INTERNATIONAL GRAMMAR SCHOOL

4 UNIT MATHEMATICS

HSC ASSESSMENT #1

NOVEMBER 1996

TIME ALLOWED: 80 MINUTES

INSTRUCTIONS

- **Approved calculators may be used**
- **Marks may be deducted for untidy work**
- **Attempt all questions**
- **Begin each question on a new page**

QUESTION 1: (8 marks)

(a) Let $Z = \dfrac{-i}{1 + i\sqrt{3}}$

(i) Sketch Z on the Argand Diagram.

(ii) Find the modulus and argument of Z.

(b) Given $w = \dfrac{2 - 3i}{1 + i}$

determine

(i) $|w|$ (i.e. the modulus of w)

(ii) $\bar{w}$ (i.e the conjugate of w)

(iii) $w + \bar{w}$

QUESTION 2: (5 marks)

1. (a) Solve the equation $Z^4 = 1$

Hence find all solutions of the equation

$$Z^4 = (Z-1)^4$$

QUESTION THREE: (9 marks)

(i) Solve the following pair of equations for z and w where z and w are complex numbers. Express your answers in the form $a + ib$.

$$2z + 3iw = 0$$

$$(1-i)z + 2w = i - 7$$

ii) Express 1, w, w^2 (the cube roots of unity) in mod-arg form. Verify that $1 + w + w^2 = 0$ and $1 . w . w^2 = 1$.

iii) Express the roots of the equation

$$z^2 + 2(1 + 2i)z - (11 + 2i) = 0$$

in the form $a + ib$ where a, b are real.

QUESTION 4 (9 marks)

(i) Find the four 4th roots of -16 and show them on a circle in an Argand diagram.

(ii) Use the principle of mathematical induction to prove De Moivre's Theorem

i.e. $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$.

for positive integral n.

(iii) Express $\cos 4\theta$ in terms of $\cos \theta$.

QUESTION 5. (9 marks)

(a) Let $w_1 = 8 - 2i$ and $w_2 = -5 + 3i$.
Find $w_1 + \overline{w}_2$.

(b) (i) Show that $(1-2i)^2 = -3 - 4i$.

(ii) Hence solve the equation
$$z^2 - 5z + (7+i) = 0.$$

(c)

(i) Find the quadratic equation whose roots are $2 + i$ and $\frac{1}{2+i}$

(ii) Solve the following for z: $\frac{1}{z} = 1 + i + \frac{2}{1-i}$

END OF PAPER

IGS 96 ASSESSMENT 4U

Complex Numbers

TOTAL = 40

Unit Answers & Marking Scale HSC Assessment #1 November 1996

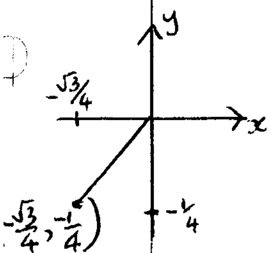
Question One

$z = \frac{-i}{1+i\sqrt3} \times \frac{1-i\sqrt3}{1-i\sqrt3}$

$= \frac{-i+i^2\sqrt3}{1-i^2.3}$ (1)

$= \frac{-i-\sqrt3}{4}$

$= \frac{-\sqrt3}{4} - \frac{i}{4}$ (1)



(1)

$r = \sqrt{\frac{3}{16}+\frac{1}{16}}$

$= \frac12$ = modulus (1)

$\tan\theta = \frac14 / \frac{\sqrt3}{4}$

$= 1/\sqrt3$

$\therefore \theta = 30^\circ$

$\therefore$ argument is -150° (1)

(or $-\frac{5\pi}{6}$)

$w = \frac{2-3i}{1+i}$

$\frac{2-3i}{1+i} \times \frac{1-i}{1-i} = \frac{2+3i^2-3i-2i}{2}$

$= \frac{-1-5i}{2}$

$|w| = \sqrt{1/4 + 25/4}$ (1)

$= \frac{\sqrt{26}}{2} = \sqrt{6.5}$

$\bar{w} = -\frac12 + \frac52 i$ (1)

Question Two

1.(a) $z^4 - 1 = 0$

$(z^2-1)(z^2+1) = 0$

$(z-1)(z+1)(z-i)(z+i) = 0$

$\therefore z = \pm1, \pm i$ (2)

(b) $z^4 = (z-1)^4$

$\frac{z^4}{(z-1)^4} = 1$ $\qquad w^4 = 1$

$\left(\frac{z}{z-1}\right)^4 = 1$ $\qquad w = 1$ $\qquad \frac{z}{z-1} = 1$ (1)

From (a) $\frac{z}{z-1} = \pm1, \pm i$

Consider all solutions :-

If $\frac{z}{z-1} = 1$, $z = z-1$ no soln (½)

If $\frac{z}{z-1} = -1$

$\therefore z = -z+1 \quad \therefore z = \frac12$ (½)

If $\frac{z}{z-1} = i$

$\therefore z = iz - i$

$z(1-i) = -i$

$z = \frac{-i}{1-i}$ (½) (or $\frac12 + \frac{i}{2}$)

If $\frac{z}{z-1} = -i$

$z = -zi + i$

$z(1+i) = i$

$z = \frac{i}{1+i}$ (½) or $(\frac12 - \frac{i}{2})$

Question Three

$2z + 3iw = 0$ —— (1)

$(1-i)z + 2w = i-7$ —— (2)

(2) × (1+i)

$(1-i)(1+i)z + 2(1+i)w = (1+i)(i-7)$

Question Three (continued)

$2z + (2+2i)w = -8-6i$ —— (3)

(1)−(3)

$(3i-2-2i)w = 8+6i$

$\therefore w = \frac{8+6i}{-2+i} \times \frac{-2-i}{-2-i}$

$w = -2-4i$ (2)

Sub into (1)

$2z + 3i(-2-4i) = 0$

$2z = -3i(-2-4i)$ (1)

$z = -6+3i$

(ii) $1 = \cos 0^\circ + i\sin 0^\circ$

$\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ (½)

$\omega^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ (½)

$= \cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)$

Hence LHS =

$1+\omega+\omega^2 = \text{cis}\,0^\circ + \text{cis}\,\frac{2\pi}{3} + \text{cis}\,\frac{4\pi}{3}$

$= 1 - \frac12 + \frac{\sqrt3}{2}i - \frac12 - \frac{\sqrt3}{2}i$ (1)

$= 0$ = RHS.

And

$1.\omega.\omega^2 = 1\left(-\frac12 + \frac{\sqrt3}{2}i\right)\left(-\frac12 - \frac{\sqrt3}{2}i\right)$

$= 1.1$

$= 1$ (1)

(iii) $z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$z = \frac{-2-4i \pm \sqrt{4(1+2i)^2 + 4(11+2i)}}{2}$

$= \frac{2(-1-2i) \pm 2\sqrt{1-4+4i+11+2i}}{2}$

$= -1-2i \pm \sqrt{8+6i}$ (2)

$= 2-i$ or $-4-3i$

(Note $\sqrt{8+6i} = 3+i$

Unit Answers (continued)

uestion 4

) $z^4 = -16$

$z^4 = 16 \text{ cis } \pi$

The four fourth roots

e given by $16^{1/4} \text{ cis}\left(\frac{\pi + 2k\pi}{4}\right)$

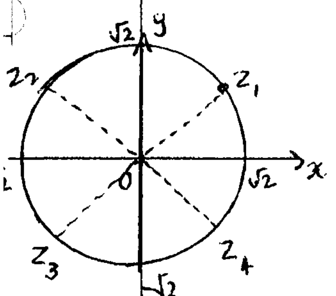
: $2 \text{ cis } \pi/4 = \sqrt{2}(1+i)$ $\left(\frac{1}{2}\right)$

$2 \text{ cis } 3\pi/4 = \sqrt{2}(-1+i)$ $\left(\frac{1}{2}\right)$

: $2 \text{ cis } 5\pi/4 = \sqrt{2}(-1-i)$ $\left(\frac{1}{2}\right)$

$= 2 \text{ cis } \frac{7\pi}{4} = \sqrt{2}(1-i)$ $\left(\frac{1}{2}\right)$

)



(1)

Induction. See the textbook

ige ______. for the complete

swer; especially note (3)

ep 3. ~~Multiply both~~

)

$\cos 4\theta$ in terms of $\cos\theta$

$\cos 4\theta + i\sin 4\theta$

$(\cos\theta + i\sin\theta)^4$ by

De Moivres theorem

$(c+is)^4$

$c^4 + 4c^3 is + 6c^2 i s^2 + 4ci^3 s^3$

$+ i^4 s^4$

$c^4 - 6c^2 s^2 + s^4 + i(4c^3 s - 4cs^3)$

quate real parts

$\cos 4\theta = c^4 - 6c^2 s^2 + s^4$

$= c^4 - 6c^2(1-c^2) + (1-c^2)^2$

$= 8\cos^4\theta - 8\cos^2\theta + 1$ (2)

Question 5

a) $w_1 = 8 - 2i$, $w_2 = -5 + 3i$

$w_1 + \overline{w_2} = 8 - 2i + (-5 - 3i)$

$= 3 - 5i$ (1)

b)(i) $(1-2i)^2 = 1 - 4i + 4i^2$

$= 1 - 4i - 4$

$= -3 - 4i$ (1)

(ii) $z = \dfrac{5 \pm \sqrt{25 - 4(7+i)}}{2}$

$= \dfrac{5 \pm \sqrt{-3-4i}}{2}$

$= \dfrac{5 \pm (1-2i)}{2}$ (2)

$\therefore z = 3 - i,\ 2 + i$.

(c)(i) $\alpha = 2 + i$ $\quad \beta = \frac{1}{2+i} = \frac{2-i}{5}$

$\therefore \alpha + \beta = 2 + i + \frac{2-i}{5} = \frac{12+4i}{5}$

$\alpha\beta = (2+i) \cdot \frac{1}{(2+i)} = 1$ (3)

$\therefore$ Equation is

$x^2 - (\alpha+\beta)x + \alpha\beta = 0$

i.e. $x^2 - \frac{1}{5}(12+4i)x + 1 = 0$

or $5x^2 - 4(3+i)x + 5 = 0$

(ii) $\frac{1}{z} = 1 + i + \frac{2}{1-i}$

$\therefore \frac{1}{z} = 1 + i + \frac{2(1+i)}{2}$

$= 1 + i + 1 + i$

$= 2 + 2i$ (2)

$\therefore z = \frac{1}{2+2i}$

$= 2 - 2i = \frac{1}{4} - \frac{i}{4}$

$\frac{1}{z} = 1 + i + \frac{2}{1-i}$

$\therefore \frac{1}{z} = \frac{(1+i)(1-i) + 2}{(1-i)}$

$(1-i) = z(1-i^2) + 2z$

$1 - i = 2z + 2z$

$\therefore z = \frac{1-i}{4}$

$= \frac{1}{4} - \frac{i}{4}$