NAME: ______________________________

TEACHER: ___________________________

GOSFORD HIGH SCHOOL

2012/2013
EXTENSION 2 MATHEMATICS
HSC ASSESSMENT TASK 1.

Time Allowed: 60 minutes (plus 5 min. reading time)

- Write using black or blue pen.
- Board-approved calculators may be used.
- Section III should be started on a new page and Section IV should be started on a new page.
- All necessary working should be shown in Section II, III and IV.

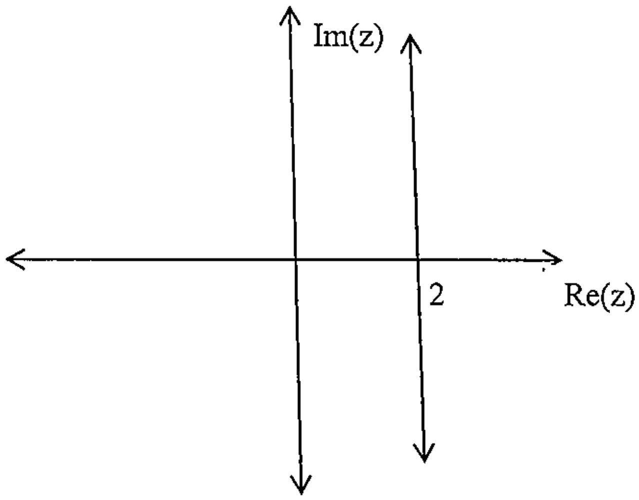
Papers are to be handed up in 3 bundles. Section I and II in one bundle, Section III in another and Section IV in another.

SECTION	QUESTION TYPE	MARKS	RESULT
I	MULTIPLE CHOICE	4	
II	EXTENDED RESPONSE	12	
III	EXTENDED RESPONSE	12	
IV	EXTENDED RESPONSE	12	
	TOTAL	40	

SECTION I. (4 marks) Answer on your own paper by writing down the correct letter.

1. If $z = 3 - i$, what is the value of $\overline{iz}$?

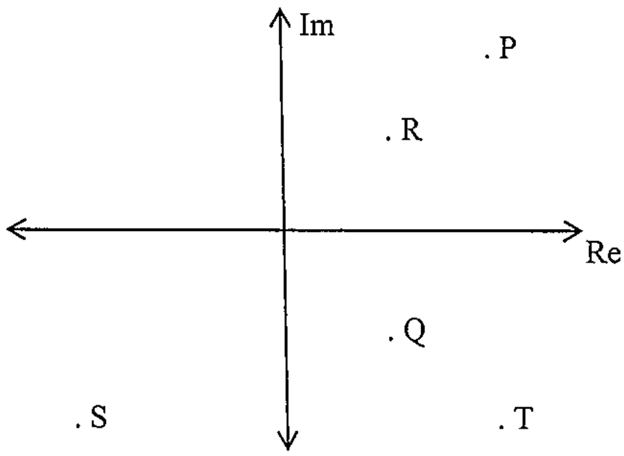
   A. $-1-3i$ B. $-1+3i$ C. $1-3i$ D. 1+3i

2. 

   Which of the following is **not** a valid algebraic description of the locus drawn above?

   A. $Re(z) = 2$ B. $|z| = |z-4|$ C. $Arg(z-4) + Arg(z) = \pi$ D. $z + \bar{z} = 4$

3. If the point P represents $\omega = 1 + i$, which of the following points best represents $\frac{1}{\omega}$?



   A. Q B. R C. S D. T

4. If $z_1 = 2\,cis\left(\frac{5\pi}{6}\right)$ and $z_2 = 4\,cis\left(\frac{\pi}{6}\right)$ then $\frac{z_1}{z_2} =$

   A. $\frac{1}{2}\,cis\left(\frac{2\pi}{3}\right)$ B. $2\,cis\left(-\frac{2\pi}{3}\right)$ C. $\frac{1}{2}\,cis\left(\frac{5\pi^2}{36}\right)$ D. $8\,cis\,(\pi)$

SECTION II. (12 marks)

1. Let $z = 3 - 2i$ and $\omega = 1 - i$. Find in the form $x + iy$, where x and y are real,

   (i) $2z + i\omega$ (1)

   (ii) $\bar{z}\,\omega$ (1)

   (iii) $\frac{4}{\omega}$ (2)

   (iv) $\left|\left(\frac{4}{\omega}\right)\right|$ (1)

2. (i) Express $\frac{1+i\sqrt{3}}{2}$ in modulus-argument form. (2)

   (ii) If $z = \frac{1+i\sqrt{3}}{2}$ show that $z^3 = -1$. (1)

   (iii) Hence evaluate z^7, expressing your answer in the form $x + iy$, where x and y are real. (1)

3. (i) Find all the 5th roots of -1 in modulus-argument form. (2)

   (ii) Sketch the 5th roots of -1 on an Argand diagram. (1)

SECTION III. (12 marks) Start a new page.

1. (i) Find the complex square roots of $-6i$, expressing your answer in the form $x + iy$, where x and y are real. (2)

(ii) Solve $z^2 + (1 + i)z + 2i = 0$, expressing the roots z_1 and z_2 in the form $x + iy$, where x and y are real. (2)

(iii) By letting $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$ it can be shown that $tan\,\frac{\pi}{12} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$. Find a similar expression for $tan\,\frac{5\pi}{12}$. (2)

(iv) Using the results of parts (ii) and (iii), prove that $arg(z_1) + arg(z_2) = \frac{\pi}{2}$. (2)

2. If $A = 1 - i$ and $B = 2 + i$ determine the cartesian equation of the locus specified by $|z - A| = |z - B|$ and draw a neat sketch on an Argand diagram to show it. (4)

SECTION IV. (12 marks) Start a new page.

1. (i) On an Argand diagram sketch the locus of $|z| = 2$ and $|z + 2| = 2$. (1)

(ii) Hence or otherwise, find in the form $x + iy$, where x and y are real, all complex numbers simultaneously satisfying $|z| = 2$ and $|z + 2| = 2$. (2)

2. Let $z = cos\theta + i\ sin\theta$

(i) Show that $z^n + z^{-n} = 2\cos n\theta$. (1)

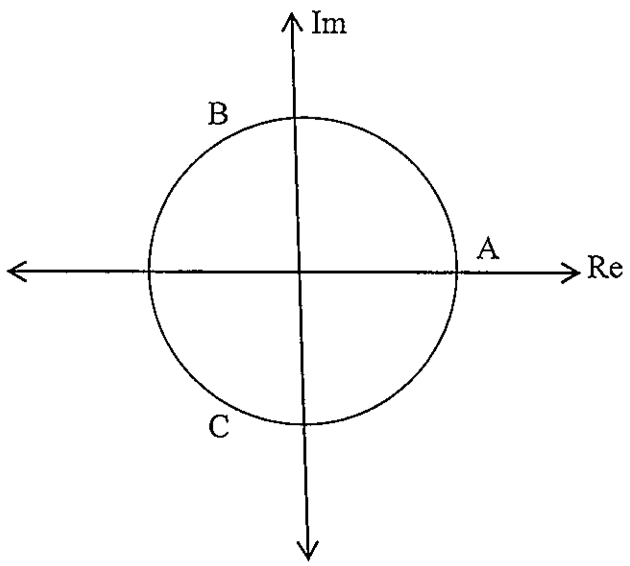
(ii) Hence, or otherwise, show that $sin^3\theta = \frac{3sin\theta}{4} - \frac{sin3\theta}{4}$. (3)

3. The roots of $z^3 - 1 = 0$ are $1, \omega$ and ω^2 where ω is one of the complex roots.

(i) Find the value of $1 + \omega + \omega^2$ (1)

(ii) Show that $z^2 + z + 1 = (z - \omega)(z - \omega^2)$ (2)

(iii) The Argand diagram shows the points A, B, C on the unit circle which correspond to the roots $1, \omega$ and ω^2 respectively.



Copy this diagram and show the vector $(1 - \omega)$ on your diagram clearly indicating its direction. (1)

(iv) Hence, or otherwise, find the product of the lengths of the chords BA and AC. (1)

2012/2013 Ext 2 Assess Task 1 Solutions

Section I

1. $iz = i(3-i)$

$= 3i - i^2$

$= 1 + 3i$

$\therefore \overline{iz} = 1 - 3i$ Hence $\underline{C}$

2. If $Re(z) = 2$

$x = 2$ ✓

If $|z| = |z-4|$

$x = 2$ ✓

If $z + \bar{z} = 4$

$x + iy + x - iy = 4$

$2x = 4$

$x = 2$ ✓

Hence $\underline{C}$

3. If $w = 1 + i$

$\frac{1}{w} = \frac{1}{1+i} \times \frac{1-i}{1-i}$

$= \frac{1-i}{2}$

$= \frac{1}{2} - \frac{1}{2}i$

Hence $\underline{A}$

4. $\frac{z_1}{z_2} = \frac{2}{4} \operatorname{cis}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right)$

$= \frac{1}{2} \operatorname{cis} \frac{4\pi}{6}$

$= \frac{1}{2} \operatorname{cis} \frac{2\pi}{3}$

Hence $\underline{A}$

Section II

1. (i) $2z + iw$

$= 2(3-2i) + i(1-i)$

$= 6 - 4i + i - i^2$

$= 7 - 3i$ (1)

(ii) $\bar{z}w$

$= (3+2i)(1-i)$

$= 3 - 3i + 2i - 2i^2$

$= 5 - i$ (1)

(iii) $\frac{4}{w}$

$= \frac{4}{1-i} \times \frac{1+i}{1+i}$

$= \frac{4(1+i)}{1-i^2}$

$= \frac{\cancel{4}^{2}(1+i)}{\cancel{2}}$

$= 2 + 2i$ (2)

(iv) $\left|\overline{\left(\frac{4}{w}\right)}\right|$

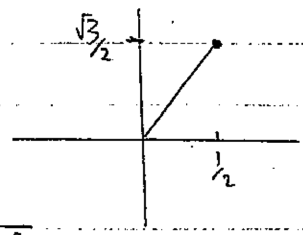
$= \sqrt{(2)^2 + (-2)^2}$ since $\overline{\left(\frac{4}{w}\right)} = 2 - 2i$

$= \sqrt{8}$

$= 2\sqrt{2}$ (1)

2 (i) $\frac{1 + i\sqrt{3}}{2}$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$



$\therefore r = \sqrt{(1/2)^2 + (\sqrt{3}/2)^2}$

$= 1.$

$\theta = \tan^{-1}\left(\dfrac{\sqrt{3}/2}{1/2}\right)$

$= \tan^{-1}(\sqrt{3})$

$= \pi/3$

$\therefore \dfrac{1+i\sqrt{3}}{2} = \text{cis}\,\pi/3$. (2)

(ii) If $z = \text{cis}\,\pi/3$

$z^3 = \text{cis}\,3(\pi/3)$

$z^3 = \text{cis}\,\pi$

$= \cos\pi + i\sin\pi$

$= -1 + i.0$

$= -1.$ (1)

(iii) $z^7 = z^6.z$

$= (z^3)^2.z$

$= (-1)^2.z$

$= z$

$= \dfrac{1+i\sqrt{3}}{2}$ or $\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$ (1)

3.(i) Let $z = \cos\theta + i\sin\theta$

If $z^5 = -1$

$\cos 5\theta + i\sin 5\theta = -1 + 0i$

$\therefore \cos 5\theta = -1$

$5\theta = \pi, 3\pi, 5\pi, 7\pi, 9\pi$

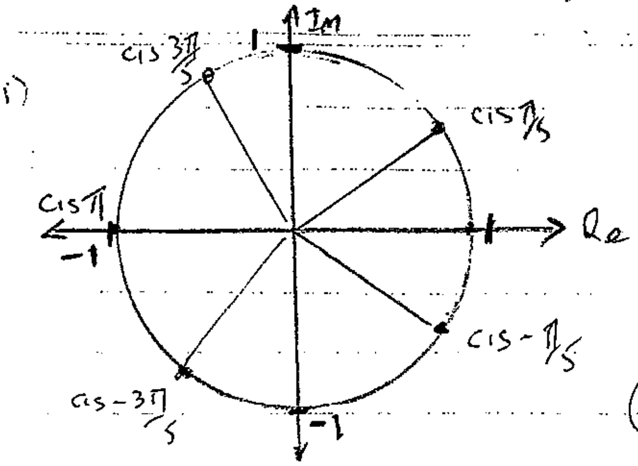
$\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$

ie $\theta = \pm\pi/5, \pm\frac{3\pi}{5}, \pi$ (2)

since $\frac{7\pi}{5} = -\frac{3\pi}{5}$ & $\frac{9\pi}{5} = -\pi/5$

The roots are $\text{cis}\,\pi/5$, $\text{cis}\,3\pi/5$, $\text{cis}\,\pi$, $\text{cis}\,{-\pi/5}$; $\text{cis}\,{-3\pi/5}$

(ii)



(1)

SECTION II

1(i) Let $\sqrt{-6i} = x+iy$

$0 - 6i = (x+iy)^2$

$0 - 6i = x^2 - y^2 + 2xyi$

$x^2 - y^2 = 0$, $2xy = -6$

$y = -\dfrac{3}{x}$

$\therefore x^2 - \left(\dfrac{-3}{x}\right)^2 = 0$

$x^2 - \dfrac{9}{x^2} = 0$

$x^4 - 9 = 0$

$(x^2-3)(x^2+3) = 0$

$x = \pm\sqrt{3}$

$y = \mp \dfrac{3}{\sqrt{3}}$

$y = \mp\sqrt{3}$ (2)

$\therefore \sqrt{-6i} = \sqrt{3} - i\sqrt{3}$ or $-\sqrt{3} + i\sqrt{3}$.

(ii) If $z^2 + (1+i)z + 2i = 0$

$z = \dfrac{-(1+i) \pm \sqrt{(1+i)^2 - 8i}}{2}$

$= \dfrac{-(1+i) \pm \sqrt{1+2i+i^2-8i}}{2}$

$$= \frac{-(1+i) \pm \sqrt{-6i}}{2}$$

$$= \frac{-(1+i) \pm (\sqrt{3} - i\sqrt{3})}{2}$$

$$= \frac{(\sqrt{3}-1)}{2} - \frac{(\sqrt{3}+1)i}{2} \quad (2)$$

$$\text{or } -\frac{(\sqrt{3}+1)}{2} + \frac{(\sqrt{3}-1)i}{2}$$

(iii) $\frac{5\pi}{12} = \frac{\pi}{6} + \frac{\pi}{4}$

$$\therefore \tan\frac{5\pi}{12} = \tan\left(\pi/6 + \pi/4\right)$$

$$= \frac{\tan\pi/6 + \tan\pi/4}{1 - \tan\pi/6 \tan\pi/4}$$

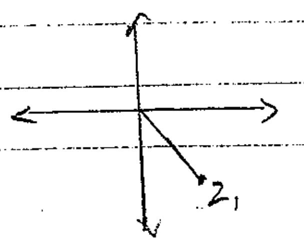
$$= \frac{\frac{1}{\sqrt{3}} + 1}{1 - 1/\sqrt{3}}$$

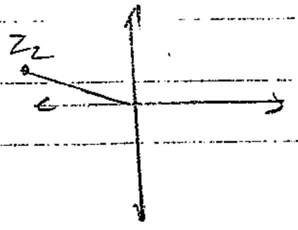
$$= \frac{\frac{\sqrt{3}+1}{3}}{\frac{\sqrt{3}-1}{3}}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1} \quad (2)$$

(iv) Let $z_1 = \frac{(\sqrt{3}-1)}{2} - \frac{(\sqrt{3}+1)i}{2}$

$$z_2 = -\frac{(\sqrt{3}+1)}{2} + \frac{(\sqrt{3}-1)i}{2}$$





$$\arg z_1 = \tan^{-1}\left[\frac{-\frac{(\sqrt{3}+1)}{2}}{\frac{(\sqrt{3}-1)}{2}}\right]$$

$$= -\tan^{-1}\left[\frac{\sqrt{3}+1}{\sqrt{3}-1}\right]$$

$$= -\frac{5\pi}{12}$$

$$\arg(z_2) = \pi - \tan^{-1}\left[\frac{\frac{(\sqrt{3}-1)}{2}}{\frac{(\sqrt{3}+1)}{2}}\right]$$

$$= \pi - \frac{\pi}{12}$$

$$= \frac{11\pi}{12}$$

$$\therefore \arg(z_1) + \arg(z_2) = -\frac{5\pi}{12} + \frac{11\pi}{12}$$

$$= \pi/2$$

2. Let $z = x + iy$

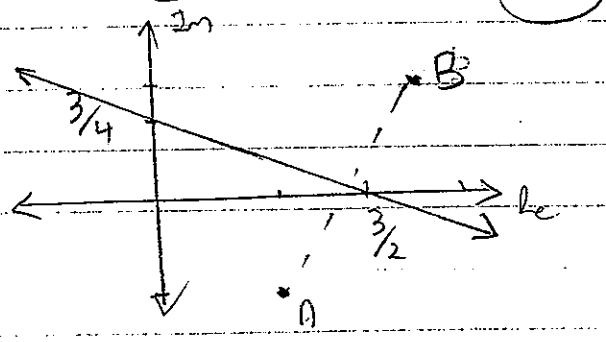
$$z - A = x + iy - (1 - i)$$
$$= x - 1 + i(y+1)$$
$$z - B = x + iy - (2 + i)$$
$$= x - 2 + i(y - 1)$$

If $|z - A| = |z - B|$

$$\sqrt{(x-1)^2 + (y+1)^2} = \sqrt{(x-2)^2 + (y-1)^2}$$

$$\therefore \cancel{x^2} - 2x + 1 + \cancel{y^2} + 2y + 1 = \cancel{x^2} - 4x + 4 + \cancel{y^2} - 2y + 1$$

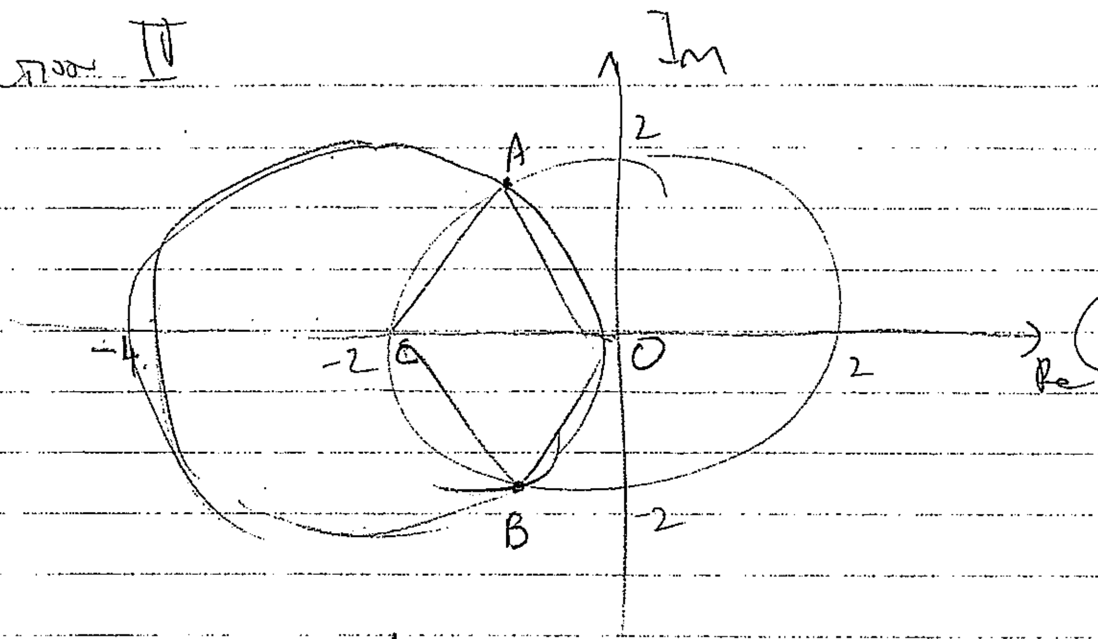
ie $2x + 4y - 3 = 0$ (3)



N.B. the locus is the perpendicular bisector of AB

Section IV

1/ (i)



(1)

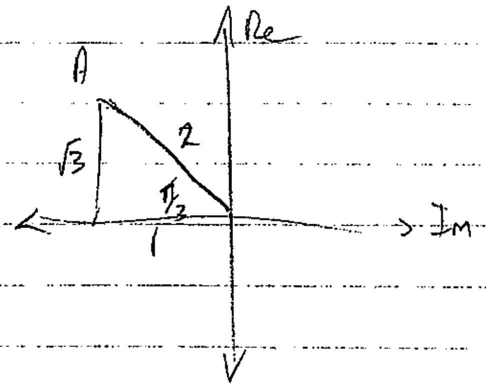
(ii) Let the points of intersection be A & B
Let the centres of the circles be O & C

$\triangle$s AOC & BOC are equilateral since $OC = CA = OB = AC = BC = 2$ units

$\therefore$ A is the point $\text{cis } \frac{2\pi}{3}$

& B is the point $\text{cis } -\frac{2\pi}{3}$

(2)



$\therefore$ A is $-1 + \sqrt{3}\, i$

& by symmetry B is $-1 - \sqrt{3}\, i$

2/ (i) If $z = \cos\theta + i\sin\theta$

$z^n = \cos(n\theta) + i\sin(n\theta)$

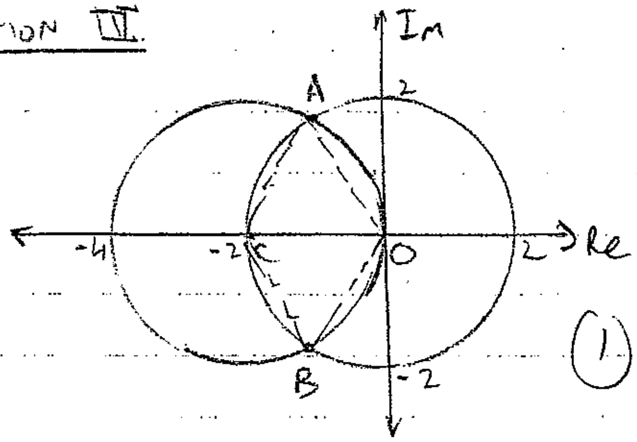
$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos(n\theta) - i\sin(n\theta)$

$\therefore z^n + z^{-n} = 2\cos n\theta$

SECTION III.

1 (i)



(1)

(ii)

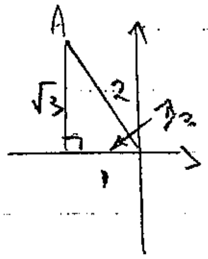
Let the points of intersection be A & B and the centres be O & C

OA = OB = OC = CA = CB = 2 units

∴ Δ AOC is equilateral

∴ A is the point $\text{cis}\,\frac{2\pi}{3}$

& B is the point $\text{cis}\,\frac{-2\pi}{3}$



A is $-1+\sqrt{3}i$

& by symmetry

B is $-1-\sqrt{3}i$

2. (i) $z^n = \cos(n\theta) + i\sin(n\theta)$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos(n\theta) - i\sin(n\theta)$

$\therefore\ z^n + z^{-n} = 2\cos(n\theta)$ (1)

(ii) If $z = \cos\theta + i\sin\theta$

$z^n - z^{-n} = 2i\sin(n\theta)$

$\therefore\ z - \frac{1}{z} = 2i\sin\theta$

Now $\left(z - \frac{1}{z}\right)^3 = (2i\sin\theta)^3$

$= -8i\sin^3\theta$

$\therefore -8i\sin^3\theta = z^3 - 3z^2\frac{1}{z} + 3z\frac{1}{z^2} - \frac{1}{z^3}$

$-8i\sin^3\theta = \left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right)$

$-8i\sin^3\theta = 2i\sin 3\theta - 3(2i\sin\theta)$

$-8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$

$\therefore \sin^3\theta = \dfrac{2i\sin 3\theta - 6i\sin\theta}{-8i}$

$= -\dfrac{\sin 3\theta}{4} + \dfrac{3\sin\theta}{4}$

$= \dfrac{3\sin\theta}{4} - \dfrac{\sin 3\theta}{4}$ (3)

3. (i) If $z^3 - 1 = 0$

$z^3 + 0z^2 + 0z - 1 = 0$

the sum of the roots is $\frac{-b}{a}$

$\therefore\ 1 + \omega + \omega^2 = 0$ (1)

(ii) $(z-\omega)(z-\omega^2)$

$= z^2 - z\omega^2 - z\omega + \omega^3$

$= z^2 - z(\omega^2 + \omega) + \omega^3$

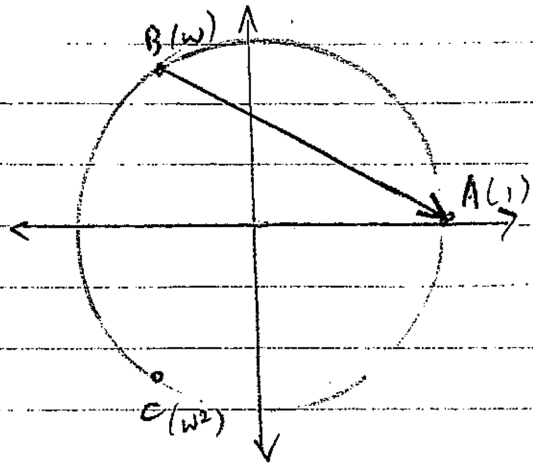
$= z^2 - z(-1) + 1$

since $\omega^2 + \omega = -1$ & if ω is a root of $z^3 - 1 = 0$, $\omega^3 = 1$

$\therefore\ (z-\omega)(z-\omega^2) = z^2 + z + 1$

(2)

ii)



(1)

v) $\overrightarrow{BA} \cdot \overrightarrow{CA} = (1-\omega)(1-\omega^2)$

$= 1 - \omega^2 - \omega + \omega^3$

$= 1 - (\omega^2 + \omega) + \omega^3$

$= 1 - (-1) + 1$

$= 3.$ (1)