Mathematics Extension 2

Student Name/Number:_________________________

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

A ⊗ B ⊗ (correct ↓) C ○ D ○

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○

Name: ______________________________ Teacher: ______________________

GOSFORD HIGH SCHOOL

MATHEMATICS EXTENSION 2

2013/2014

HSC Assessment Task #1

(Complex Numbers)

Time Allowed: 60 minutes (plus 5 minutes reading)

- Answer **SECTION I** (multiple choice) on the answer sheet provided – no working needs to be shown.
- **SECTION II:** Should be attempted on your own paper, starting each question on a new sheet of paper – all necessary working **MUST** be shown.
- Write using black or blue pen and board approved calculators may be used.

SECTION	QUESTIONS	MARKS	RESULT
I	Question 1–5: (Multiple Choice)	5	
II	Question 6: (Extended Response)	12	
	Question 7: (Extended Response)	12	
	Question 8: (Extended Response)	13	
	TOTAL	42	

SECTION I: (5 marks)

- **Attempt Questions 1 – 5**
- **Allow about 8 minutes for this section**
- **Use the multiple choice answer sheet for Questions 1 – 5**

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1. Given $z = \sqrt{2}\,(\cos\frac{\pi}{3} + isin\frac{\pi}{3})$

   $z^{-4} =$

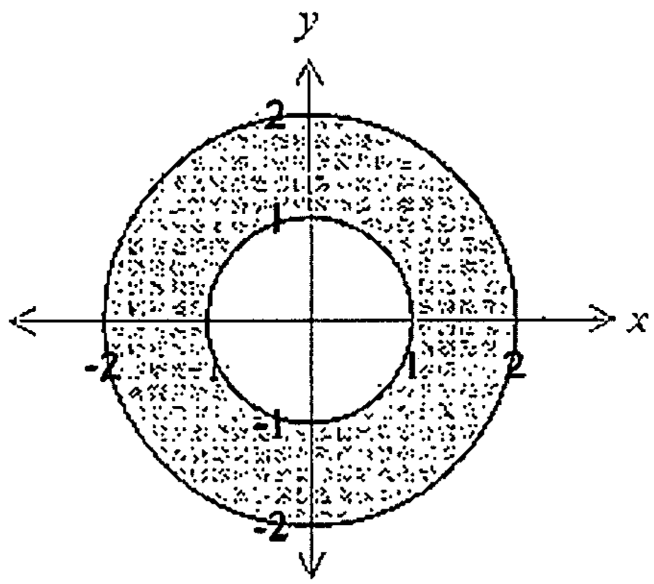
   (A) $4(\cos\left(\frac{-4\pi}{3}\right) + isin\left(\frac{-4\pi}{3}\right))$

   (B) $\frac{1}{4}(\cos\left(\frac{4\pi}{3}\right) + isin\left(\frac{4\pi}{3}\right))$

   (C) $4(\cos\left(\frac{2\pi}{3}\right) + isin\left(\frac{2\pi}{3}\right))$

   (D) $\frac{1}{4}(\cos\left(\frac{2\pi}{3}\right) + isin\left(\frac{2\pi}{3}\right))$

2. Consider the Argand diagram below:



   The inequality that represents the shaded area is:

   (A) $0 \le |z| \le 2$

   (B) $1 \le |z| \le 2$

   (C) $0 \le |z - 1| \le 2$

   (D) $1 \le |z - 1| \le 2$

3. Let $z = x + iy$ where x and y are real and non-zero. Which of the following is **NOT** true.

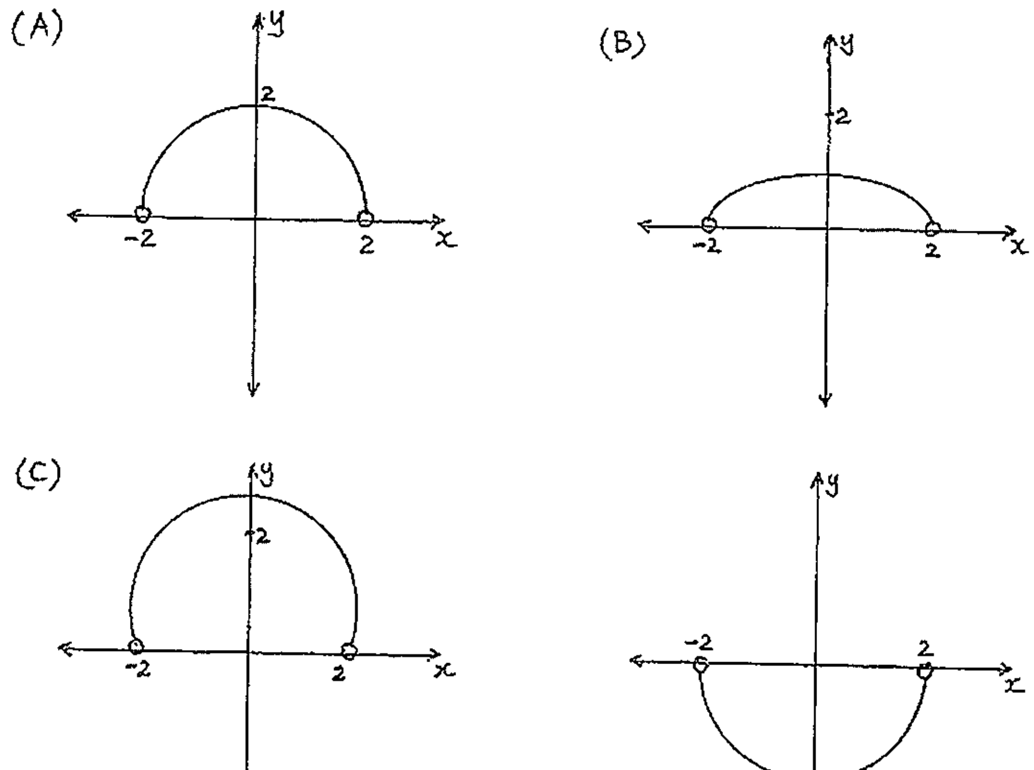
(A) $z + \bar{z}$ is real

(B) $z\,\bar{z}$ is real and positive

(C) $z^2 - (\bar{z})^2$ is real

(D) z^2 is non-real

4. The locus of z *if* $\arg(z-2) - \arg(z+2) = \frac{\pi}{4}$ is best shown as:



5. If w is an imaginary cube root of unity, then $(1 + w - w^2)^7$ is equal to :

(A) $128\,w$

(B) $-128\,w$

(C) $128\,w^2$

(D) $-128\,w^2$

END OF SECTION I

SECTION II: (37 marks)

- **Attempt Questions 6 – 8**
- **Allow about 52 minutes for this section**
- Start each question on a new sheet of paper.
- In Questions 6 – 8, your responses should include relevant mathematical reasoning and/or calculations

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Question 6 (12 Marks) *Use a new sheet of paper.*

a. Let $z = -5 - 12i$ and $w = -2 + i$, find in the form $x + iy$:

(i) $\bar{z} - w$ 1

(ii) zw 1

(iii) $\frac{5}{iw}$ 2

b. The complex numbers z and w have a modulus of 2 and the arguments of z and w are $\frac{4\pi}{9}$ *and* $\frac{7\pi}{9}$ respectively. On the Argand diagram the point A represents z, the point B represents w and the point C represents $z + w$.

(i) Sketch vectors representing z, w *and* $z + w$ on an Argand diagram and explain why OACB is a rhombus. 2

(ii) Find arg $(z + w)$ 1

(iii) Evaluate $|z + w|$ 1

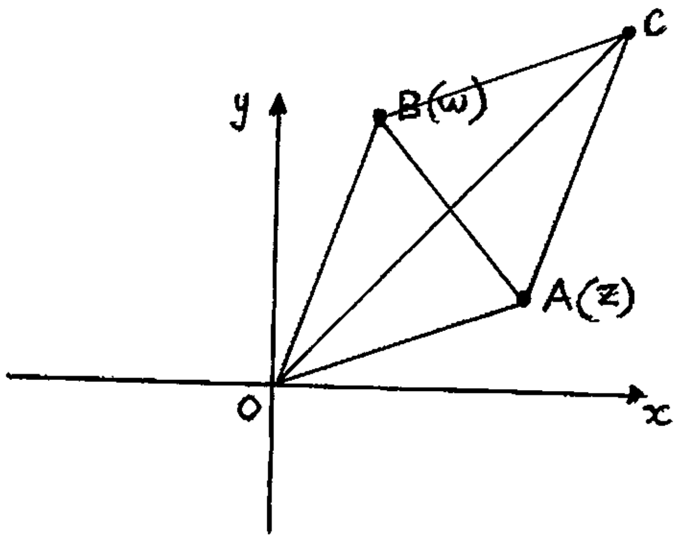
c. (i) $(x + iy)^2 = 5 - 12i$, where x, y are real, find values of x and y . 2

(ii) Hence, solve: 2

$$z^2 + (1 - 2i)z + (2i - 2) = 0$$

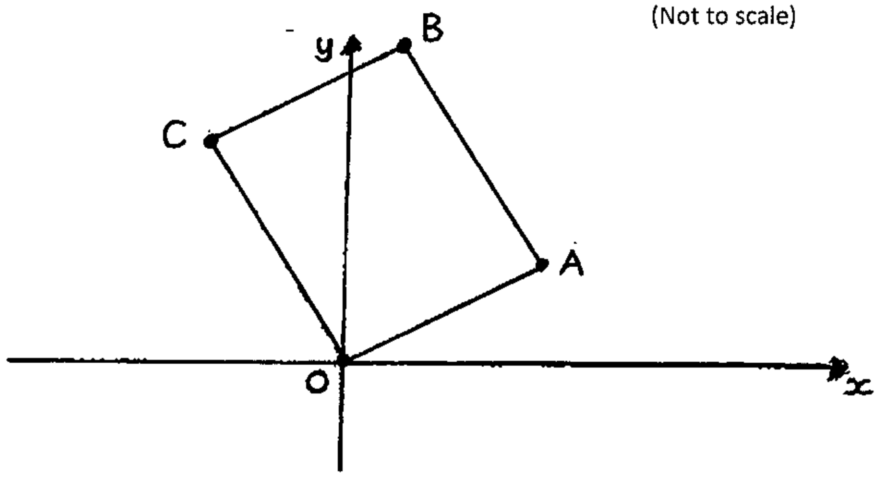
Question 7 (12 Marks) Start *a new sheet of paper.*

a.



Given the above Argand diagram with OACB a parallelogram. Vector $\overrightarrow{OA}$ represents complex number z and vector $\overrightarrow{OB}$ represents complex number w. Name the vector representing complex number $z - w$. 1

b. *OABC* is a rectangle in an Argand diagram where *O* is the Origin and point A corresponds to the complex number $2 + i$



Given that the length of the rectangle is twice its breadth and *OA* is one of the shorter sides, find the complex number representing *C*. 1

c. (i) Write $(1 + i\sqrt{3})(1 + i)$ in Cartesian form. 1

(ii) By first expressing $1 + i\sqrt{3}$ and $1 + i$ in mod-arg form, find $(1 + i\sqrt{3})(1 + i)$ in mod-arg form. 2

(iii) Hence, find the exact value of $\tan\frac{7\pi}{12}$. 2

d. Describe the locus of z that can be represented by the equation:

$$(z + \overline{z})^2 - (z - \overline{z})^2 = 16$$ 2

e. Given $|3z + 4iz| = 10$, evaluate $|z|$ 2

Question 8 (13 Marks) Start *a new sheet of paper.*

a. On separate one-third page Argand diagrams sketch the locus of points z such that:

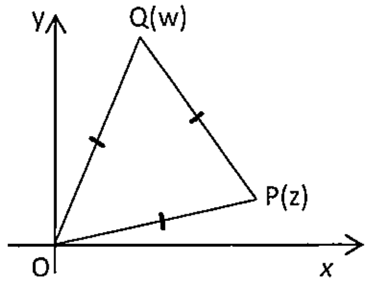
(i) $|z-1| = |z+i|$ 1

(ii) $\text{Arg}\,(z+3-i) = \frac{3\pi}{4}$ 1

b. The three roots of $P(z) = 0$ are $z = -3,\ 4+2\sqrt{3}i$ and $4-2\sqrt{3}i$. Express $P(z)$ as a product of its factors in simplest form over the real field. 2

c. Given that $z = -1$ is the only real root of $z^5 + 1 = 0$, write down the other roots and indicate all roots on an Argand diagram. *(Leave answers in simplest mod-arg form.)* 2

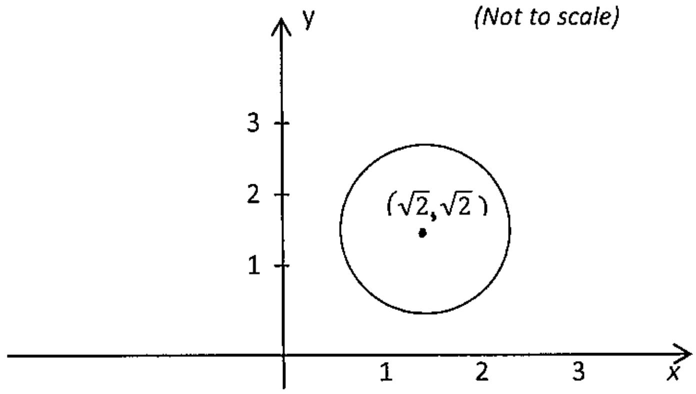
d. 2



In the Argand diagram, *OPQ* is an equilateral triangle. *P* represents the complex number z and *Q* represents the complex number w.

Show that $w^3 + z^3 = 0$

e. Given the sketch of $\left|z - \sqrt{2} - \sqrt{2i}\right| = 1$ below: 2



Find the minimum value of arg z.

f. The complex number z is a function of the real number r given by the rule

$$z = \frac{r - i}{r + i}\ , \ 0 \le r \le 1$$

(i) Evaluate $|z|$ 1

(ii) Hence, describe the locus of z as r varies from 0 to 1. 2

END OF ASSESSMENT TASK

MATHS EXT 2 Assessment Task #1 Solution 2013/14

Section I (Multiple Choice)

1. D 2. B 3. C 4. C 5. D

1/ $\left[\sqrt{2}\,\text{cis}\,\frac{\pi}{3}\right]^{-4}$

$= \left(\frac{1}{\sqrt{2}}\right)^4 \cdot \text{cis}\,\frac{-4\pi}{3}$

$= \frac{1}{4}\,\text{cis}\,\frac{2\pi}{3}$ $\therefore$ D.

2/ B.

3/ $z + \bar{z} = 2x$ (real)

$z\bar{z} = x^2 + y^2$ (real, >0)

$z^2 - (\bar{z})^2 = (x+iy)^2 - (x-iy)^2$

$= x^2 + 2xyi - y^2 - (x^2 - 2xyi - y^2)$

$= 4xy\,i$ (not real)

$z^2 = x^2 + 2xyi - y^2$ (not real)

$\therefore$ C

4/ Major arc of a circle obtains an acute subtended angle $\therefore$ C

5/ Since $1 + w + w^2 = 0$

$1 + w = -w^2$

$\therefore (1 + w - w^2)^7 = (-w^2 - w^2)^7$

$= (-2w^2)^7$

$= -128\,w^{14}$

$= -128\,w^2$ $\therefore$ D.

Section II

Question 6

(a) (i) $\bar{z} - w = -5 + 12i - (-2 + i)$

$= -5 + 12i + 2 - i$

$= -3 + 11i$

(ii) $zw = (-5 - 12i)(-2 + i)$

$= 10 - 5i + 24i + 12$

$= 22 + 19i$

(iii) $\frac{5}{iw} = \frac{5}{i(-2+i)}$

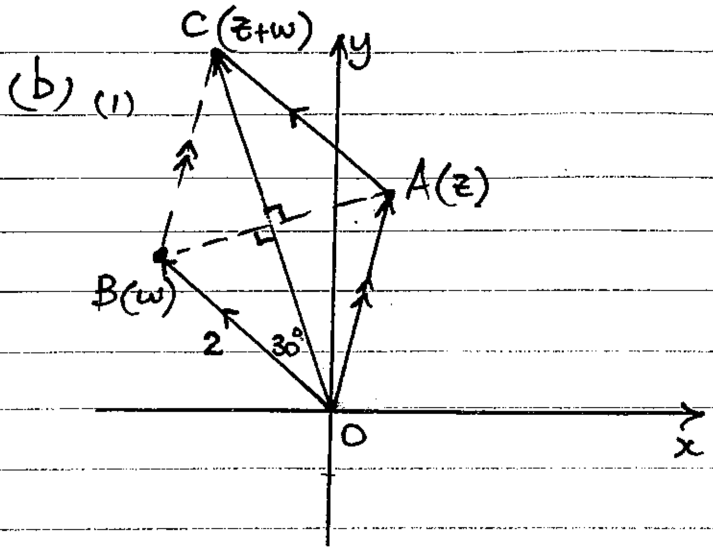
$= \frac{5}{-1-2i} \times \frac{-1+2i}{-1+2i}$

$= \frac{-5 + 10i}{1 + 4}$

$= \frac{-5 + 10i}{5}$

$= -1 + 2i$

(b) (i)



OACB is a parallelogram with adjacent sides OB and OA equal ∴ Rhombus

(ii) $\arg z = \frac{4\pi}{9}$

$\hat{COA} = \frac{\pi}{6}$ (diagonal bisects angles of a rhombus)

$\therefore \arg(z+w) = \frac{4\pi}{9} + \frac{\pi}{6}$

$= \frac{11\pi}{18}$

(iii) Since diagonals bisect at right angles

$\frac{\frac{1}{2}OC}{2} = \cos 30°$

$\frac{1}{4}OC = \frac{\sqrt{3}}{2}$

$\therefore OC = 2\sqrt{3}$.

$\therefore |z+w| = 2\sqrt{3}$

(c) (i) $x^2 + 2xiy - y^2 = 5 - 12i$

Equating real and imaginary parts :

$x^2 - y^2 = 5 \quad — ①$

$2xy = -12$

$xy = -6 \quad — ②$.

From ② $y = -\frac{6}{x}$

Sub in ① $x^2 - \left(-\frac{6}{x}\right)^2 = 5$

$x^2 - \frac{36}{x^2} = 5$

$x^4 - 36 = 5x^2$

$x^4 - 5x^2 - 36 = 0$

$(x^2 - 9)(x^2 + 4) = 0$

Since x is real

$x = \pm 3$.

$y = \mp 2$

∴ when $x = 3$, $y = -2$

$x = -3$, $y = 2$.

(ii) to solve use the quadratic formula

i.e $z = \frac{-(1-2i) \pm \sqrt{(1-2i)^2 - 4(2i-2)}}{2}$.

$\therefore z = \dfrac{-1+2i \pm \sqrt{5-12i}}{2}$

Since $\pm$ use either answer from (1) say $3-2i$

$\therefore z = \dfrac{-1+2i \pm (3-2i)}{2}$

$= \dfrac{-1+2i+3-2i}{2}, \dfrac{-1+2i-3+2i}{2}$

$= 1, \dfrac{-4+4i}{2}$

$= 1, -2+2i$

Question 7

(a) $\overrightarrow{BA}$

(b) Since $C\hat{O}A = 90^\circ$ (angle in rectagle)

OC $\Rightarrow$ rotating OA through $\frac{\pi}{2}$ and doubling its length

$\therefore C = 2 \times (2+i) \times i$

$= 4i - 2$

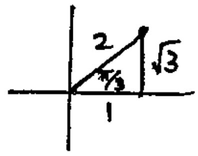
$= -2 + 4i$

(c) (i) $(1+i\sqrt{3})(1+i)$

$= 1 + i + i\sqrt{3} - \sqrt{3}$

$= (1-\sqrt{3}) + i(1+\sqrt{3})$

(ii) $1+i\sqrt{3} \Rightarrow$



$\therefore 1+i\sqrt{3} = 2 \text{ cis } \frac{\pi}{3}$

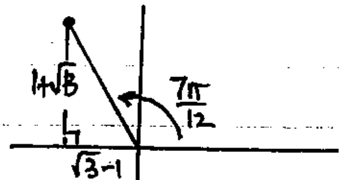
Similarly

$1+i = \sqrt{2} \text{ cis } \frac{\pi}{4}$

$\therefore (1+i\sqrt{3})(1+i) = \left(2 \text{ cis } \frac{\pi}{3}\right)\left(\sqrt{2} \text{ cis } \frac{\pi}{4}\right)$

$= 2\sqrt{2} \text{ cis } \frac{7\pi}{12}$

(iii) $\therefore 2\sqrt{2} \text{ cis } \frac{7\pi}{12}$



$\therefore$ From (i) $2\sqrt{2} \text{ cis } \frac{7\pi}{12} = (1-\sqrt{3}) + i(1+\sqrt{3})$

From Δ: $\tan \frac{5\pi}{12} = \dfrac{1+\sqrt{3}}{\sqrt{3}-1}$

$\therefore \tan \frac{7\pi}{12} = -\dfrac{(1+\sqrt{3})}{\sqrt{3}-1}$

$\therefore \tan \frac{7\pi}{12} = \dfrac{1+\sqrt{3}}{1-\sqrt{3}}$

OR Easier just to say

$\tan \frac{7\pi}{12} = \dfrac{\sin \frac{7\pi}{12}}{\cos \frac{7\pi}{12}} = \dfrac{1+\sqrt{3}}{1-\sqrt{3}}$ from (i)

(d) $(z+\bar{z})^2 - (z-\bar{z})^2 = 16$

let $x+iy = z$
$x-iy = \bar{z}$

$\therefore$ LHS $= (2x)^2 - (x+iy-x+iy)^2$

$= 4x^2 - (2yi)^2$

$= 4x^2 - (-4y^2)$

$= 4x^2 + 4y^2$

$\therefore$ locus $\Rightarrow$ $4x^2+4y^2 = 16$
$x^2+y^2 = 4$

$\therefore$ locus is circle centre $(0,0)$ and radius 2 units

(e) $|3z+4iz| = 10$

LHS $= |z(3+4i)|$

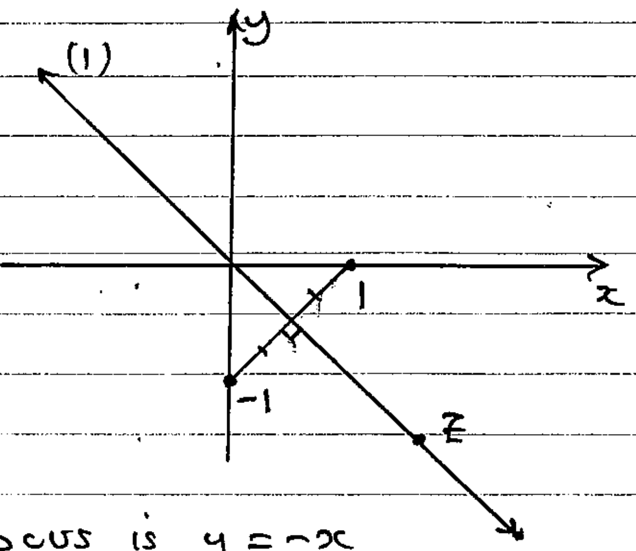
$= |z||3+4i|$

$= 5|z|$

$\therefore$ $5|z| = 10$
$|z| = 2$

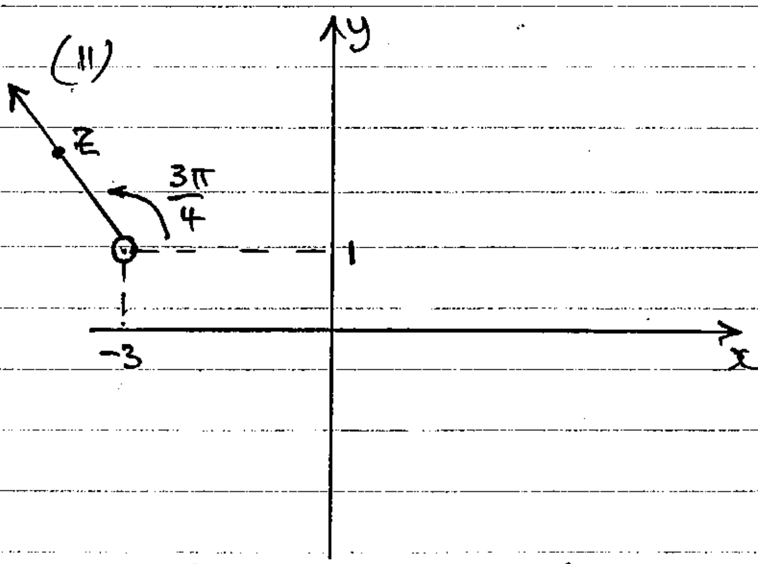
Question 8.

(a) (i)



Locus is $y = -x$ the perpendicular bisector of the line joining $(1,0)$ and $(0,-1)$

(ii)



$\arg(z+3-i) = \arg(z-(-3+i))$

(b) $P(z) = (z+3)(z-(4+2\sqrt{3}i))(z-(4-2\sqrt{3}i))$

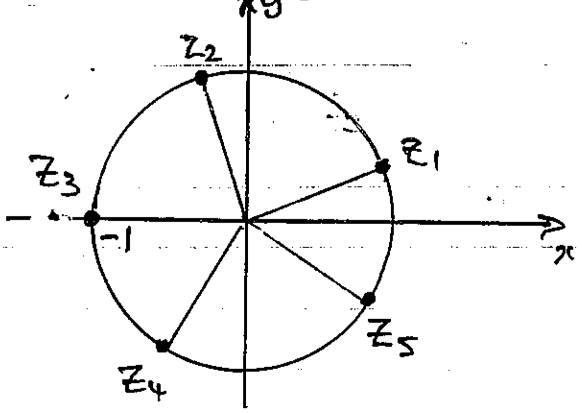
$= (z+3)(z^2-8z+28)$ over R.

using $(z-z_1)(z-\bar{z}_1)$

$= (z^2 - 2\text{Re}\, z_1 z + z_1\bar{z}_1)$

(c) other complex roots are even spaced about Argand Diagram i.e $\frac{2\pi}{5}$ apart on a unit circle

i.e



i.e $z_1 = \text{cis}\frac{\pi}{5}$

$z_2 = \text{cis}\frac{3\pi}{5}$

given: $z_3 = -1$ $(\text{cis}\,\pi)$

$z_4 = \text{cis}\left(-\frac{3\pi}{5}\right)$

$z_5 = \text{cis}\left(-\frac{\pi}{5}\right)$

(d) $w = z \times \text{cis}\frac{\pi}{3}$

$\therefore w^3 = z^3 \times \left(\text{cis}\frac{\pi}{3}\right)^3$

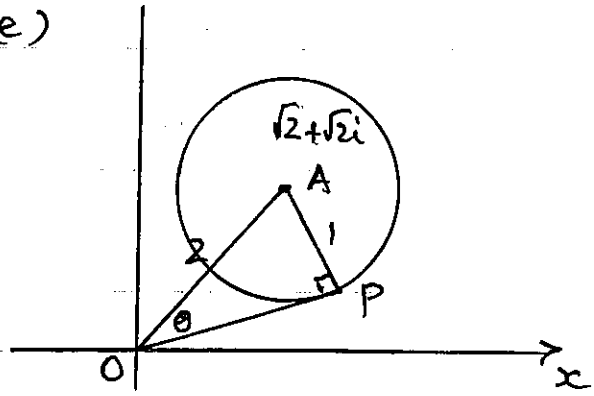
$\therefore w^3 = z^3 \times (\text{cis}\,\pi)$

$w^3 = z^3 \times (-1)$

$\therefore w^3 = -z^3$

$w^3 + z^3 = 0$

(e)



$x\hat{O}P$ would represent the minimum arg z

Using $\Delta AOP \Rightarrow \sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

$\therefore$ Min arg $z = \arg(\sqrt{2}+\sqrt{2}i) - \frac{\pi}{6}$

$= \frac{\pi}{4} - \frac{\pi}{6}$

$= \frac{\pi}{12}$.

(f)(i) $|z| = \frac{|r-i|}{|r+i|}$

$= \frac{\sqrt{r^2+1}}{\sqrt{r^2+1}}$

$= 1$

$\therefore |z| = 1$

(ii) When $r=0$ $\quad z = \frac{-i}{i} = -1$

$r=1 \quad z = \frac{1-i}{1+i} \times \frac{1-i}{1-i}$

$= \frac{-2i}{2}$

$= -i$

$\therefore$ locus $\quad |z| = 1$

from -1 to $-i$

Also test $r = \frac{1}{2}$

$z = \frac{\frac{1}{2}-i}{\frac{1}{2}+i} \times \frac{\frac{1}{2}-i}{\frac{1}{2}-i}$

$= \frac{-\frac{3}{4} - i}{\frac{5}{4}}$

$\therefore Re(z) < 0$

Hence the locus must be the part of the unit circle $x^2+y^2=1$ in the 3rd Quadrant (i.e quarter of a circle)

Alternatively, realise the denominator from the beginning

i.e $\quad z = \frac{r-i}{r+i}$

$= \frac{r-i}{r+i} \times \frac{r-i}{r-i}$

$= \frac{r^2-2i-1}{r^2+1}$

$= \frac{r^2-1}{r^2+1} - \frac{2i}{r^2+1}$

Considering $0 \le r \le 1$,

$\therefore Re(z) = \frac{r^2-1}{r^2+1} < 0$

$\therefore$ locus as discussed earlier.