GOSFORD HIGH SCHOOL

2010/2011 HIGHER SCHOOL CERTIFICATE

MATHEMATICS EXTENSION 2

ASSESSMENT TASK 1

Time Allowed – 60 minutes

- All necessary working should be shown.
- Full marks may not be awarded for unnecessarily untidy work or work that is poorly organised.
- Students must begin each new question on a new sheet of paper.
- Questions will be collected separately at the conclusion of the assessment task.
- All questions are to be attempted and are of equal value.

Question 1 - (15 marks) – Use a separate sheet of paper

Marks

a. Let $z = 3 + 4i$ $and\ w = -1 + i$

Express the following in the form $a + ib$, where a, b are real numbers:

i.	$z - w$	1
ii.	iz	1
iii.	$z + \overline{z}$	1
iv.	$\frac{z}{w}$	2
v.	w^4	2

b. If $\propto = \dfrac{5\ (\cos\frac{\pi}{3} + \ isin\ \frac{\pi}{3})}{2\ (\cos\frac{2\pi}{3} + \ isin\frac{2\pi}{3})}$ 1

Express $\propto$ in mod - arg form.

c. Let $\beta = -2 + 2i\sqrt{3}$

i. Find the exact value of $|\beta|$ and $arg\beta$. 2

ii. Hence, express β^5 in the form $a + ib$, where a, b real numbers. 2

d. Find the complex square roots of $7 + 6\sqrt{2}\,i$. 3

End of Question 1

Question 2 - (15 marks) – Use a separate sheet of paper

Marks

a.

i. On an Argand diagram show vectors representing z_1, z_2, $z_1 + z_2$ *and* $z_1 - z_2$. 1

ii. If z_1 *and* z_2 are two complex numbers such that 2

$\frac{z_1 + z_2}{z_1 - z_2} = 2i$ show that $|z_1| = |z_2|$.

b. On an Argand diagram, shade the region defined by: 2

$Im(z) \leq 2$ AND $\frac{\pi}{3} \leq \arg(z + i) \leq \frac{\pi}{2}$

c.

i. Sketch the graph on the Argand diagram specified by: 2

$|z - 1 - \sqrt{3}i| = 2$

ii. Hence, find the maximum value of $|z|$. 1

d. Find the three cube roots of $8i$ (express answers in the form $a + ib$). 2

e.

i. Find all solutions of the equation $z^5 = 1$, giving your answers in modulus – argument form. 2

ii. Show on a diagram that the points representing the five roots of this equation, on an Argand diagram, form the vertices of a regular pentagon. 1

iii. Hence, show that the perimeter of this regular pentagon is $10 \sin\frac{\pi}{5}$ *units*. 2

End of Question 2

Question 3 - (15 marks) – Use a separate sheet of paper

Marks

a. Sketch the region in the complex plane where the inequalities $|z| > 2$ and $2 \leq z + \bar{z} \leq 6$ hold simultaneously. 3

b. Given $z = 3 + 4i$, find all possible co-ordinates of w, so that the origin $0, z$ *and* w form a right angled isosceles triangle (right angled at z) on the Argand diagram. 3

c. The point P on the Argand diagram represents the complex number $|z - 1| = Re\ (z)$

i. Find the equation of the locus of P in terms of x *and* y. 3

ii. Describe the locus geometrically, giving at least <u>two</u> features of the curve.

d. Find the locus in cartesian form of z if: 3

$\arg(z + 4) = \arg(z - 4) - \frac{\pi}{4}$.

e. Find the Cartesian equation of the locus of w if $w = \frac{z-2}{z}$ and given that $|z| = 2$. Describe the locus geometrically. 3

End of Assessment Task

1a. (i) $z - w = 3+4i - (-1+i)$

$= 4+3i$

(ii) $iz = i(3+4i)$

$= -4+3i$

(iii) $z + \bar{z} = 3+4i+3-4i$

$= 6$

(iv) $\dfrac{z}{w} = \dfrac{3+4i}{-1+i} \times \dfrac{-1-i}{-1-i}$

$= \dfrac{-3-3i-4i+4}{1+1}$

$= \dfrac{1-7i}{2}$

$= \frac{1}{2} - \frac{7}{2}i$

(v) $w^4 = (w^2)^2$

$= \left[(-1+i)^2\right]^2$

$= (1-2i-1)^2$

$= (-2i)^2$

$= 4i^2$

$= -4$.

b. $\alpha = \frac{5}{2}\,\text{cis}\left(\frac{\pi}{3} - \frac{2\pi}{3}\right)$

$= \frac{5}{2}\,\text{cis}\left(-\frac{\pi}{3}\right)$

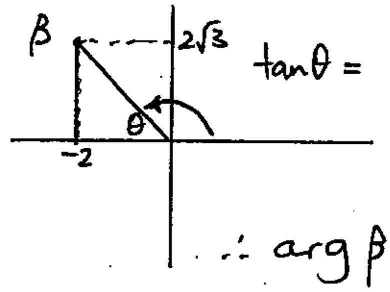
c. (i) $\beta = -2+2\sqrt{3}\,i$

$|\beta| = \sqrt{(-2)^2+(2\sqrt{3})^2}$

$= \sqrt{4+12}$

$= \sqrt{16}$

$= 4$



$\tan\theta = \dfrac{2\sqrt{3}}{2} = \sqrt{3}$

$\therefore \theta = \dfrac{\pi}{3}$

$\therefore \arg\beta = \dfrac{2\pi}{3}$

(ii) $\beta = 4\,\text{cis}\left(\frac{2\pi}{3}\right)$

$\beta^5 = \left[4\,\text{cis}\left(\frac{2\pi}{3}\right)\right]^5$

$= 1024\,\text{cis}\left(\frac{10\pi}{3}\right)$

$= 1024\,\text{cis}\left(-\frac{2\pi}{3}\right)$

$= -512 - 512\sqrt{3}\,i$

d. let $z = \sqrt{7+6\sqrt{2}\,i}$

$z^2 = 7+6\sqrt{2}i$

$\therefore (x+iy)^2 = 7+6\sqrt{2}\,i$

$x^2 - y^2 + 2xyi = 7+6\sqrt{2}i$

Equating real, imaginary parts

$x^2 - y^2 = 7 \quad -(1)$

$2xy = 6\sqrt{2} \quad -(2)$

From (2) $y = \dfrac{3\sqrt{2}}{x}$

In (1) $x^2 - \left(\dfrac{3\sqrt{2}}{x}\right)^2 = 7$

$x^2 - \dfrac{18}{x^2} = 7$

$x^4 - 7x^2 - 18 = 0$

$(x^2+2)(x^2-9) = 0$

Since x is real: $x = \pm 3$, $y = \pm\sqrt{2}$

the square roots of $7+6\sqrt{2}i$ are

$3+\sqrt{2}i$, $-3-\sqrt{2}i$

d) By inspection:

one root is $-2i$ (since $(-2i)^3 = 8i$)

$-2i = 2\,\text{cis}\left(-\frac{\pi}{2}\right)$ and the two other roots are evenly spaced around Argand diagram

i.e $z_1 = -2i$

$z_2 = 2\,\text{cis}\left(\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

$= \sqrt{3} + i$

$z_3 = 2\,\text{cis}\left(\frac{5\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

$= -\sqrt{3} + i$

$\therefore$ 3 roots are: $-2i,\ \sqrt{3}+i,\ -\sqrt{3}+i$

e)(i) If $z^5 = 1$, by inspection

$z_1 = 1$ and other 4 are evenly spaced around Argand Diagram.

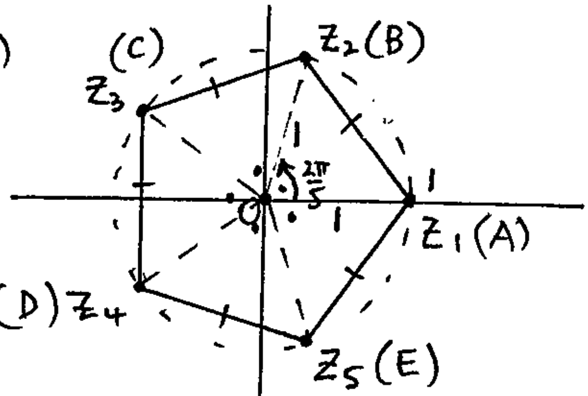
i.e $z_2 = \text{cis}\,\frac{2\pi}{5}$

$z_3 = \text{cis}\,\frac{4\pi}{5}$

$z_4 = \text{cis}\left(-\frac{4\pi}{5}\right)$

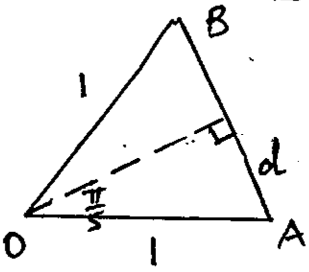
$z_5 = \text{cis}\left(-\frac{2\pi}{5}\right)$

(ii)



$\therefore$ A regular pentagon is formed

(iii) Consider Δ AOB:



$\frac{d}{1} = \sin\frac{\pi}{5}$

$\therefore d = \sin\frac{\pi}{5}$

$\therefore AB = 2\sin\frac{\pi}{5}$

$\therefore$ Perimeter of pentagon $= 5 \times AB$

$= 10\sin\frac{\pi}{5}$ units as req.

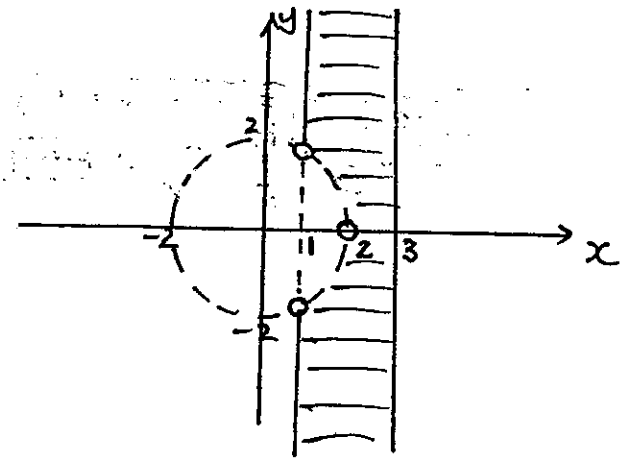
3) a) $|z| > 2$ → outside circle centre $(0,0)$ radius 2 units

$2 \le z + \bar{z} \le 6$ → $z + \bar{z} = x+iy+x-iy$

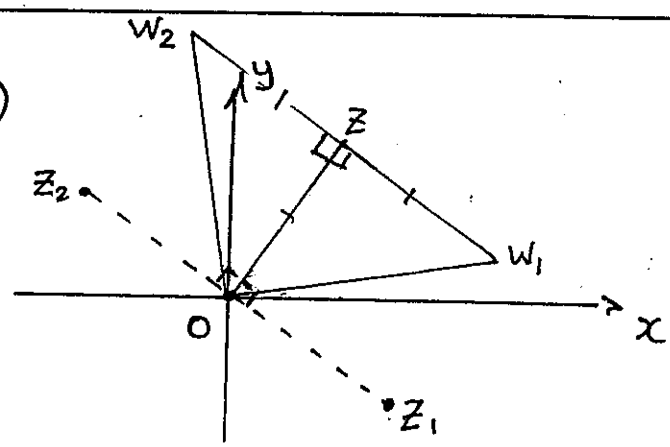
$= 2x$

i.e $2 \le 2x \le 6$

$1 \le x \le 3$

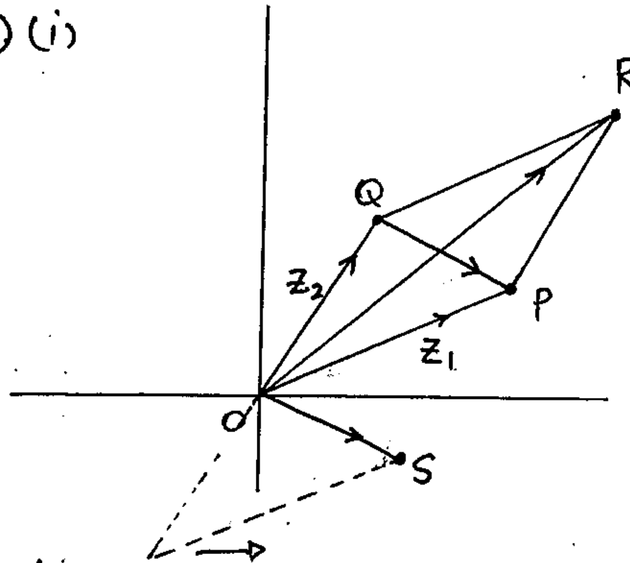


b)



Q2/

a) (i)



Let $z_1 = \overrightarrow{OP}$

$z_2 = \overrightarrow{OQ}$

$\therefore\ z_1 + z_2 = \overrightarrow{OR}$

$z_1 - z_2 = \overrightarrow{OS} = \overrightarrow{QP}$

(ii) $\dfrac{z_1 + z_2}{z_1 - z_2} = 2i$

$\therefore\ z_1 + z_2 = 2i \times (z_1 - z_2)$

So $\overrightarrow{OR}$ is obtained by rotating $\overrightarrow{QP}$ $90°$ in an anticlockwise direction including an enlargement by a factor of 2.

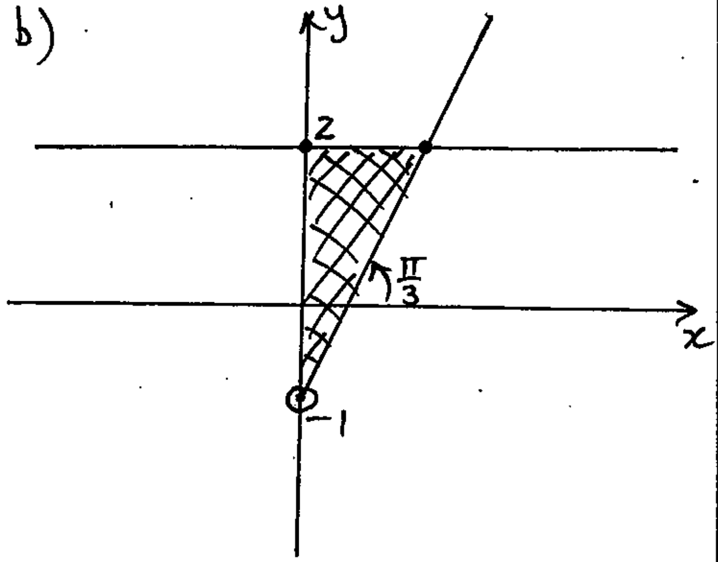
$\therefore$ diagonals of parallelogram OPRQ are at right angles to each other

$\therefore$ OPRQ is a rhombus

$\therefore\ \overrightarrow{OP} = \overrightarrow{OQ}$

$\therefore\ |z_1| = |z_2|$ as required

b)

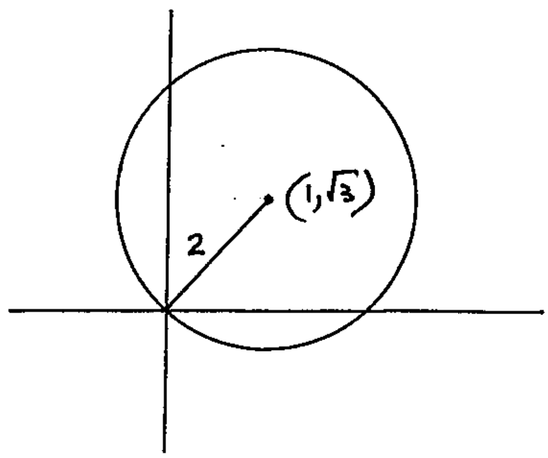


c) (i) $|z - (1+\sqrt{3}i)| = 2$

Circle radius 2 units, centre $(1, \sqrt{3})$ and

$|1+\sqrt{3}i| = \sqrt{1^2 + (\sqrt{3})^2} = 2$

$\therefore$ Circle passes through $(0,0)$



(ii) Max value of $|z|$ will be the diameter of circle

i.e. $2 \times 2 = 4$ units.

There are two possible positions for $w \to w_1, w_2$

$w_1 = z + z_1$

where $z_1 = z \times -i$

$= (3+4i) \times -i$

$= -3i + 4$

$= 4 - 3i$

$\therefore w_1 = 3+4i + 4-3i$

$= 7+i$

$w_2 = z + z_2$

where $z_2 = z \times i$

$= (3+4i) \times i$

$= 3i - 4$

$= -4 + 3i$

$\therefore w_2 = 3+4i - 4 + 3i$

$= -1 + 7i$

$\therefore w = 7+i$ or $-1+7i$

c) (i) $|z-1| = \mathrm{Re}(z)$

let $z = x+iy$

$\therefore \sqrt{(x-1)^2+y^2} = x$

$(x-1)^2+y^2 = x^2$

$x^2-2x+1+y^2 = x^2$

$y^2 = 2x-1$

(ii) This is the equation of a parabola.

$y^2 = 4.\frac{1}{2}(x-\frac{1}{2})$

with Vertex $= (\frac{1}{2},0)$
focus $= (1,0)$
Directrix $\Rightarrow x = \frac{1}{2}$
Axis : $y=0$

only 2 required

d) $\arg(z+4) = \arg(z-4) - \frac{\pi}{4}$

$\frac{\pi}{4} = \arg(z-4) - \arg(z+4)$

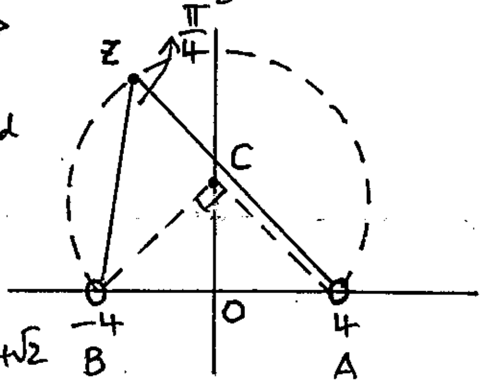
$\therefore \arg\left(\frac{z-4}{z+4}\right) = \frac{\pi}{4}$

and use algebraic approach starting $\mathrm{Re}(z+4) = \mathrm{Im}(z-4)$

or better to use geometric approach

$\arg(z-4) - \arg(z+4) = \frac{\pi}{4}$

implies locus is major arc of circle $\to$



Since $\triangle OCB$ is right angled isosceles

$OC = 4$

$\therefore C = (0,4)$

$\therefore CB = \sqrt{32} = 4\sqrt{2}$

$\therefore$ Locus is major arc of circle

$x^2+(y-4)^2 = 32$ excluding $(-4,0)(4,0)$

e) $w = \frac{z-2}{z}$

$zw = z-2$

$zw - z = -2$

$z(w-1) = -2$

$z = \frac{-2}{w-1}$

$z = \frac{2}{1-w}$

$\therefore |z| = \left|\frac{2}{1-w}\right|$

$\therefore 2 = \left|\frac{2}{1-w}\right|$ (from data)

$2 = \frac{2}{|1-w|}$

$|1-w| = 1$

$\therefore |w-1| = 1$

and locus of w is circle centre $(1,0)$ radius 1 unit

$\therefore (x-1)^2+y^2 = 1$