Q1-5

1. a) $3-4i+i(4-3i)$
$= 3-4i+4i+3$
$= 6$ ①

b) $(3-4i)(4-3i)$
$= 12-9i-16i-12$
$= -25i$ ①

c) $(4-3i)^2-(3-4i)^2$
$= 16-24i-9-(9-24i-16)$
$= 7-24i+7+24i$
$= 14$ ①

d) $\frac{3-4i}{4-3i} \times \frac{4+3i}{4+3i}$
$= \frac{12+9i-16i+12}{16+9}$
$= \frac{24}{25} - \frac{7i}{25}$ ①

2. let $z^2 = -8-15i$
$\therefore (x+iy)^2 = -8-15i$
$x^2-y^2+2xyi = -8-15i$
$\therefore x^2-y^2 = -8 \quad -(1)$
$2xy = -15 \quad -(2)$
From (2) $y = \frac{-15}{2x}$
In (1) $x^2 - \frac{225}{4x^2} = -8$
$\therefore 4x^4 - 225 = -32x^2$
$4x^4 + 32x^2 - 225 = 0$
$(2x^2+25)(2x^2-9) = 0$
$\therefore x^2 = \frac{-25}{2}, \frac{9}{2}$
(not real.)

$\therefore x = \pm\frac{3}{\sqrt{2}}$ are only real solns.
$\therefore x = \pm\frac{3\sqrt{2}}{2}$ $y = \mp\frac{5\sqrt{2}}{2}$
$\therefore z = \frac{3\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i, \ \frac{-3\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i$

OR use mod arg approach but be careful to write answers in exact form
i.e $z = \sqrt{17}\,cis\left(-90^\circ + \frac{1}{2}\tan^{-1}\left(\frac{15}{8}\right)\right)$
or $\sqrt{17}\,cis\left(90^\circ + \frac{1}{2}\tan^{-1}\left(\frac{15}{8}\right)\right)$

3. Easiest way is to use factor theorem (try to avoid long division!)
Let $P(z) = z^3-2-2i$
if $z+1-i$ is a factor then $P(-1+i) = 0$
$P(-1+i) = (-1+i)^3-2-2i$
$= (-1+i)^2(-1+i)-2-2i$
$= (1-2i-1)(-1+i)-2-2i$
$= 2i+2-2-2i$
$= 0$ as req.
$\therefore z+1-i$ is a factor

OR could solve $z^3 = 2+2i$
best way use mod-arg and verify that $-1+i$ is a root
$\therefore z+1-i$ is a factor.

4. Many methods used but best way
$\frac{2z}{1+i} - \frac{2z}{i} = \frac{5}{2+i}$
Realise denom^s individually
$\frac{2z}{1+i} \times \frac{1-i}{1-i} - \frac{2z}{i} \cdot \frac{i}{i} = \frac{5}{2+i} \times \frac{2-i}{2-i}$
$\frac{2z-2zi}{2} + 2zi = \frac{10-5i}{5}$
$z - zi + 2zi = 2-i$
$z + zi = 2-i$
$z(1+i) = 2-i$
$z = \frac{2-i}{1+i} \times \frac{1-i}{1-i}$
$= \frac{2-2i-i-1}{2}$
$= \frac{1-3i}{2}$
$= \frac{1}{2} - \frac{3}{2}i$
$\therefore x = \frac{1}{2}, \ y = \frac{-3}{2}$.

5. a) $\frac{z-i}{z+1} = \frac{x+iy-i}{x+iy+1}$
$= \frac{x+i(y-1)}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$
On simplifying: ...
$= \frac{x^2+x+y^2-y}{(x+1)^2+y^2} + \frac{i(y-x-1)}{(x+1)^2+y^2}$

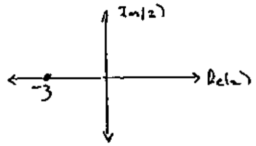
b) If real (purely): then $\text{Im}(z) = 0$
$\therefore y-x-1 = 0$
$y = x+1$ is the locus – but excluding $(1,0)$ and $(0,1)$

PART B Q6-10

6. $z_1z_2 = r_1r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$
$\therefore \overline{z_1z_2} = r_1r_2\left[\cos(\theta_1+\theta_2) - i\sin(\theta_1+\theta_2)\right]$

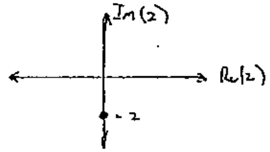
$\overline{z_1}.\overline{z_2} = r_1\left[\cos\theta_1 - i\sin\theta_1\right].r_2\left[\cos\theta_2 - i\sin\theta_2\right]$
$= r_1r_2\left[\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 - i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2\right]$
$= r_1r_2\left[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 - i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right]$
$= r_1r_2\left[\cos(\theta_1+\theta_2) - i\sin(\theta_1+\theta_2)\right]$
$= \overline{z_1z_2}$.

7. a)



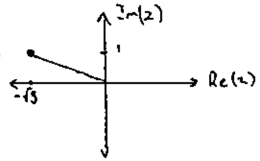
Let $z = -3+0i$
$|z| = 3$
$\text{Arg}(z) = \pi$ or 180°

b)



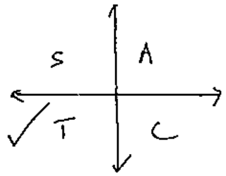
Let $z = 0-2i$
$|z| = 2$
$\text{Arg}(z) = -\frac{\pi}{2}$ or -90°

c)

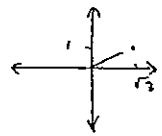


Let $z = -\sqrt{3}+i$
$|z| = \sqrt{\sqrt{3}^2+1^2} = 2$
$\text{Arg}(z) = \frac{5\pi}{6}$ or 150°

8. $2\,cis\left(-\frac{5\pi}{6}\right) = 2\left(\cos -\frac{5\pi}{6} + i.\sin -\frac{5\pi}{6}\right)$
or $2(\cos -150^\circ + i.\sin -150^\circ)$
$= 2\left(-\frac{\sqrt{3}}{2} + i.-\frac{1}{2}\right)$
$= -\sqrt{3} - i$.



9. If $z^4 = 8(\sqrt{3}+i)$
$= 8.\ 2\,cis\frac{\pi}{6}$
$= 2^4\,cis\frac{\pi}{6}$ or $2^4\,cis\,30^\circ$



Let the roots be of the form $r\,cis\,\theta$.

$\therefore r^4 = 2^4$ & $4\theta = \frac{\pi}{6} + 2k\pi \quad k = 0,1,2,3$

$r = 2$.

or $4\theta = 30^\circ + 360^\circ . k$.

$\therefore \theta = \frac{\pi}{24} + \frac{2k\pi}{6}$ or $7.5^\circ + 90^\circ . k \quad k=0,1,2,3$

$\theta = \frac{\pi}{24}, \frac{13\pi}{24}, \frac{25\pi}{24}, \frac{37\pi}{24}$

or $7.5^\circ, 97.5^\circ, 187.5^\circ, 277.5^\circ$

Since $-\pi \le \theta \le \pi$ or $-180^\circ \le \theta \le 180^\circ$

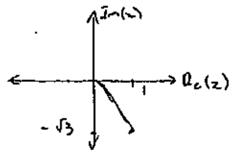
$\theta = \frac{\pi}{24}, \frac{13\pi}{24}, -\frac{23\pi}{24}, -\frac{11\pi}{24}$

or $7.5^\circ, 97.5^\circ, -172.5^\circ, -82.5^\circ$.

$\therefore$ Roots are $2\,\text{cis}\,\frac{\pi}{24}$, $2\,\text{cis}\,\frac{13\pi}{24}$, $2\,\text{cis}\,-\frac{23\pi}{24}$, $2\,\text{cis}\,-\frac{11\pi}{24}$

or $2\,\text{cis}\,7.5^\circ$, $2\,\text{cis}\,97.5^\circ$, $2\,\text{cis}\,-172.5^\circ$, $2\,\text{cis}\,-82.5^\circ$,

10 a)



$1 - i\sqrt{3} = 2\,\text{cis}\,-\frac{\pi}{3}$.

$\therefore (1 - i\sqrt{3})^9 = \left(2\,\text{cis}\,-\frac{\pi}{3}\right)^9$

$= 2^9\,\text{cis}\,-3\pi$

$= 2^9\,\text{cis}\,-\pi$

$= 2^9 . (-1 + 0i)$

$= 512 . -1$

$= -512$.

b) $(\text{cis}\,\theta)^3 = \text{cis}\,3\theta$

$= \cos 3\theta + i \sin 3\theta \quad (1)$

Also $(\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta\, i^2\sin^2\theta + i^3\sin^3\theta$

$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta) \quad (2)$

Equating the real parts of (1) & (2)

$\therefore \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$

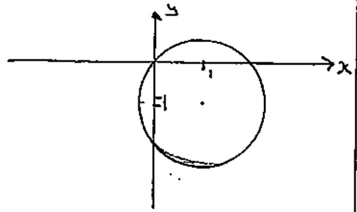
$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

PART C Q11-14

Q11) a) $|z - 1 + i| = \sqrt{2}$

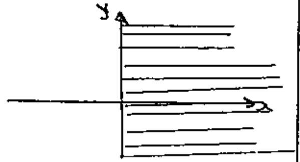
$|z - (1 - i)| = \sqrt{2}$



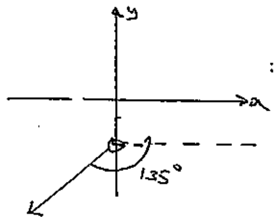
b) $\left|\frac{z-1}{z+1}\right| \le 1$

$\frac{|z-1|}{|z+1|} \le 1$

$|z-1| \le |z+1|$



c) $\arg(z + i) = -135^\circ$



Q12) a) $z^3 = 1$

roots evenly spaced around the unit circle

$z_1 = 1$

$z_2 = \text{cis}\,\frac{2\pi}{3}$

$z_3 = \text{cis}\,-\frac{2\pi}{3}$

b) i) let $\omega = \text{cis}\,\frac{2\pi}{3}$

$\therefore \omega^2 = \left(\text{cis}\,\frac{2\pi}{3}\right)^2$

$= \text{cis}\,\frac{4\pi}{3}$

but $\frac{4\pi}{3} = -\frac{2\pi}{3}$

$\therefore \omega^2 = \text{cis}\,-\frac{2\pi}{3}$ which is the other root.

ii) let roots be $1, \omega, \omega^2$

Sum of the roots $= \frac{-b}{a}$

$\therefore 1 + \omega + \omega^2 = 0$

$\omega + \omega^2 = -1$

iii) $(1 + 2\omega + 3(\omega^2))(1 + 2\omega^2 + 3\omega) = 3$

L.H.S.

$= (1 + 2\omega + 3(-1-\omega))(1 + 2(-1-\omega) + 3\omega)$

$= (-2 - \omega)(-1 + \omega)$

$= 2 - 2\omega + \omega - \omega^2$

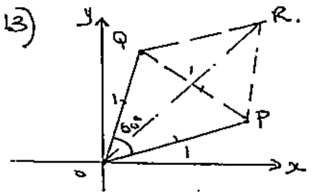
$= 2 - 2\omega + \omega - (-1 - \omega)$

$= 2 - 2\omega + \omega + 1 + \omega$

$= 3$

$=$ RHS.

13)



a) $PQ = z_2 - z_1$

b) $|z_1 + z_2| = OR$

$OR^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos 120^\circ$

$= 3$.

$|z_1 + z_2| = \sqrt{3}$

c) $\frac{z_1 + z_2}{z_1 - z_2} = k$

$z_1 + z_2 = k(z_1 - z_2)$

ie OR $= k$ PQ

ie $\sqrt{3} = k$.

but OPRQ is a rhombus

$\therefore$ the diagonals intersect at right angles

$\therefore$ PQ $= i$ OR

a rotation through 90°

ie $k = i\sqrt{3}$.

14) a) $\frac{2}{z+1} = \frac{1 + \cos\theta - i\sin\theta}{1 + \cos\theta}$

LHS $= \frac{2}{1 + \cos\theta + i\sin\theta} \times \frac{(\cos\theta + 1) - i\sin\theta}{(\cos\theta + 1) - i\sin\theta}$

$= \frac{2(\cos\theta + 1 - i\sin\theta)}{\cos^2\theta + 2\cos\theta + 1 + \sin^2\theta}$

$= \frac{2(\cos\theta + 1 - i\sin\theta)}{2(1 + \cos\theta)}$

$= \frac{1 + \cos\theta - i\sin\theta}{1 + \cos\theta} = \text{RHS}$

b) $\frac{2}{z+1} = 1 - i\tan\frac{\theta}{2}$

LHS $= \frac{1 + \cos\theta - i\sin\theta}{1 + \cos\theta}$

$= \frac{1 + 2\cos^2\frac{\theta}{2} - 1 - 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1 + 2\cos^2\frac{\theta}{2} - 1}$

$= \frac{2(\cos^2\frac{\theta}{2} - i\sin\frac{\theta}{2}\cos\frac{\theta}{2})}{2\cos^2\frac{\theta}{2}}$

$= \frac{\cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} - \frac{i\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}$

$= 1 - i\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$

$= 1 - i\tan\frac{\theta}{2}$