HURLSTONE AGRICULTURAL HIGH SCHOOL

MATHEMATICS – EXTENSION TWO

2006 HSC

ASSESSMENT TASK 1

Examiners ~ G Rawson, Z Pethers

GENERAL INSTRUCTIONS

- Reading Time – 3 minutes.
- Working Time – 40 MINUTES.
- Attempt **all** questions.
- **All** necessary working should be shown in every question.
- This paper contains two (2) questions.
- Marks may not be awarded for careless or badly arranged work.
- Board approved calculators may be used.
- **Each question is to be started on a new piece of paper.**
- This examination paper must **NOT** be removed from the examination room.

STUDENT NAME: ______________________________

TEACHER: ______________________________

QUESTION ONE **20 marks** *Start a SEPARATE sheet*

(a) If $z = 2 - i$ and $w = 1 + 2i$, find

(i) $z + w$ **1**

(ii) $w - z$ **1**

(iii) zw **1**

(iv) $z\overline{w}$ **1**

(v) $\dfrac{z}{w}$ **2**

(b) Find all pairs of integers x and y that satisfy $(x + iy)^2 = 24 + 10i$ **3**

(c) Consider the equation $z^2 + az + (1 + i) = 0$.

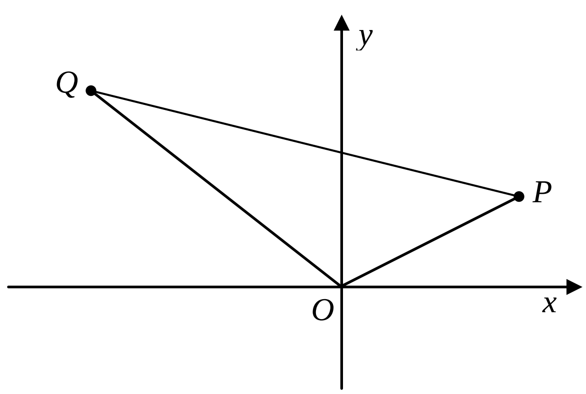
Find the complex number a, given that i is a root of the equation **2**

(d) It is given that $2 + i$ is a root of $P(z) = z^3 + rz^2 + sz + 20$, where r and s are real numbers

(i) State why $2 - i$ is also a root of $P(z)$ **1**

(ii) Factorise $P(z)$ over the real numbers **2**

(e) The diagram below show a complex plane with origin O.
The points P and Q represent arbitrary non-zero complex numbers z and w respectively.
Thus the length of PQ is $|z - w|$



(i) Copy the diagram, and use it to show that $|z - w| \le |z| + |w|$ **2**

(ii) Construct the point R representing $z + w$.
What type of quadrilateral is $OPRQ$? **2**

(iii) If $|z - w| = |z + w|$, what can be said about the complex number $\dfrac{w}{z}$? **2**

QUESTION TWO **20 marks** *Start a SEPARATE sheet*

(a) Given $z = \sqrt{6} - \sqrt{2}\, i$, find

(i)	$\text{Re}(z^2)$	**1**
(ii)	$(\text{Im}\, z)^2$	**1**
(iii)	$\lvert z \rvert$	**1**
(iv)	$\arg z$	**2**
(v)	z^4 in the form $x + iy$	**2**

(b) Let $\alpha = 1 + \sqrt{3}i$ and $\beta = 1 + i$

(i) Find $\dfrac{\alpha}{\beta}$ in the form $x + iy$ **2**

(ii) Express α in modulus-argument form **2**

(iii) Given that β has modulus-argument form $\beta = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$,

Find the modulus-argument form of $\dfrac{\alpha}{\beta}$ **2**

(iv) Hence, find the exact value of $\sin\dfrac{\pi}{12}$ **2**

(c) Sketch the region in the complex plane where the inequalities

$$\lvert z - 1 - i \rvert < 2 \text{ and } 0 < \arg(z - 1 - i) < \frac{\pi}{4}$$

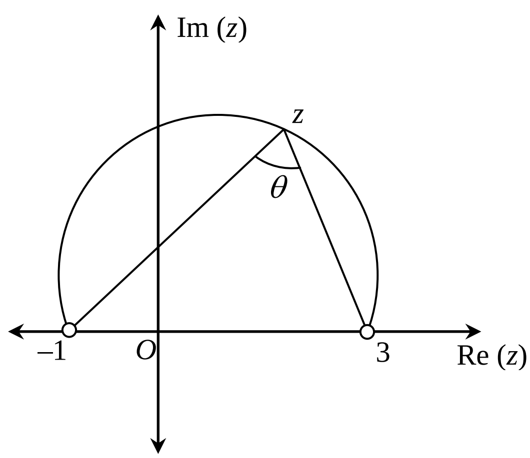
hold simultaneously. **3**

(d) The diagram below shows the locus of points z in the complex plane such that

$\arg(z - 3) - \arg(z + 1) = \dfrac{\pi}{3}$. This locus is part of a circle.

The angle between the lines from –1 to z and from 3 to z is θ, as shown.

Explain why $\theta = \dfrac{\pi}{3}$ **2**



Mathematics Ext 2 Assessment Task 1 (2006 HSC) – Q1

(a) $z = 2 - i$, $w = 1 + 2i$

(i) $z + w = 2 - i + 1 + 2i$
$= 3 + i$

(ii) $w - z = (1 + 2i) - (2 - i)$
$= -1 + 3i$

(iii) $zw = (2 - i)(1 + 2i)$
$= 2 + 4i - i - 2i^2$
$= 4 + 3i$

(iv) $z\overline{w} = (2 - i)(\overline{1 + 2i})$
$= (2 - i)(1 - 2i)$
$= 2 - 4i - i + 2i^2$
$= -5i$

(v) $\dfrac{z}{w} = \dfrac{2 - i}{1 + 2i} \times \dfrac{1 - 2i}{1 - 2i}$
$= \dfrac{-5i}{1 + 4}$
$-i$

(b) $(x + iy)^2 = 24 + 10i$
$x^2 - y^2 + 2xyi = 24 + 10i$

$$\therefore \begin{cases} x^2 - y^2 = 24 & \ldots(1) \\ 2xy = 10 & \ldots(2) \end{cases}$$

Method 1:
x, *y* must be integers, so by inspection, $x = 5$, $y = 1$ or $x = –5$, $y = –1$

Method 2:
Substituting $y = \dfrac{5}{x}$ from (2) into (1):

$x^2 - \dfrac{25}{x^2} - 24 = 0$
$x^4 - 24x^2 - 25 = 0$
$(x^2 - 25)(x^2 + 1) = 0$
$x = \pm 5 \ldots\ldots(x \text{ is real})$
$\therefore x = 5,\ x = 1 \quad \text{or} \quad x = -5,\ y = -1$

(c) $z^2 + az + (1 + i) = 0$
Substitute $z = i$, since i is a root
$i^2 + ai + 1 + i = 0$
$-1 + ai + 1 + i = 0$
$i(a + 1) = 0$
$a = -1$

(d) (i) Since all the coefficients of $P(z)$ are real, complex roots occur in conjugate pairs. So if $2 + i$ is a root, $2 - i$ must also be a root.

(ii) **Method 1:**

Let the other root be α , and using product of roots…

$$(2+i)(2-i)\alpha = -\frac{20}{1}$$

$$(4-i^2)\alpha = -20$$

$$5\alpha = -20$$

$$\alpha = -4$$

$$\therefore P(z) = \left[z-(-4)\right]\left[z-(2+i)\right]\left[z-(2-i)\right]$$

$$= (z+4)(z^2 - 4z + 5)$$

Method 2:

Combining the known factors:

$$(z-2-i)(z-2+i) = (z-2)^2 + 1 = z^2 - 4z + 5$$

$$\therefore P(z) = (z^2 - 4z + 5)(z+4)$$

where the factor $z + 4$ has been found by equating the coefficient of z^3 and the constant.

(e) (i) The sum of any two sides of a triangle is together greater than the third side

so $PQ < OQ + OP$ $\therefore |z-w| < |w| + |z|$ …①

If POQ is straight,

then $PQ = OQ + OP$ $\therefore |z-w| = |w| + |z|$ …②

From ① and ② $|z-w| \le |w| + |z|$ ***NOTE:*** Must explain the < ***and*** the =.

(ii) Parallelogram

(iii) $|z-w| = |z+w|$ means diagonals are =, so rectangle.

Now, $\arg\frac{w}{z} = \arg w - \arg z = \angle QOP = \frac{\pi}{2}$ (rectangle)

$\therefore \frac{w}{z}$ is imaginary

Mathematics Ext 2 Assessment Task 1 (2006 HSC) – Q2

(a) $z = \sqrt{6} - \sqrt{2}i$

(i)
$$z^2 = \left(\sqrt{6} - \sqrt{2}i\right)\left(\sqrt{6} - \sqrt{2}i\right)$$
$$= 6 - 2 - 2\sqrt{12}i$$
$$= 4 - 4\sqrt{3}i$$
$$\text{Re}\left(z^2\right) = 4$$

(ii)
$$|z| = \sqrt{\left(\sqrt{6}\right)^2 + \left(\sqrt{2}\right)^2}$$
$$= 2\sqrt{2}$$

(iii)
$$\arg z = \tan^{-1}\left(\tfrac{-\sqrt{2}}{\sqrt{6}}\right)$$
$$= \tan^{-1}\left(-\sqrt{3}\right)$$
$$= -\frac{\pi}{6}$$

(iv)
$$z^4 = \left(z^2\right)^2 = \left(4 - 4\sqrt{3}i\right)^2$$
$$= 16 - 48 - 32\sqrt{3}i$$
$$= -32 - 32\sqrt{3}i$$

or

$$z = 2\sqrt{2}cis\left(-\frac{\pi}{6}\right)$$
$$z^4 = \left(2\sqrt{2}\right)^4 cis\left(-\frac{4\pi}{6}\right)$$
$$= 64\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

Marks: (i) correct answer – **1 mark** (ii) correct answer – **1 mark**

(iii) correct answer – **2 marks**; correct method with incorrect values – **1 mark** only

(iv) correct answer – **2 marks**; correct method with incorrect values – **1 mark** only

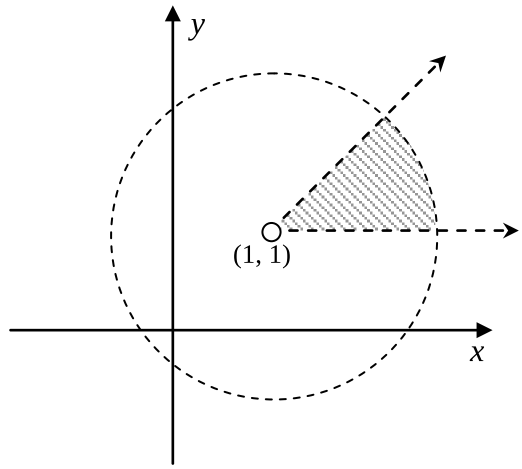
(b) (i)
$$\frac{1+\sqrt{3}\,i}{1+i} \times \frac{1-i}{1-i}$$
$$= \frac{1 - i + \sqrt{3}\,i + \sqrt{3}}{1+1}$$
$$= \frac{1+\sqrt{3}}{2} + i\,\frac{\sqrt{3}-1}{2}$$

(ii)
$$|\,\alpha\,| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$$
$$\arg\alpha = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$$
$$\therefore \alpha = 2\,\text{cis}\frac{\pi}{3}$$

(iii)
$$\frac{\alpha}{\beta} = \frac{2\,\text{cis}\,\frac{\pi}{3}}{\sqrt{2}\,\text{cis}\,\frac{\pi}{4}}$$
$$= \frac{2}{\sqrt{2}}\,\text{cis}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$
$$= \sqrt{2}\,\text{cis}\frac{\pi}{12}$$

(iv) From (i) and (iii),
$$\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) = \frac{1+\sqrt{3}}{2} + i\,\frac{\sqrt{3}-1}{2}$$
so,
$$\sqrt{2}\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2}$$
$$\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(c)



(d) Let arg(z – 3) = α and arg(z + 1) = β

$\alpha - \beta = \Theta$ since the exterior angle of a triangle is equal to the sum of the interior opposite angles.

But we are told $\alpha - \beta = \dfrac{\pi}{3}$,

$\therefore \theta = \dfrac{\pi}{3}$