# HURLSTONE AGRICULTURAL HIGH SCHOOL

**MATHEMATICS – EXTENSION TWO**

# 2010 HSC

ASSESSMENT TASK 1

**Examiner ~ G Rawson**

**GENERAL INSTRUCTIONS**

- Reading Time – 3 minutes.
- Working Time – 40 MINUTES.
- Attempt **all** questions.
- **All** necessary working should be shown in every question.
- This paper contains two (2) questions.
- Marks may not be awarded for careless or badly arranged work.
- Board approved calculators may be used.
- **Each question is to be started on a new piece of paper.**
- This examination paper must **NOT** be removed from the examination room.

STUDENT NAME: ______________________________

TEACHER: ______________________________

**QUESTION ONE** **15 marks** *Start a SEPARATE sheet*

(a) The ellipse $\mathcal{E}$ has equation $\frac{x^2}{25}+\frac{y^2}{16}=1$.

(i) State the intercepts with the axes. **1**

(ii) Determine the eccentricity of $\mathcal{E}$ **1**

(iii) State the coordinates of the two foci. **1**

(iv) Write down the equation of the auxiliary circle. **1**

(b) (i) Sketch the function $f(x)=x^2-c^2$, where $|c|>1$,
clearly showing its vertex and intercepts **1**

(ii) Hence, without using calculus, draw separate sketches, at least $\frac{1}{3}$ of a page, for each of the following curves.
For each sketch, show the original function with a dotted line, and clearly indicate turning points.

(A) $y=|f(x)|$ **2**

(B) $y=\frac{1}{f(x)}$ **2**

(C) $y=\sqrt{f(x)}$ **2**

(D) $y=[f(x)]^2$ **2**

(E) $y=[f(x)]^3$ **2**

**QUESTION TWO** **15 marks** *Start a SEPARATE sheet*

(a) The points $P\left(cp, \dfrac{c}{p}\right)$ and $Q\left(cq, \dfrac{c}{q}\right)$ are points on the rectangular hyperbola $xy = c^2$.
Tangents to the rectangular hyperbola at *P* and *Q* intersect at the point *R*.

(i) Show that the tangent to the rectangular hyperbola at any point $T\left(ct, \dfrac{c}{t}\right)$ has equation $x + t^2 y - 2ct = 0$. **1**

(ii) Find the coordinates of *R*. **2**

(iii) If *P* and *Q* are variable points on the rectangular hyperbola which move so that $p^2 + q^2 = 2$, show that the equation of the locus of *R* is given by $xy + y^2 = 2c^2$. **3**

(b) The points $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$ lie on the same branch of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and *PQ* is a focal chord, passing through $S(ae,\ 0)$.
Use the gradients of *PS* and *QS* to show that $e = \dfrac{\sin\theta - \sin\phi}{\sin(\theta - \phi)}$. **4**

(c) (i) Determine the real values of $\lambda$ for which the equation

$$\frac{x^2}{4-\lambda} + \frac{y^2}{2-\lambda} = 1 \text{ defines}$$

(A) an ellipse **1**

(B) a hyperbola **1**

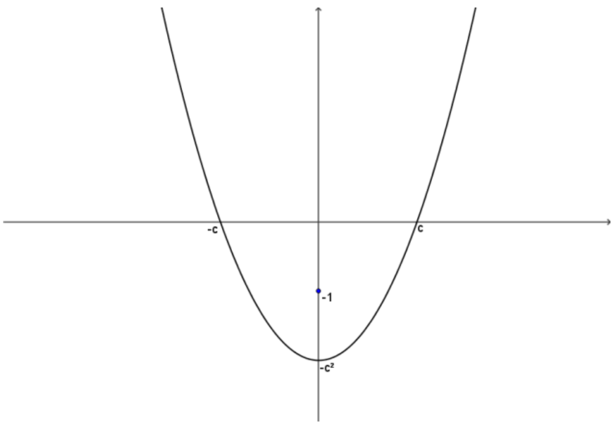
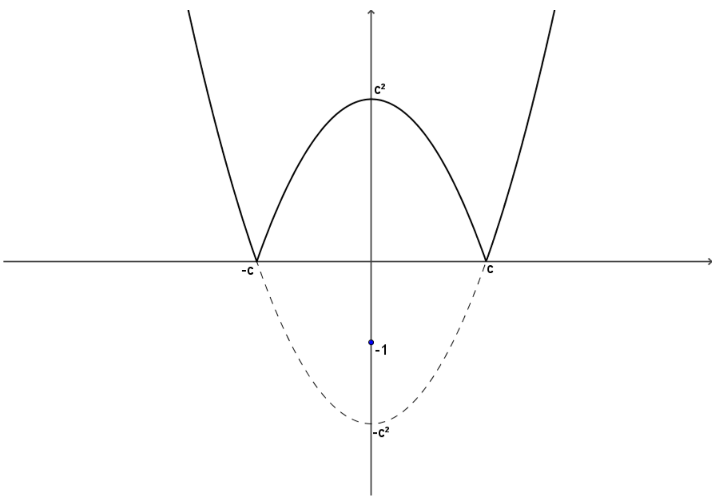
(ii) Describe how the shape of this curve changes as $\lambda$ increases from 1 towards 2. What is the limiting position of the curve as $\lambda$ approaches 2? **3**

| Year 12 | Mathematics Extension 2 | Ass Task 1 2010 HSC |
|---|---|---|
| Question No. 1 | Solutions and Marking Guidelines | |

**Outcomes Addressed in this Question**

E4 - uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections

E6 - combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions

| Part / Outcome | Solutions | Marking Guidelines |
|---|---|---|
| **(a) E4** | (i) (5, 0), (–5, 0), (0, 4), (0, –4) | **<u>1 mark :</u>** correct solution |
| | (ii) $b^2 = a^2\left(1-e^2\right)$<br>$16 = 25\left(1-e^2\right)$<br>$e^2 = \dfrac{9}{25}$<br>so $e = \dfrac{3}{5}$ | **<u>1 mark :</u>** correct solution |
| | (iii) Foci: $\left(\pm ae, 0\right)$ ie $\left(3,0\right)$ and $\left(-3,0\right)$ | **<u>1 mark :</u>** correct solution |
| | (iv) auxiliary circle: $x^2+y^2=25$ | **<u>1 mark :</u>** correct solution |
| **(b)** (i) **E6** |  | **<u>1 mark :</u>** correct graph with correct intercepts shown |
| (ii)(A) **E6** |  | **<u>2 marks :</u>** correct graph with correct intercepts shown<br><br>**<u>1 mark :</u>** partially correct |

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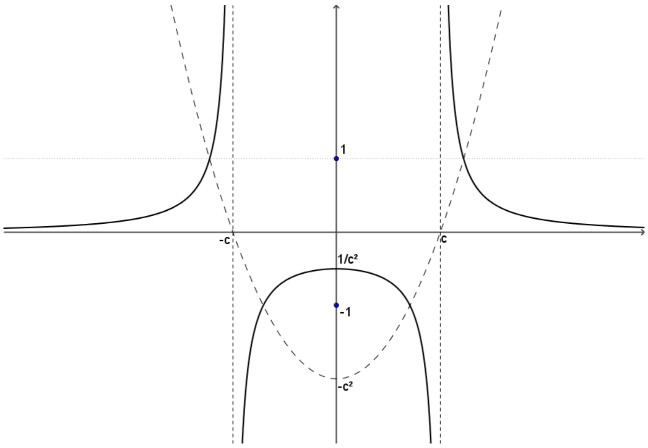
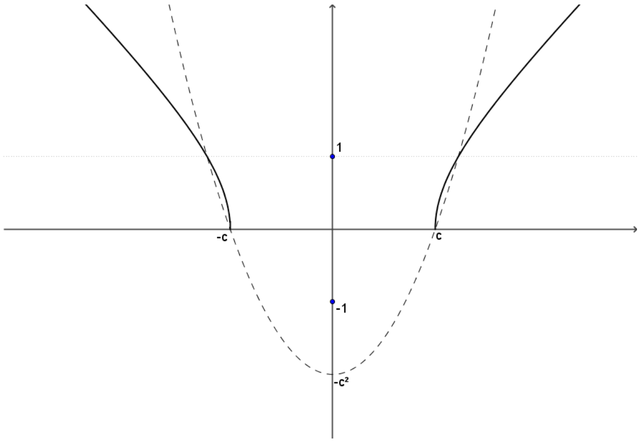
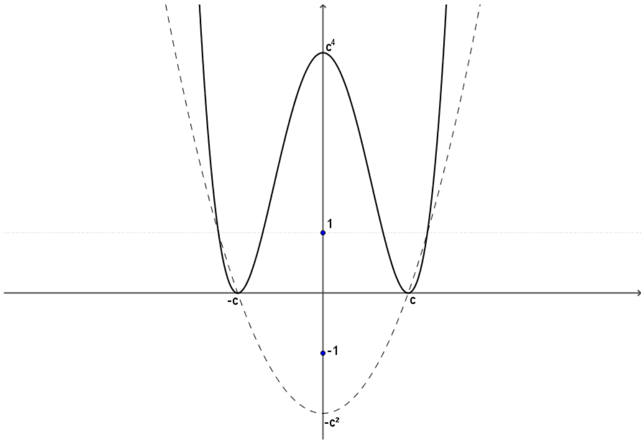
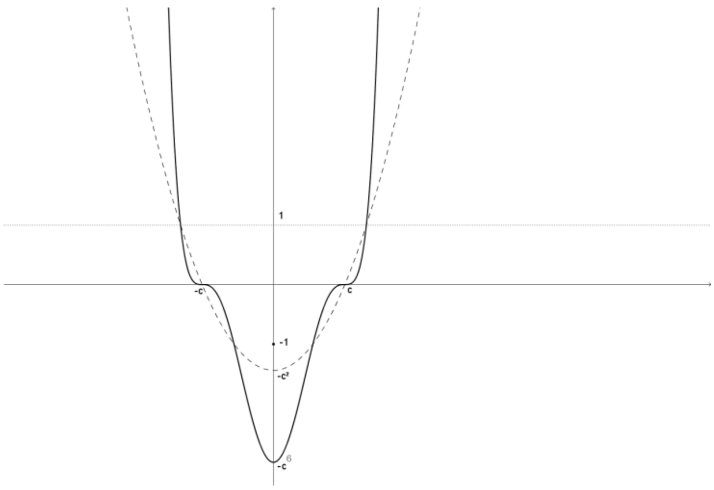
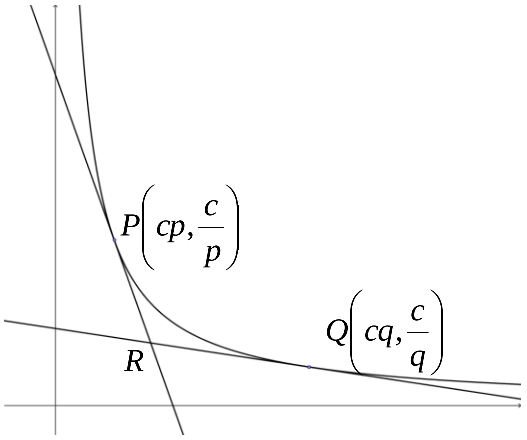
|---|---|---|
| (ii)(B) **E6** |   | **<u>2 marks :</u>** correct graph with correct intercepts shown<br><br>**<u>1 mark :</u>** partially correct |
| (ii)(C) **E6** |   | **<u>2 marks :</u>** correct graph with correct intercepts shown<br><br>**<u>1 mark :</u>** partially correct |
| (ii)(D) E6 |   | **<u>2 marks :</u>** correct graph with correct intercepts shown<br><br>**<u>1 mark :</u>** partially correct |
| (ii)(E) E6 |   | **<u>2 marks :</u>** correct graph with correct intercepts shown<br><br>**<u>1 mark :</u>** partially correct |

| Year 12 | Mathematics Extension 2 | Ass Task 1 2010 HSC |
|---|---|---|
| Question No. 2 | Solutions and Marking Guidelines | |

**Outcomes Addressed in this Question**

E4 - uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections

| Part / Outcome | Solutions | Marking Guidelines |
|---|---|---|
| **(a)** |  | |
| **(a)(i)** | $xy = c^2$<br>$y = \frac{c^2}{x}$<br>$y' = -\frac{c^2}{x^2}$<br>$m = -\frac{1}{t^2}$ at $x = ct$<br>$\therefore$ tangent is<br>$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$<br>$t^2 y - ct = -x + ct$<br>$x + t^2 y - 2ct = 0$ | **<u>1 mark :</u>** correct solution |
| **(a)(ii)** | tangent at $P$: $x + p^2 y - 2cp = 0$ ...(1)<br>tangent at $Q$: $x + q^2 y - 2cq = 0$ ...(2)<br>$(1)-(2)$ $\quad (p^2 - q^2)y = 2c(p - q)$<br>$y = \frac{2c}{p+q} \quad (p \neq q)$<br>sub into (1) $\quad x = 2cp - p^2\left(\frac{2cp}{p+q}\right)$<br>$= \frac{2cp^2 + 2cpq - 2cp^2}{p+q}$<br>$= \frac{2cpq}{p+q}$<br>$\therefore R$ is $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$ | **<u>2 marks :</u>** correct solution<br><br>**<u>1 mark :</u>** significant progress towards correct solution |

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|---|---|---|
| **(a)(iii)** | $x=\dfrac{2cpq}{p+q} \quad ...(1)$<br>$y=\dfrac{2c}{p+q} \quad ...(2)$<br>$(1)\div(2) \quad \dfrac{x}{y}=pq$<br>and from $(2)$ $\quad p+q=\dfrac{2c}{y}$<br>Now, given that $p^2+q^2=2$<br>$(p+q)^2-2pq=2$<br>$\left(\dfrac{2c}{y}\right)^2-2\left(\dfrac{x}{y}\right)=2$<br>$4c^2-2xy=2y^2$<br>$xy+y^2=2c^2$<br><br>***OR*** sub $R\left(\dfrac{2cpq}{p+q},\dfrac{2c}{p+q}\right)$ into $xy+y^2=2c^3$ to show true | **<u>3 marks :</u>** correct solution<br><br>**<u>2 marks :</u>** substantial progress towards correct solution<br><br>**<u>1 mark :</u>** partial progress towards correct solution |
| **(b)** | $S$ lies on $PQ$<br>so $\quad m_{PS}=m_{QS}$<br>$\dfrac{b\tan\theta-0}{a\sec\theta-ae}=\dfrac{b\tan\phi-0}{a\sec\phi-ae}$<br>$\dfrac{\tan\theta}{\sec\theta-e}=\dfrac{\tan\phi}{\sec\phi-e}$<br>$\tan\theta\sec\phi-e\tan\theta=\tan\phi\sec\theta-e\tan\phi$<br>$e(\tan\theta-\tan\phi)=\tan\theta\sec\phi-\tan\phi\sec\theta$<br>$e=\dfrac{\tan\theta\sec\phi-\tan\phi\sec\theta}{\tan\theta-\tan\phi}\times\dfrac{\cos\theta\cos\phi}{\cos\theta\cos\phi}$<br>$=\dfrac{\sin\theta-\sin\phi}{\sin\theta\cos\phi-\cos\theta\sin\phi}$<br>$=\dfrac{\sin\theta-\sin\phi}{\sin(\theta-\phi)}$ | **<u>4 marks :</u>** correct solution<br><br>**<u>3 marks :</u>** substantial progress towards correct solution<br><br>**<u>2 marks :</u>** partial progress towards correct solution<br><br>**<u>1 mark :</u>** recognising that $m_{PS}=m_{QS}$ |

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|---|---|---|
| **(c)(i)(A)** | $$\frac{x^2}{4-\lambda}+\frac{y^2}{2-\lambda}=1$$ for ellipse, $4-\lambda>0$ and $2-\lambda>0$<br>$\lambda<4$ and $\lambda<2$<br>ie $\underline{\lambda<2}$ | **1 mark :** correct solution |
| **(c)(i)(B)** | for hyperbola, $4-\lambda>0$ and $2-\lambda<0$<br>$\lambda<4$ and $\lambda>2$<br>ie $2<\lambda<4$<br>*OR*<br>for hyperbola, $4-\lambda<0$ and $2-\lambda>0$<br>$\lambda>4$ and $\lambda<2$<br>ie no solution<br><br>$\therefore \underline{2<\lambda<4}$ | **1 mark :** correct solution |
| **(c)(ii)** | As $\lambda$ increases from 1 to 2, the curve remains as an ellipse, with both the major and minor axes decreasing. The ellipse becomes flatter, or more 'cigar' shaped.<br><br>As $\lambda$ approaches 2, the ellipse approaches a line segment joining $\left(-\sqrt{2}, 0\right)$ to $\left(\sqrt{2}, 0\right)$. | **3 marks :** correct and complete explanation<br><br>**2 marks :** reasonably correct explanation<br><br>**1 mark :** partially correct |