HSC Mathematics Extension 2
Task 1 2016-2017

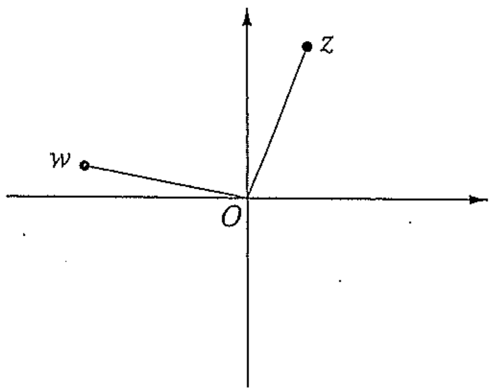
Time Allowed - 1 hour + 5minutes reading

Instructions: Calculators may be used in any parts of the task. For 1 Mark Questions, the correct answer is sufficient to receive full marks. For Questions worth more than 1 Mark, necessary working MUST be shown to receive full marks.

Multiple Choice	/4
Question 5	/12
Question 6	/14
Question 7	/14
Total	/44

2016 - 2017 Extension 2 Assessment Task 1
Answer on the multiple choice answer sheet (1 mark each)

1 The Argand diagram shows the complex numbers z and w, where z lies in the first quadrant and w lies in the second quadrant.



Which complex number could lie in the 3rd quadrant?

(A) $-w$

(B) $2iz$

(C) $\overline{z}$

(D) $w-z$

2 Given $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, which expression is equal to $\left(\overline{z}\right)^{-1}$?

(A) $\frac{1}{2}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

(B) $2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

(C) $\frac{1}{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(D) $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

3 The complex number z satisfies $|z-1| = 1$.

What is the greatest distance that z can be from the point i on the Argand diagram?

(A) 1

(B) $\sqrt{5}$

(C) $2\sqrt{2}$

(D) $\sqrt{2}+1$

4. The cartesian equation of the locus specified by $|z|^2 = z + \bar{z}$ is

A. $(x-1)^2 + y^2 = 1$.

B. $(x+1)^2 + y^2 = 1$

C. $x^2 + (y-1)^2 = 1$

D. $x^2 + (y+1)^2 = 1$

Question 5 (12 Marks) Begin a New Sheet of Paper

a) Given $A = 3 + 4i$ and $B = 1 - i$, express the following in the form $x + iy$ **Marks**

(i) AB 1

(ii) $\dfrac{A}{iB}$ 1

b) Let $\alpha = -\sqrt{3} + i$

(i) Find the exact value of $|\alpha|$ and $\arg \alpha$ 2

(ii) Find the exact value of α^5 in the form $a + ib$ where a and b are real. 2

c) Sketch the locus of the point representing the complez number z on the Argand diagram and find its Cartesian equation:

(i) $|z + i| = |z - 1|$ 2

(ii)

$$\text{Arg}\left(\frac{z+1}{z-i}\right) = \frac{\pi}{2}$$ 3

(d) Multiplying a non-zero complex number by $\dfrac{1-i}{1+i}$ results in a rotation about the origin on an Argand diagram.

What is the rotation? 1

Question 6 (14 Marks) Begin a New Sheet of Paper **Marks**

(a) Let $z_1 = \cos\theta + i\sin\theta$ and $z_2 = \cos\phi + i\sin\phi$, where θ and ϕ are real.

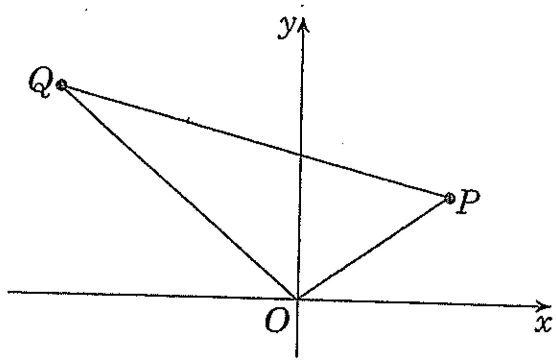
Show that: (i) $\frac{1}{z_1} = \cos\theta - i\sin\theta$ **1**

(ii) $z_1 z_2 = \cos(\theta+\phi) + i\sin(\theta+\phi)$ **1**

(b) (i) Find all pairs of integers x and y such that $(x+iy)^2 = -3-4i$. **2**

(ii) Using (i), or otherwise, solve the quadratic equation $z^2 - 3z + (3+i) = 0$. **2**

c)



The diagram shows a complex plane with origin O. The points P and Q represent arbitrary non-zero complex numbers z_1 and z_2 respectively.

(i) Use the diagram to show that

$$|z_1 - z_2| \leq |z_1| + |z_2|.$$ **2**

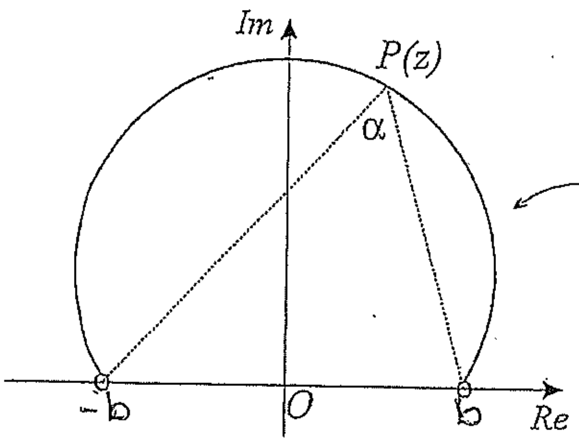
(ii) $$|z_1 - z_2| \geq |z_1| - |z_2|.$$ **2**

(iii) Construct the point R representing $z_1 + z_2$. What type of quadrilateral is $OPRQ$? **2**

(iv) If $|z_1 - z_2| = |z_1 + z_2|$, what can be said about the complex number $\frac{z_1}{z_2}$? **2**

Question 7 (14 Marks) Begin a New Sheet of Paper **Marks**

(a)



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In the diagram above,the locus of the point P representing the complex number z is graphed. Write down a possible equation in terms of z,b and α for the locus of P. Note that constants b and α are real.

b) (i) Use De Moivre's Theorem, to find expressions for $\cos 3\theta$ and $\sin 3\theta$ 3

(ii) Hence prove that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$ 2

c) It is given that $z^5 = 1$ where $z \neq 1$

(i) Show that $z^2 + z + 1 + z^{-1} + z^{-2} = 0$ 2

(ii) Show that $z + z^{-1} = 2\cos\dfrac{2k\pi}{5}$ $k = 1,2,3,4$. 2

(iii) By letting $x = z + z^{-1}$ reduce the equation in (i) above to a quadratic equation in x. 2

(iv) Hence deduce that $\cos\dfrac{\pi}{5}.\cos\dfrac{2\pi}{5} = \dfrac{1}{4}$ 2

1) D. 2) C

3) D 4) A

Q5.

a) $A = 3+4i \quad B = 1-i$

i) $AB = (3+4i)(1-i)$
$= 3 - 3i + 4i + 4$
$= 7 + i$ (1)

ii) $\frac{A}{iB} = \frac{3+4i}{1+i} \times \frac{1-i}{1-i}$
$= \frac{7+i}{2}$ (1)

b) $\alpha = -\sqrt{3} + i$

i) $|\alpha| = \sqrt{3+1}$
$= 2$ (1)

$\tan\beta = \frac{-1}{\sqrt{3}}$

$\therefore \arg \alpha = \frac{5\pi}{6}$ (1)

ii) $\alpha = 2 \text{ cis } \frac{5\pi}{6}$

$\alpha^5 = 2^5 \left(\text{cis } \frac{25\pi}{6}\right)$
$= 32 \text{ cis } \frac{\pi}{6}$ (1)
$= 32 \left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$
$= 32\left(\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}\right)$
$= 16\sqrt{3} + 16i$ (1)

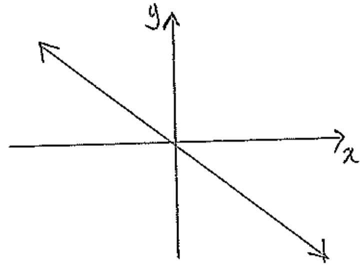
c) i) $|z+i| = |z-1|$

$|x + (y+1)i| = |(x-1) + iy|$

$x^2 + y^2 + 2y + 1 = x^2 - 2x + 1 + y^2$

$2y = -2x$.

$\therefore y = -x$ (1)



(1)

ii) $\text{Arg}\left(\frac{z+1}{z-i}\right) = \frac{\pi}{2}$ → undefined for -1 and i

ie $\text{Arg}(z+1) - \text{Arg}(z-i) = \frac{\pi}{2}$

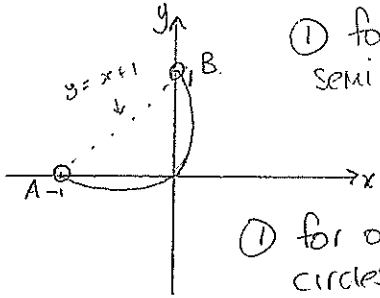
let $\text{Arg}(z+1) = \alpha$ and $\text{Arg}(z-i) = \beta$

$\therefore \alpha - \beta = \frac{\pi}{2}$

$\alpha = \frac{\pi}{2} + \beta$

ie $\alpha > \beta$.

$\angle AZB = \frac{\pi}{2}$



(1) for semi circle

(1) for open circles at $(-1,0)$ and $(0,$

$\therefore$ AB is diameter of circle, centre $\left(-\frac{1}{2}, \frac{1}{2}\right)$ radius $\frac{\sqrt{2}}{2}$ (1) for eqn

ie $\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{1}{2}$, $y < x+1$

d) $\frac{1-i}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-2i-1}{1+1}$
$= -i$

$\therefore$ clockwise by $\frac{\pi}{2}$ (1)
or $-\frac{\pi}{2}$

Q6

a) $z_1 = \text{cis}\,\theta$, $z_2 = \text{cis}\,\phi$

i) $\frac{1}{z_1} = (z_1)^{-1}$

$= (\cos\theta + i\sin\theta)^{-1}$

$= \cos(-\theta) + i\sin(-\theta)$

(by De Moivre)

$= \cos\theta - i\sin\theta$

$=$ RHS //

ii) $z_1 z_2 = \cos(\theta+\phi) + i\sin(\theta+\phi)$

LHS $= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$

$= \cos\theta\cos\phi + i\sin\phi\cos\theta$

$+ i\sin\theta\cos\phi - \sin\theta\sin\phi$

$= (\cos\theta\cos\phi - \sin\theta\sin\phi)$

$+ i(\sin\theta\cos\phi + \sin\phi\cos\theta)$

$= \cos(\theta+\phi) + i\sin(\theta+\phi)$

b i) $(x+iy)^2 = -3-4i$

$x^2 - y^2 = -3$

$2xy = -4 \rightarrow y = \frac{-2}{x}$

$\therefore x^2 - \frac{4}{x^2} = -3$

$x^4 + 3x^2 - 4 = 0$

$(x^2+4)(x^2-1) = 0$

$x = \pm 1$ $\quad x, y \in \mathbb{R}$

$x = 1$, $y = -2$ | $x = -1$, $y = 2$

ii) $z^2 - 3z + (3+i) = 0$

$z = \frac{3 \pm \sqrt{9 - 4(3+i)}}{2}$

$= \frac{3 \pm \sqrt{9-12-4i}}{2}$

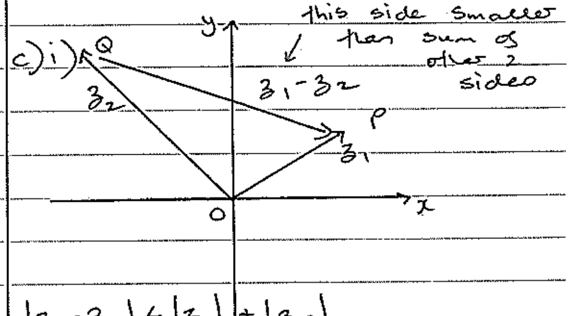
$= \frac{3 \pm \sqrt{-3-4i}}{2}$

$= \frac{3 \pm (1-2i)}{2}$

$= \frac{3+1-2i}{2}, \frac{3-1+2i}{2}$

$= 2-i, \ 1+i$

c) i)



$|z_1 - z_2| \le |z_1| + |z_2|$

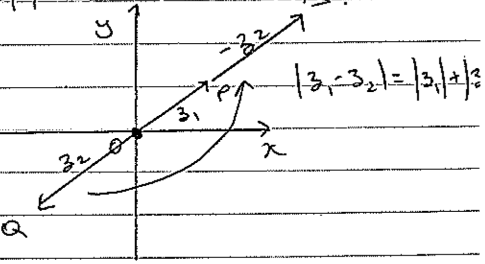
$|z_1 - z_2|$ is the side $\vec{QP}$ (ie length)

$|z_1|$ is the side $\vec{OP}$ (length)

$|z_2|$ is the side $\vec{OQ}$ (length)

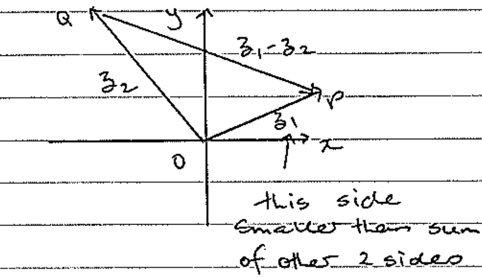
$\therefore \vec{QP}$ has length less than the sum of the other 2 sides in the triangle.

Equality when $\vec{OP}$ (z_1) & $\vec{OQ}$ (z_2) are parallel in opposite directions.



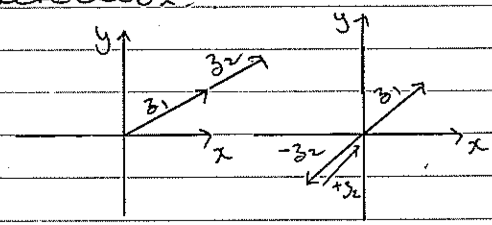
ii) $|z_1 - z_2| \ge |z_1| - |z_2|$

ie $|z_1 - z_2| + |z_2| \ge |z_1|$



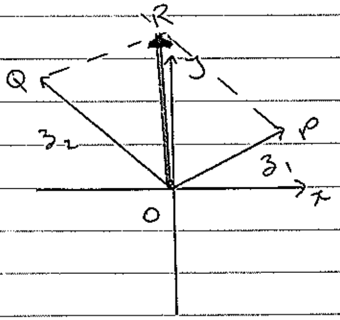
$|z_1|$ is smaller than the sum of the other 2 sides.

Equality when z_1 & z_2 are parallel & in the same direction.



$|z_1 - z_2| + |z_2| = |z_1|$

iii)



$\vec{OR}$ represents $z_1 + z_2$

$\therefore$ OPRQ is a parallelogram

iv) If $|z_1 - z_2| = |z_1 + z_2|$

diagonals equal

$\therefore$ OPRQ is rectangle.

$\arg z_1 - \arg z_2 = \pm\frac{\pi}{2}$

$\therefore \frac{z_1}{z_2} = \pm ki \rightarrow$ Purely Imaginary

7) a) $\arg(z - -b) + \alpha = \arg(z-b)$

$$\alpha = \arg(z-b) - \arg(z+b)$$
$$= \arg\left(\frac{z-b}{z+b}\right)$$

b) i) $(\cos\theta + i\sin\theta)^3 = (\text{cis}\,\theta)^3$ by de Moivre's theorem
$$= \text{cis}\,3\theta$$
$$= \cos 3\theta + i\sin 3\theta$$

also
$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta$$
$$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

equating real parts
$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

equating imaginary parts
$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

ii) $\tan 3\theta = \dfrac{\sin 3\theta}{\cos 3\theta}$
$$= \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$$

÷ top & bottom by $\cos^3\theta$
$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

c) i) $z^5 = 1$
$$z^5 - 1 = 0$$
$$(z-1)(z^4 + z^3 + z^2 + z + 1) = 0$$
since $z \neq 1$ ÷ both sides by $z - 1$
$$z^4 + z^3 + z^2 + z + 1 = 0$$
since $z \neq 0$ ÷ both sides by z^2
$$z^2 + z + 1 + z^{-1} + z^{-2} = 0$$

ii) $z^5 = 1$
$$z^5 = \text{cis}(0 + 2k\pi) \quad \text{for } k \in \mathbb{Z}$$
$$z = \text{cis}\,\frac{2k\pi}{5} \quad \text{but } k \neq 0, 5, 10, \ldots$$
since $z \neq 1$

$$z^{-1} = \left(\text{cis}\,\frac{2k\pi}{5}\right)^{-1}$$
$$= \text{cis}\left(\frac{-2k\pi}{5}\right)$$
$$z + z^{-1} = \text{cis}\left(\frac{2k\pi}{5}\right) + \text{cis}\left(-\frac{2k\pi}{5}\right)$$
$$= 2\cos\frac{2k\pi}{5}$$

iii) $z^2 + z + 1 + z^{-1} + z^{-2} = 0$
$$z^2 + 2 + z^{-2} + z + z^{-1} + 1 - 2 = 0$$
$$(z + z^{-1})^2 + z + z^{-1} - 1 = 0$$
$$x^2 + x - 1 = 0$$

iv) $x^2+x-1=0$

let the roots be α, β

$\Delta=(1)^2-4\times1\times-1$

$=1+4$

$=5$

$\therefore \alpha,\beta \in \mathbb{R}$ & $\alpha \neq \beta$

$x = z+z^{-1} = 2\cos\frac{2k\pi}{5}$ for $k=1,2,3,4$

Note that for $k=4$ $x=2\cos\frac{8\pi}{5}$

$=2\cos\frac{2\pi}{5}$

which is the same as $k=1$

for $k=3$ $x=2\cos\frac{6\pi}{5}$

$=2\cos\frac{4\pi}{5}$

which is the same as $k=2$

$\therefore \alpha = 2\cos\frac{2\pi}{5}$ & $\beta = 2\cos\frac{4\pi}{5}$ or vice versa.

by product of roots $\alpha\beta = \frac{-1}{1}$

$2\cos\frac{2\pi}{5}\times 2\cos\frac{4\pi}{5} = -1$

$4\cos\frac{2\pi}{5}\times -\cos\frac{\pi}{5} = -1$

$\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{4}$