NAME: ______________________________

TEACHER: ___________________________

GOSFORD HIGH SCHOOL

2015/2016
EXTENSION 2 MATHEMATICS
HSC ASSESSMENT TASK 1.

Time Allowed: 60 minutes (plus 5 min. reading time)

- Write using black or blue pen.
- Board-approved calculators may be used.
- Section III should be started on a new page and Section IV should be started on a new page.
- All necessary working should be shown in Section II, III and IV.

SECTION	QUESTION TYPE	MARKS	RESULT
I	MULTIPLE CHOICE	4	
II	EXTENDED RESPONSE	18	
III	EXTENDED RESPONSE	18	
	TOTAL	40	

SECTION I. (4 marks) Answer on the multiple choice answer sheet.

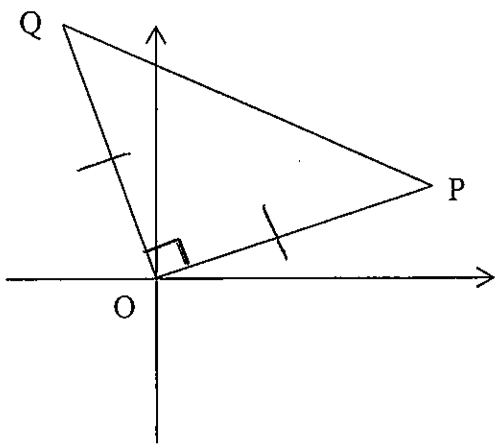
1. If $w = -1 - \sqrt{3}\, i$ what is the value of $\arg(w)$?

   A. $\frac{\pi}{3}$ B. $\frac{-\pi}{3}$ C. $\frac{2\pi}{3}$ D. $\frac{-2\pi}{3}$

2. On an Argand diagram, the points A and B represent the complex numbers $3 + 2i$ and $2 - i$ respectively.The point P is such that $OAPB$ is a parallelogram. What complex number does P represent?

   A. $5 + i$ B. $5 - i$ C. $-1 - 3i$ D. $1 + 3i$

3. The points P and Q in the complex plane correspond to the complex numbers z and w respectively. The triangle OPQ is isosceles with $OP = OQ$ and $\angle POQ$ is right angle.



   Which of these statements is correct?

   A. $z^2 + w^2 = 1$. B. $z^2 - w^2 = 1$. C. $w^2 - z^2 = 1$. D. $z^2 + w^2 = 0$.

4. The cartesian equation of the locus specified by $|z|^2 = z + \bar{z}$ is

   A. $(x-1)^2 + y^2 = 1$. B. $(x+1)^2 + y^2 = 1$

   C. $x^2 + (y-1)^2 = 1$ D. $x^2 + (y+1)^2 = 1$

SECTION II. (18 marks)

1. If $z = 2 + i$ and $\omega = 1 - 2i$. Find in the form $x + iy$, where x and y are real,

   (i) $2z + iw$ (1)

   (ii) $z\overline{w}$ (1)

   (iii) $\dfrac{z}{\omega}$ (2)

2. (i) Express $1 - \sqrt{3}i$ in modulus-argument form. (2)

   (ii) Express $(1 - \sqrt{3}i\,)^5$ in modulus-argument form. (1)

   (iii) Hence, express $(1 - \sqrt{3}i\,)^5$ in the form $x + iy$, where x and y are real. (2)

3. On separate Argand diagrams sketch each of the following regions.

   (i) $Re(z) \geq -1$ and $1 < Im(z) < 2$. (1)

   (ii) $|z + 2i| \leq 2$. (1)

   (iii) $z + \bar{z} > 6$, $|z - 1| \leq 4$ and $|\arg z| < \dfrac{\pi}{6}$. (2)

4. Given $z = r\,(cos\theta + i\,sin\theta\,)$, where $z \neq 0$:

   (i) Show that $z\bar{z}$ is real. (1)

   (ii) Use De Moivre's Theorem to show that $z^n + \bar{z}^n$ is real for all integers $n \geq 1$. (1)

   (iii) Show that $\dfrac{z}{\bar{z}} + \dfrac{\bar{z}}{z}$ is real. (2)

   (iv) Hence, show that $-2 \leq \dfrac{z}{\bar{z}} + \dfrac{\bar{z}}{z} \leq 2$. (1)

SECTION III. (18 marks) Start a new page.

1. (i) Find all pairs of integers x and y such that $(x + iy)^2 = -3 - 4i$. (2)

   (ii) Hence, solve $z^2 - 3z + (3 + i) = 0$, expressing the roots in the form

   $a + ib$, where a and b are real. (2)

2. (i) Solve the equation $z^3 - 1 = 0$ expressing your answers in modulus-argument form. (1)

   (ii) Let w be the root of $z^3 - 1 = 0$ with the smallest positive argument.

   Simplify $(1 + w^2)^4$. (2)

3. Find the three cube roots of $2 - 2i$. (4)

4. Sketch the locus of z and find the cartesian equation of the locus if:

$$|z - 3 - 2i| = |z - 5 + 3i|.$$ (3)

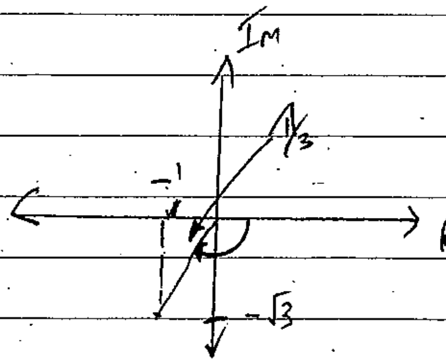
5. If $z = cos\theta + i\,sin\theta$ and using the expansion of $(z - \frac{1}{z})^4$, show that

$$sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3).$$ (4)

SOLUTIONS

SECTION 1.

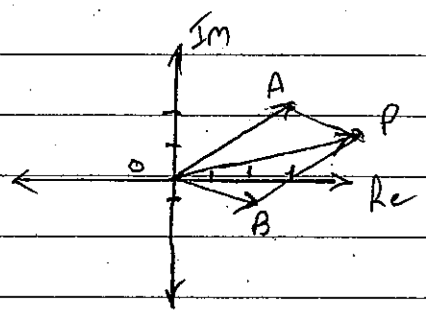
1/



$\arg(w) = -\frac{2\pi}{3}$

D

2/



$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB}$

$= 3+2i + 2-i$

$= 5+i$

P is $5+i$

A

3/ $w = iz$

$w^2 = i^2 z^2$

$= -z^2$

$\therefore z^2 + w^2 = 0$

D

4/ $|z| = \sqrt{x^2+y^2}$

$z + \bar{z} = x+iy + x-iy$

$= 2x$

If $|z|^2 = z + \bar{z}$

$x^2+y^2 = 2x$

$x^2 - 2x + y^2 = 0$

$x^2 - 2x + 1 + y^2 = 1$

A

$(x-2)^2 + y^2 = 1$

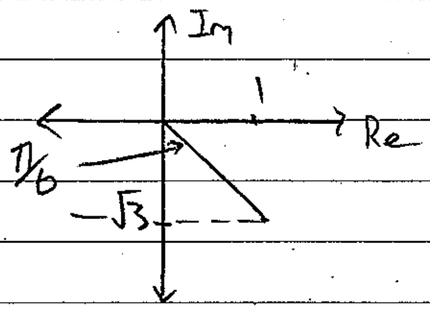
SECTION II

1/ (i) $2z + iw = 2(2+i) + i(1-2i)$
$= 4 + 2i + i - 2i^2$
$= 6 + 3i$ (1)

(ii) $z\overline{w} = (2+i)(1+2i)$
$= 2 + 4i + i + 2i^2$
$= 0 + 5i$ (1)

(iii) $\dfrac{z}{w} = \dfrac{(2+i)}{(1-2i)} \times \dfrac{(1+2i)}{(1+2i)}$
$= \dfrac{0+5i}{1-4i^2}$
$= \dfrac{0+5i}{5}$
$= 0 + i$ (2)

2/ (i)



$|1-\sqrt{3}i| = \sqrt{1^2 + \sqrt{3}^2}$
$= 2$
$\arg(1-\sqrt{3}i) = -\frac{\pi}{3}$

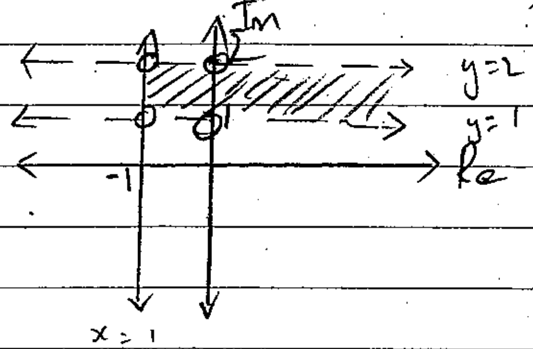
$\therefore \quad 1-\sqrt{3}i = 2 \operatorname{cis}\left(-\frac{\pi}{3}\right)$ (2)

(ii) $(1-\sqrt{3}i)^5 = 2^5 \operatorname{cis}\left(-\frac{5\pi}{3}\right)$ (1)
$= 32 \operatorname{cis}\frac{\pi}{3}$

(iii) $(1-\sqrt{3}i)^5 = 32\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ (2)
$= 32\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$
$= 16 + 16\sqrt{3}i$

3/ (i) If $Re(z) \geq -1$, $x \geq -1$

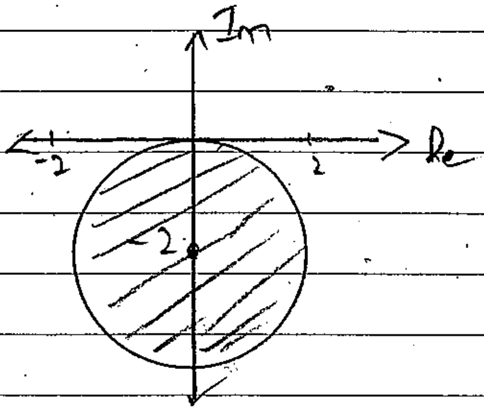
If ~~Im~~ $1 < Im(z) < 2$, $1 < y < 2$



(1)

(ii) $|z+2i| \leq 2$

$|z-(0-2i)| \leq 2$



(1)

(iii) $z+\bar{z} > 6$

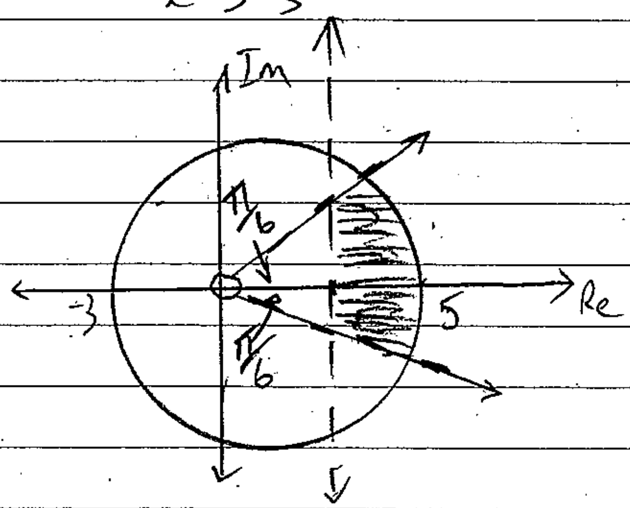
$2x > 6$

$x > 3$

$|z-1| \leq 4$

$|z-(1+0i)| \leq 4$

$-\frac{\pi}{6} < arg(z) < \frac{\pi}{6}$



(2)

4/ (i) $z = r(\cos\theta + i\sin\theta)$

$\bar{z} = r(\cos\theta - i\sin\theta)$

$$z\bar{z} = r(\cos\theta + i\sin\theta) \times r(\cos\theta - i\sin\theta)$$

$$= r^2(\cos^2\theta - i^2\sin^2\theta)$$

$$= r^2(\cos^2\theta + \sin^2\theta)$$ (1)

$$= r^2$$

which is real.

(ii) $z^n = r^n(\cos n\theta + i\sin n\theta)$

$\bar{z}^n = r^n(\cos n\theta - i\sin n\theta)$

$$z^n + \bar{z}^n = r^n(2\cos n\theta)$$ (1)

$$= 2r^n\cos n\theta$$

which is real.

(iii) $\dfrac{z}{\bar{z}} + \dfrac{\bar{z}}{z}$

$$= \frac{z^2 + \bar{z}^2}{z\bar{z}}$$ (2)

$$= \frac{2r^2\cos 2\theta}{r^2}$$ from (ii) from (i)

$$= 2\cos 2\theta$$

which is real.

(iv) Since $-1 \leq \cos 2\theta \leq 1$ (1)

$$-2 \leq 2\cos 2\theta \leq 1$$

$$\therefore -2 \leq \frac{z}{\bar{z}} + \frac{\bar{z}}{z} \leq 2$$

SECTION III

1 (i) If $(x+iy)^2 = -3-4i$

$x^2-y^2+2xyi = -3-4i$

$\therefore\ x^2-y^2 = -3$

& $2xy = -4$

$y = -\frac{2}{x}$

$\therefore\ x^2 - \left(\frac{-2}{x}\right)^2 = -3$

$x^2 - \frac{4}{x^2} = -3$

$x^4+3x^2-4=0$

$(x^2+4)(x^2-1)=0$

$\therefore\ x^2 = 1$

$x = \pm 1$ 2

$y = \mp 2.$

(ii) If $z^2-3z+(3+i)=0$

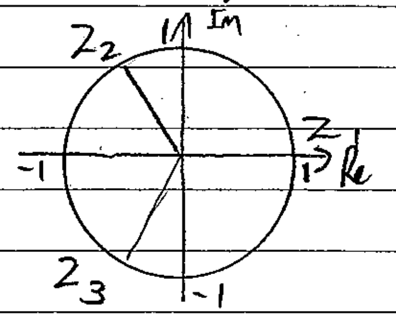
$z = \frac{3 \pm \sqrt{9-4(3+i)}}{2}$

$= \frac{3 \pm \sqrt{-3-4i}}{2}$

$= \frac{3 \pm (1-2i)}{2}$ 2

$= 2-i$ or $1+i$

2/ (i)



If $z^3 = 1$

$z = 1,\ \text{cis}\frac{2\pi}{3},\ \text{cis}-\frac{2\pi}{3}$ 1

(ii) Let the roots be $1, \omega, \omega^2$

$$\sum \text{roots} = -\frac{b}{a}$$

$$\therefore 1 + \omega + \omega^2 = 0$$
$$1 + \omega^2 = -\omega$$

$$\text{So } (1+\omega^2)^4 = (-\omega)^4$$
$$= \omega^4$$
$$= \omega^3 \times \omega$$
$$= 1 \times \omega$$
$$= \omega$$

2

3. If $z^3 = 2-2i$
$$|z|^3 = \sqrt{2^2+(-2)^2}$$
$$= 2\sqrt{2}$$
$$\therefore |z| = \sqrt{2}$$

Let the roots be of the form $\sqrt{2}\operatorname{cis}\phi$

$$\text{So } z^3 = (\sqrt{2}\operatorname{cis}\phi)^3$$
$$z^3 = 2\sqrt{2}\operatorname{cis}3\phi$$
$$\therefore 2\sqrt{2}(\cos 3\phi + i\sin 3\phi) = 2-2i$$

Equating real ~~&~~ imaginary parts

$$2\sqrt{2}\cos 3\phi = 2 \qquad 2\sqrt{2}\sin 3\phi = -2$$
$$\cos 3\phi = \frac{1}{\sqrt{2}} \qquad \sin 3\phi = -\frac{1}{\sqrt{2}}$$

S	A
T	C ✓

$$\therefore 3\phi = \frac{7\pi}{4}, \frac{15\pi}{4}, \frac{23\pi}{4}$$

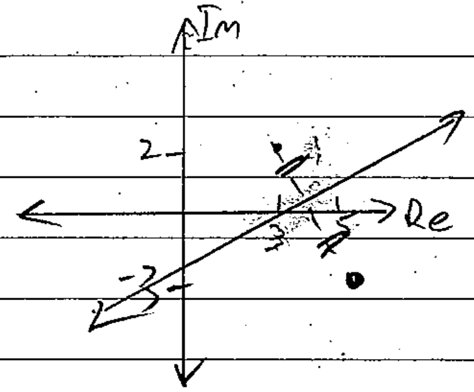
$$\phi = \frac{7\pi}{12}, \frac{15\pi}{12}, \frac{23\pi}{12}$$

$\therefore$ the 3 cube roots are 4

$$z = \sqrt{2}\operatorname{cis}\frac{7\pi}{12},\ \sqrt{2}\operatorname{cis}-\frac{9\pi}{12},\ \sqrt{2}\operatorname{cis}-\frac{\pi}{12}$$

ie $z = \sqrt{2}\operatorname{cis}\frac{7\pi}{12},\ \sqrt{2}\operatorname{cis}-\frac{3\pi}{4},\ \sqrt{2}\operatorname{cis}-\frac{\pi}{12}$

4/



If $|z-3-2i| = |z-5+3i|$

$|z-(3+2i)| = |z-(5-3i)|$

The locus is the perp. bisector of the interval joining $(3,2)$ & $(5,-3)$ on the complex plane

Let $z = x+iy$

$$|x+iy-3-2i| = |x+iy-5+3i|$$

$$\sqrt{(x-3)^2+(y-2)^2} = \sqrt{(x-5)^2+(y+3)^2}$$ 3

$$\cancel{x^2}-6x+\cancel{9}+\cancel{y^2}-4y+4 = \cancel{x^2}-10x+25+\cancel{y^2}+6y+\cancel{9}$$

$$\therefore\ 4x-10y-21=0.$$

5/ $\left(z-\frac{1}{z}\right)^4 = z^4 - 4.z^3.\frac{1}{z} + 6.z^2.\frac{1}{z^2} - 4.z.\frac{1}{z^3} + \frac{1}{z^4}$

$$= \left(z^4+\frac{1}{z^4}\right) - 4\left(z^2+\frac{1}{z^2}\right) + 6$$

Since $z^n + \frac{1}{z^n} = 2\cos n\theta$

$$\left(z-\frac{1}{z}\right)^4 = 2\cos 4\theta - 8\cos 2\theta + 6$$

Also $z^n - \frac{1}{z^n} = 2i \sin n\theta$

$\therefore \left(z - \frac{1}{z}\right)^4 = (2i \sin\theta)^4$

$= 16\, i^4 \sin^4\theta$

$= 16 \sin^4\theta$

4

$\therefore 16 \sin^4\theta = 2\cos 4\theta - 8\cos 2\theta + 6$

$\sin^4\theta = \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}$

$= \frac{1}{8}\left[\cos 4\theta - 4\cos 2\theta + 3\right]$