

GOSFORD HIGH SCHOOL.
Extension 2 Mathematics.
Assessment Task December 2011.

Question 1

- (a) Solve for x over the complex field $5x^2 - 12x + 17 = 0$. 2
- (b) Find the complex factors for $x^2 + 4x + 5$. 1
- (c) Given the complex numbers $A = 3 - 4i$ and $B = 1 + i$,
determine the following in the form $x + iy$.
- (i) $A - B$ 1
- (ii) $A\bar{B}$ 2
- (iii) B^2 2
- (iv) $\frac{A}{B}$ 2
- (d) Find real numbers x and y such that $3x + 2iy - ix + 5y = 7 + 5i$. 2

Question 2

- (a) Given $C = 1 + i$
- (i) Write C in mod-arg form. 2
- (ii) Hence, using De Moivre's theorem find C^4 . 1
- (b) If $z_1 = 3(\cos 40^\circ + i \sin 40^\circ)$ and $z_2 = 4(\cos 80^\circ + i \sin 80^\circ)$
Evaluate $z_1 \cdot z_2$. 2
- (c) Given that $1, \omega, \omega^2$ are the cube roots of unity show that:
 $(1 + 2\omega + 3\omega^2)(1 + 2\omega^2 + 3\omega) = 3$ 2
- (d) If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ show that $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$. 2
- (e) Find the real numbers a and b such that $(a + ib)^2 = 9 + 40i$ 3

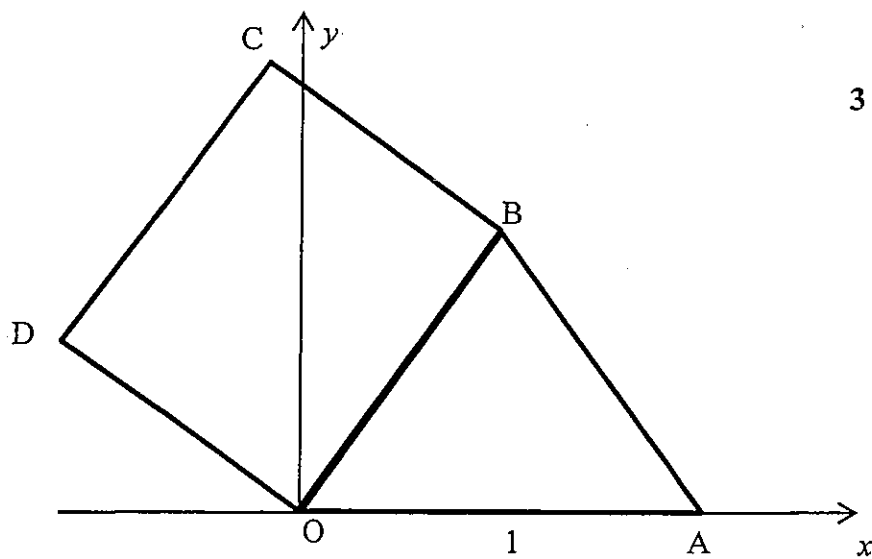
Question 3

Draw a neat sketch of the following.

- a) $\text{Im}(z) = -4$ 1
- b) $|z| \leq 4$ 1
- c) $|z - 3 + 4i| = 5$ 2
- d) $\arg(z+1) = 120^\circ$ 2
- e) $\arg(z-1) - \arg(z+1) = 45^\circ$ 3
- f) $z\bar{z} - 4(z + \bar{z}) = 10$ 3

Question 4

(a)



OAB is an equilateral triangle of side length 1 unit. OBCD is a square.

Find the complex numbers represented by the points B, D and C, in exact terms in the form $x+iy$.

(b) Solve

(i) $z^4 = -1$ writing answers in simplest cartesian form

3

(ii) $z^3 = 2 + 2i$ answers in mod-arg

3

(c) (i) Show that if $z = (\cos \theta + i \sin \theta)$ then $z^n + z^{-n} = 2 \cos n\theta$.

2

(ii) Hence or otherwise solve $z^4 + 2z^3 + 3z^2 + 2z + 1 = 0$.

3

$$\begin{aligned} \frac{1}{(a)} x &= \frac{12 \pm \sqrt{144 - 340}}{10} \\ &= \frac{12 \pm \sqrt{-196}}{10} \\ &= \frac{12 \pm 14i}{10} \\ &= \frac{6 \pm 7i}{5} \end{aligned}$$

$$\begin{aligned} (b) x^2 + 4x + 4 + 1 \\ &= (x+2)^2 + 1 \\ &= (x+2-i)(x+2+i) \end{aligned}$$

$$\begin{aligned} (c) (i) 3-4i - (1+i) \\ &= 2-5i \end{aligned}$$

$$\begin{aligned} (ii) (3-4i)(1-i) \\ &= 3 - 3i - 4i - 4 \\ &= -1 - 7i \end{aligned}$$

$$\begin{aligned} (iii) (1+i)^2 \\ &= 1 + 2i - 1 \\ &= 2i \end{aligned}$$

$$\begin{aligned} (iv) \frac{3-4i}{1+i} \times \frac{1-i}{1-i} \\ &= \frac{3-3i-4i-4}{2} \end{aligned}$$

$$= \frac{-1-7i}{2} = -\frac{1}{2} - \frac{7}{2}i$$

$$\begin{aligned} (d) (3x+5y) + i(2y-x) \\ &= 7 + 5i \end{aligned}$$

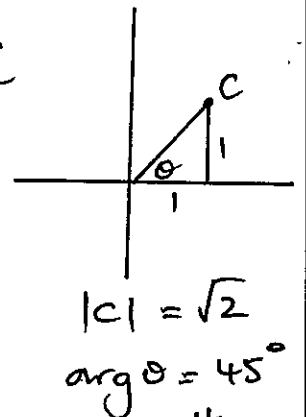
$$\begin{aligned} \therefore 3x+5y &= 7 \quad \text{--- (1)} \\ 2y-x &= 5 \quad \text{--- (2)} \end{aligned}$$

$$\text{(2)} \times 3 \Rightarrow 6y - 3x = 15 \quad \text{--- (3)}$$

$$\begin{aligned} \text{(1)} + \text{(3)} \quad 11y &= 22 \\ y &= 2 \\ x &= -1 \end{aligned}$$

$$\frac{2}{(a)} (i) C = 1+i$$

$$\therefore C = \sqrt{2} \text{cis } 45^\circ$$



$$\begin{aligned} (ii) C^4 &= (\sqrt{2} \text{cis } 45^\circ)^4 \\ &= 4 \text{cis } 180^\circ \\ &= -4 \end{aligned}$$

$$\begin{aligned} (b) z_1 z_2 &= (3 \text{cis } 40^\circ)(4 \text{cis } 80^\circ) \\ &= 12 \text{cis } 120^\circ \\ &= 12 \left[-\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} \right] \\ &= -6 + 6\sqrt{3}i \end{aligned}$$

(c) Since $1, \omega, \omega^2$ are roots of $z^3 - 1 = 0$

$$\therefore 1 + \omega + \omega^2 = 0$$

$$\begin{aligned} \therefore (1 + 2\omega + 3\omega^2) &= (1 + \omega + \omega^2) + (\omega + \omega^2) + \omega^2 \\ &= 0 + (-1) + \omega^2 \\ &= \omega^2 - 1 \end{aligned}$$

$$\begin{aligned} (1 + 2\omega^2 + 3\omega) &= (1 + \omega + \omega^2) + (\omega + \omega^2) + \omega \\ &= 0 + (-1) + \omega \\ &= \omega - 1 \end{aligned}$$

$$\begin{aligned} \therefore \text{LHS} &= (\omega^2 - 1)(\omega - 1) \\ &= \omega^3 - \omega^2 - \omega + 1 \\ &= \omega^3 - (\omega^2 + \omega) + 1 \\ &= 1 - (-1) + 1 \\ &= 3 \\ &= \text{RHS as req.} \end{aligned}$$

$$\begin{aligned} \text{(d) LHS} &= \overline{(x_1 + x_2) + i(y_1 + y_2)} \\ &= (x_1 + x_2) - i(y_1 + y_2) \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \overline{x_1 + iy_1} + \overline{x_2 + iy_2} \\ &= x_1 - iy_1 + x_2 - iy_2 \\ &= (x_1 + x_2) - iy_1 - iy_2 \\ &= (x_1 + x_2) - i(y_1 + y_2) \\ &= \text{RHS} \\ &\text{as req.} \end{aligned}$$

$$\text{(e) } (a + ib)^2 = 9 + 40i$$

$$a^2 - b^2 + 2abi = 9 + 40i$$

$$\therefore a^2 - b^2 = 9 \quad \textcircled{1}$$

$$2ab = 40 \quad \textcircled{2}$$

$$\textcircled{2} \Rightarrow b = \frac{20}{a}$$

$$\text{Sub in } \textcircled{1}: a^2 - \left(\frac{20}{a}\right)^2 = 9$$

$$a^4 - 400 = 9a^2$$

$$a^4 - 9a^2 - 400 = 0$$

$$(a^2 - 25)(a^2 + 16) = 0$$

$$a^2 = 25, -16$$

Since a is real

$$a^2 = 25$$

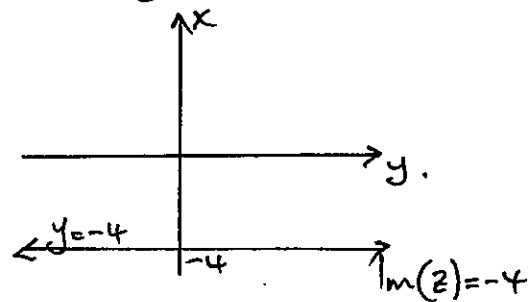
$$\therefore a = \pm 5, b = \pm 4$$

$$\therefore \text{when } a = 5, b = 4$$

$$a = -5, b = -4$$

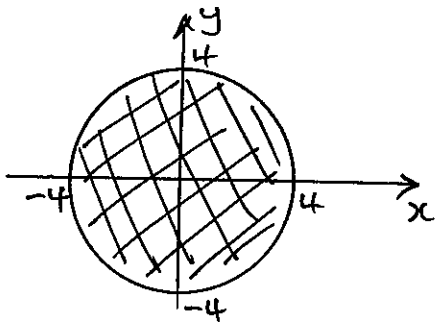
$$3, \text{ a) } \text{Im}(z) = -4$$

$$\therefore y = -4$$



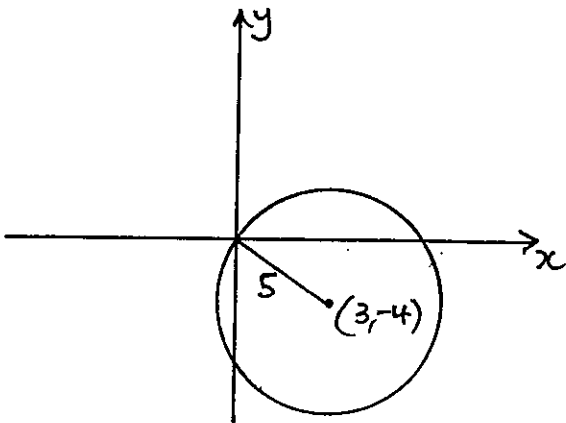
$$\text{b) } |z| \leq 4$$

$$\sqrt{x^2 + y^2} \leq 4 \quad \therefore x^2 + y^2 \leq 16$$



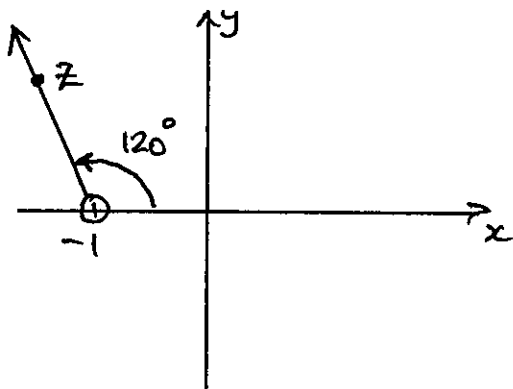
$$(c) |z - (3 - 4i)| = 5$$

Circle Centre $(3, -4)$ Radius = 5

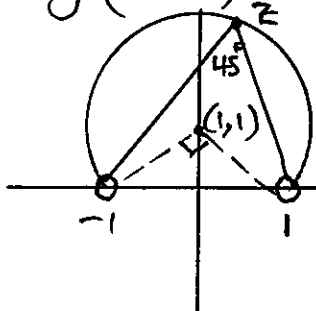


$$(d) \arg(z + 1) = 120^\circ$$

$$\arg(z - (-1)) = 120^\circ$$



$$(e) \arg(z - 1) - \arg(z + 1) = 45^\circ$$



Major arc
of circle
centre
 $(1, 1)$ radius
 $\sqrt{2}$.

$$(f) z\bar{z} - 4(z + \bar{z}) = 10$$

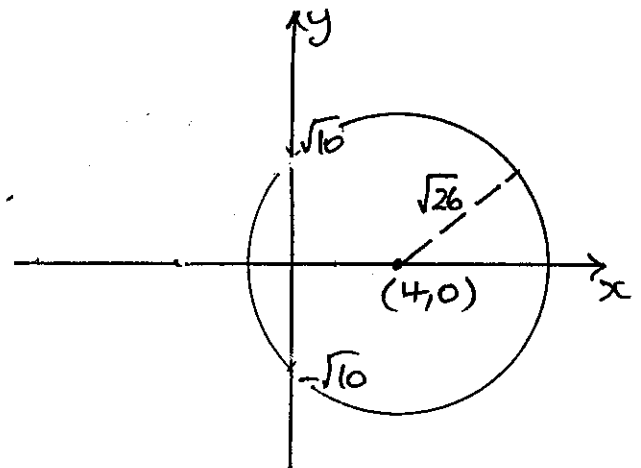
$$(x + iy)(x - iy) - 4(2x) = 10$$

$$x^2 + y^2 - 8x = 10$$

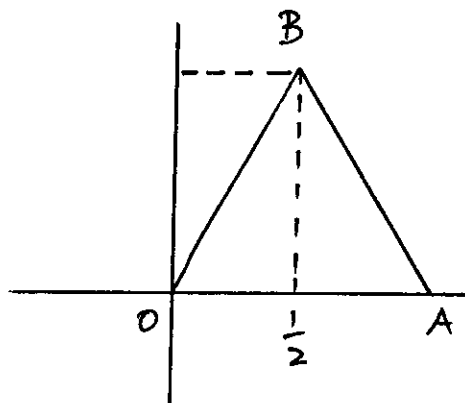
$$x^2 - 8x + 16 + y^2 = 10 + 16$$

$$(x - 4)^2 + y^2 = 26$$

\therefore Circle centre $(4, 0)$
and radius $\sqrt{26}$ units.



4 (a)



$$OB^2 = 1^2 - \left(\frac{1}{2}\right)^2$$

$$= \frac{3}{4}$$

$$\therefore OB = \frac{\sqrt{3}}{2}$$

$$\therefore B = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$\therefore B = \frac{1}{2} + \frac{\sqrt{3}i}{2}$$

Now $OD = i \times OB$

$$\begin{aligned} \therefore D &= i \times \left(\frac{1}{2} + \frac{\sqrt{3}i}{2} \right) \\ &= \frac{1}{2}i - \frac{\sqrt{3}}{2} \\ &= -\frac{\sqrt{3}}{2} + \frac{i}{2} \end{aligned}$$

and $C = B + D$.

$$\begin{aligned} &= \frac{1}{2} + \frac{\sqrt{3}i}{2} + \left(-\frac{\sqrt{3}}{2} + \frac{i}{2} \right) \\ &= \frac{1-\sqrt{3}}{2} + i \left(\frac{\sqrt{3}+1}{2} \right) \end{aligned}$$

(b) (i) $z^4 = -1$

$$\begin{aligned} \therefore (r \operatorname{cis} \theta)^4 &= \operatorname{cis} 180^\circ \\ r \operatorname{cis} 4\theta &= \operatorname{cis} 180^\circ \end{aligned}$$

$$\therefore r=1 \quad \theta = 45^\circ \quad \therefore z_1 = \operatorname{cis} 45^\circ$$

since the roots are equally spaced around unit circle

$$z_2 = \operatorname{cis} 135^\circ$$

$$z_3 = \operatorname{cis} -135^\circ$$

$$z_4 = \operatorname{cis} -45^\circ$$

$$\therefore z_1 = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \quad z_2 = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

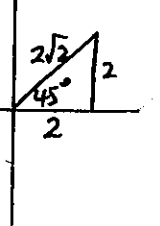
$$z_3 = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \quad z_4 = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

(ii) $z^3 = 2 + 2i$

$$(r \operatorname{cis} \theta)^3 = 2\sqrt{2} \operatorname{cis} 45^\circ$$

$$r^3 \operatorname{cis} 3\theta = 2\sqrt{2} \operatorname{cis} 45^\circ$$

$$\begin{aligned} \therefore r &= \sqrt{2} \quad 3\theta = 45^\circ \\ \therefore \theta &= 15^\circ \end{aligned}$$



other 2 roots are evenly spaced on Argand Diagram

$$z_1 = \sqrt{2} \operatorname{cis} 15^\circ$$

$$z_2 = \sqrt{2} \operatorname{cis} 135^\circ$$

$$z_3 = \sqrt{2} \operatorname{cis} -105^\circ$$

(c) (i) $z^n = (\cos \theta + i \sin \theta)^n$

$$= \cos n\theta + i \sin n\theta$$

$$z^{-n} = \cos -n\theta + i \sin -n\theta$$

$$= \cos n\theta - i \sin n\theta$$

$$\therefore z^n + z^{-n} = 2 \cos n\theta \text{ as req.}$$

otherwise:

$$(ii) z^4 + 2z^3 + 3z^2 + 2z + 1 = 0$$

$$z^2 \left(z^2 + 2z + 3 + \frac{2}{z} + \frac{1}{z^2} \right) = 0$$

$$z^2 \left(z^2 + \frac{1}{z^2} + 2 \left(z + \frac{1}{z} \right) + 3 \right) = 0$$

$$\text{let } z + \frac{1}{z} = A \quad \therefore z^2 \left(\left(z + \frac{1}{z} \right)^2 + 2A + 1 \right) = 0$$

$$\left(z + \frac{1}{z} \right)^2 = z^2 + \frac{1}{z^2} + 2 \quad \therefore z^2 (A^2 + 2A + 1) = 0$$

$$z^2 (A+1)^2 = 0$$

$$z^2 \left(z + \frac{1}{z} + 1 \right)^2 = 0$$

$$\left[z \left(z + \frac{1}{z} + 1 \right) \right]^2 = 0$$

$$\left(z^2 + z + 1 \right)^2 = 0$$

$$\therefore \text{Double roots } z = \frac{-1 \pm \sqrt{3}i}{2}$$