Question 1. **Marks**

(a) Let $z = 5 - 2i$ and $w = 3 + 4i$.

(i) Find $2w - \overline{w}$. **2**

(ii) Find $z\left(2w - \overline{w}\right)$. **2**

(b) Given $\alpha = 1 - i\sqrt{3}$,

(i) Find the exact value of $|\alpha|$ and $Arg(\alpha)$. **2**

(ii) Hence or otherwise, find the exact value of α^8. **3**
[express in the form $a + ib$, where a and b are real numbers]

(c) Find the new complex number when $1 + i$ is rotated 45^0 anticlockwise about the origin O, in the Argand plane. **2**

(d) Find x and y, when $(x + iy)(2 + 3i) = 5 + 6i$. **2**

(e) Given $\psi^2 = 24 - 70i$, **2**

Find ψ in the form $a + ib$, where a and b are real.

Question 2. **[START A NEW PAGE]**

(a) Shade the region of the Argand plane consisting of those variable points z for which:

(i) $\text{Re}(z) < 2$ and $\text{Im}(z) > -1$. **1**

(ii) $|z - 1 - i| \le 1$ and $0 \le \arg z \le \frac{\pi}{4}$. **2**

(b) Find the equation of the locus of z, if $|z - i| = \text{Im}(z)$. **3**
Sketch the locus on an Argand plane.

(c) Find the equation of the locus of z when $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$. **2**

Marks

Q 2 *continued*

(d) Given that $z = x + iy$ is a variable point on the Argand plane such that

$$z\overline{z} - 2(z + \overline{z}) = 21.$$

(i) Find the locus of z. **2**

(ii) Hence determine the maximum value of: $|z - 4|$. **2**

(e) Given 1, ω and ω^2 are the cube roots of unity and each are represented by the points A_1, A_2 and A_3 respectively on an Argand plane. **3**

Find the value of $A_1A_2 \times A_1A_3$, where 'A_1A_2' represents the length A_1A_2.

Question 3. **[START A NEW PAGE]**

(a) Point A represents the complex number α and the point P represents the complex number z. **2**
Point P is rotated about the point A through a right angle in an anticlockwise direction to take up the new position, B, representing the complex number β.

Find β (in terms of α and z).

(b) Given $z = \cos\theta + i\sin\theta$,

(i) Use De Moivre's theorem, to show that **2**

$$\cos n\theta = \frac{1}{2}\left(z^n + z^{-n}\right).$$

(ii) Hence deduce that: $\cos\theta\cos 2\theta = \frac{1}{2}\left(\cos\theta + \cos 3\theta\right)$ **3**

(c) Given the complex number z, such that $z = k(\cos\theta + i\sin\theta)$, where k is a real number and $0 < \theta < \pi$. **3**

Show that: $\arg(z + k) = \frac{1}{2}\theta$.

Marks

Q 3. *continued*

(d) Given $z = x + iy$ is a variable point and $\alpha = a + ib$ is a fixed point on the Argand plane,

(i) Show that $z\alpha - \bar{z}\bar{\alpha} = 0$ represents a straight line through the origin *O*. **2**

(ii) Suppose that z_1 and z_2 are the solutions to the simultaneous equations: $z\alpha - \bar{z}\bar{\alpha} = 0$ and $|z - \beta| = k$, where $\beta = p + iq$, and where *p*, *q* and *k* are positive real numbers, **3**

show that $|z_1||z_2| = |p^2 + q^2 - k^2|$.

Question 4. **[START A NEW PAGE]**

(a) Let α, β and γ be the zeros of the polynomial function:

$$P(x) = x^3 + 2x^2 + 19x + 18.$$

(i) Find $\alpha + \beta + \gamma$. **1**

(ii) Find $\alpha^2 + \beta^2 + \gamma^2$. **2**

(iii) Hence, or otherwise determine how many of the zeros are real. Give reasons. **2**

(b) Consider the polynomial equation: $z^5 - i = 0$,

(i) Show that $z = i$ is a solution to the equation, and hence show that $1 - iz - z^2 + iz^3 + z^4 = 0$, for $z \neq i$. **2**

(ii) Hence or otherwise, find all the roots of $z^5 - i = 0$. [you may leave the roots in $cis\theta$ form] **2**

(iii) Show that $(z - i)[z^2 - 2i\sin\frac{\pi}{10}z - 1][z^2 + 2i\sin\frac{3\pi}{10}z - 1] = 0$. **4**

(iv) Hence show that: $\sin\frac{\pi}{10}\sin\frac{3\pi}{10} = \frac{1}{4}$. **2**

THE END

☺ 😐 ☹

JRAHS SOLUTIONS AT1 M.E.2
TERM 4, 2004.

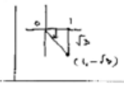
Q1 (a) $z = 5 - 2i$, $w = 3 + 4i$

(i) $2w - \bar{w} = 6 + 8i - (3 - 4i)$
$= \underline{3 + 12i}$ ② | 1 $3-4i$ | 1 $3+12i$

(ii) $z(2w - \bar{w}) = (5 - 2i)(3 + 12i)$
$= 15 + 60i - 6i + 24$ | 1
$= \underline{39 + 54i}$ ② | 1

(b) $\alpha = 1 - i\sqrt{3}$

(i) $|\alpha| = \sqrt{1^2 + (-\sqrt{3})^2}$
$|\alpha| = \underline{2}$



$\text{Arg}\,\alpha = \underline{-\frac{\pi}{3}}$ ② | 1 each

(ii) $\alpha^8 = [2 \text{ cis}(-\frac{\pi}{3})]^8$
$= 2^8 \text{ cis}(-\frac{8\pi}{3})$ (de Moivre's thm)
$= 256 \text{ cis}(-\frac{2\pi}{3})$ | 1
$= 256 \text{ cis} \frac{4\pi}{3}$
$= 256(\cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3})$
$= 256(-\frac{1}{2} - i\frac{\sqrt{3}}{2})$ ③ | 1
$\alpha^8 = \underline{-128 - 128\sqrt{3}\,i}$ | 1

(c) New complex no. $= (1+i) \text{ cis} \frac{\pi}{4}$
$= (1+i)\frac{1}{\sqrt{2}}(1+i)$
$= \underline{i\sqrt{2}}$ ②

(d) $(x+iy)(2+3i) = 5+6i$
$(x+iy) = \frac{(5+6i)}{(2+3i)} \times \frac{(2-3i)}{(2-3i)}$
$= \frac{10 - 15i + 12i + 18}{4+9}$
$= \frac{28 - 3i}{13} = \frac{28}{13} - \frac{3}{13}i$

$\therefore \begin{cases} x = \frac{28}{13} \\ y = -\frac{3}{13} \end{cases}$ ② | 1 | 1

(e) $\psi^2 = 24 - 70i$

METHOD 1
$24 - 70i = 24 - 2 \times 35i$
$= 24 - 2 \times 7 \times 5i$
$= (7 - 5i)^2$ or $(-7 + 5i)^2$
$\therefore \underline{\psi = 7 - 5i \text{ or } -7 + 5i}$ ②

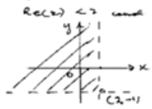
METHOD 2 Let $\psi = x + iy$, $x, y \in \mathbb{R}$
$\psi^2 = x^2 - y^2 + 2xyi$
$\therefore x^2 - y^2 = 24$ — (1)
$xy = -35$ — (2)
$x^2y^2 = (-35)^2 = 1225$
$\therefore x^2 \times (1)$ $x^4 - x^2y^2 = 24x^2$
i.e. $x^4 - 1225 = 24x^2$
$x^4 - 24x^2 - 1225 = 0$
$(x^2 - 49)(x^2 + 25) = 0$
$\therefore x^2 = 49$ or $x^2 = -25$
$\therefore x = \pm 7$ or no real soln poss.
so $y = \mp 5$ from (2)
$\therefore \underline{\psi = 7 - 5i \text{ or } -7 + 5i}$

METHOD 3 $\psi = x + iy$
$|\psi| = \sqrt{x^2 + y^2}$
$|\psi|^2 = |\psi^2| = x^2 + y^2$
but $|\psi^2| = |24 - 70i| = \sqrt{24^2 + (-70)^2} = \sqrt{5476}$
i.e. $|\psi^2| = 74$

$\therefore x^2 + y^2 = 74$ — (1)
$x^2 - y^2 = 24$ — (2)
$2x^2 = 98$
$x^2 = 49$
$x = \pm 7$
so $y^2 = 25 \Rightarrow y = \pm 5$
care now: $\underline{\psi = 7 - 5i \text{ or } -7 + 5i}$

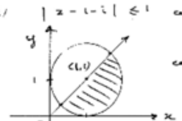
Q2 (a)

(i) $\text{Re}(z) < 2$ and $\text{Im}(z) > -1$
$x < 2$ and $y < -1$



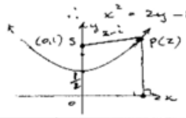
①

(ii) $|z - 1 - i| \le 1$ and $0 \le \arg z \le \frac{\pi}{4}$
$(x-1)^2 + (y-1)^2 \le 1$
and $0 \le \arg z \le \frac{\pi}{4}$



②

(b) Let $z = x + iy$
$\text{Im}(z) = y$, $|z - i| = \sqrt{x^2 + (y-1)^2}$
$\therefore \sqrt{x^2 + (y-1)^2} = y$ | 1
For $y \ge 0$ $x^2 + (y-1)^2 = y^2$
$\vdots$
$\therefore x^2 = 2y - 1 = 4(\frac{1}{2})(y - \frac{1}{2})$ | 1
A parabola
$V = (0, \frac{1}{2})$

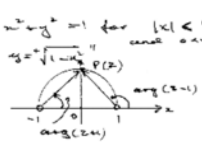


③ | 1

(c) $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$, Let $z = x + iy$ ②

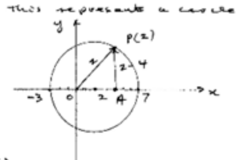
Many methods:
- Geometry/Angles
- Trig.
- Algebraic

$x^2 + y^2 = 1$ for $|x| < 1$ and $0 < y < 1$
"$y = +\sqrt{1 - x^2}$"



Q2 (d) $z\bar{z} - 2(z + \bar{z}) = 21$

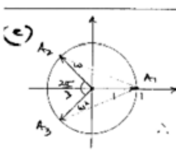
(i) Let $z = x + iy$, $\therefore \bar{z} = x - iy$
$\therefore x^2 + y^2 - 2.2x = 21$
$\Rightarrow (x-2)^2 + y^2 = 25$
This represents a circle $C = (2, 0)$ $r = 5$ | 2



②

(ii) Let $|z - 4| = AP$ where $A = (4, 0)$
max AP when P is at $(-3, 0)$ ②
$\therefore AP_{max} = \max|z - 4| = 3 + 4 = 7$ | 2

(e)

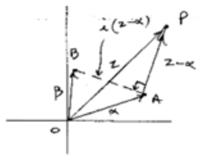


METHOD 1 $z^3 = 1$, $|z| = 1$
$\overrightarrow{A_1A_2} = \omega - 1$ $A_1A_2 = |\omega - 1|$
$\overrightarrow{A_1A_3} = \omega^2 - 1$ $A_1A_3 = |\omega^2 - 1|$

$\therefore A_1A_2 \times A_1A_3 = |\omega - 1||\omega^2 - 1|$
$= |\omega^3 - \omega - \omega^2 + 1|$
$= |1 - (\omega + \omega^2) + 1|$
$= |1 - (-1) + 1|$
$= |3|$
$= \underline{3}$ ③

METHOD 2. use Cosine Rule ×2
✓ 3. Basic trig with equil. Δ.
4.

Q3 (a)



$\vec{AP} = z - \alpha$

rotate $\frac{\pi}{2}$ about A $\therefore \vec{AB} = i(z-\alpha)$

$\therefore \vec{OB} = \beta = \alpha + i(z-\alpha)$.

$= (1-i)\alpha + iz$ (2)

(b) (i) $z = \operatorname{cis}\theta$

$z^n = (\operatorname{cis}\theta)^n = \operatorname{cis} n\theta$ (de M. Thm)

ie $z^n = \cos n\theta + i\sin n\theta$.

$z^{-n} = (\operatorname{cis}\theta)^{-n} = \operatorname{cis}(-n\theta)$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

ie $z^{-n} = \cos n\theta - i\sin n\theta$ as $\cos(-A) = \cos A$, $\sin(-A) = -\sin A$

so $z^n + z^{-n} = 2\cos n\theta$

$\therefore \cos n\theta = \frac{1}{2}(z^n + z^{-n})$ qed. (2)

(ii) Now $\cos\theta = \frac{1}{2}(z+z^{-1})$

$\cos 2\theta = \frac{1}{2}(z^2+z^{-2})$

$\therefore$ LHS $= \cos\theta \cdot \cos 2\theta$

$= \frac{1}{4}(z+z^{-1})(z^2+z^{-2})$

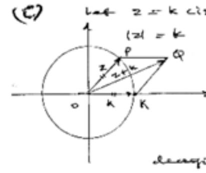
$= \frac{1}{4}(z^3+z^{-1}+z^1+z^{-3})$

$= \frac{1}{4}[z+z^{-1}+z^3+z^{-3}]$

$= \frac{1}{4}(2\cos\theta + 2\cos 3\theta)$ (3)

$= \frac{1}{2}(\cos\theta + \cos 3\theta)$ qed.

(c) Let $z = k\operatorname{cis}\theta$



$\operatorname{Arg} z = \theta$, $\vec{OK} = k$

Form parallelogram of vectors

$\vec{OQ} = z + k$

OPQK is a rhombus

OQ is a diagonal of the rhombus

diagonal of rhombus bisects interior $\angle KOP$

$\therefore \angle KOP = \arg(z+k) = \frac{1}{2}\theta$.

or Algebraic: $z+k = k(1+\operatorname{cis}\theta)$

$= k(1+\cos\theta + i\sin\theta)$

$= k(2\cos^2\frac{1}{2}\theta + i.2\sin\frac{\theta}{2}\cos\frac{\theta}{2})$

$0 \le \theta < \pi$ $\quad z+k = 2k\cos\frac{1}{2}\theta[\cos\frac{1}{2}\theta + i\sin\frac{1}{2}\theta]$

$\therefore \arg(z+k) = 0 + \frac{1}{2}\theta$

$= \frac{1}{2}\theta$

(3)

Q3(d) (i) $z = x+iy$, $\alpha = a+ib$

$z\alpha - \bar{z}\bar{\alpha} = (x+iy)(a+ib) - (x-iy)(a-ib)$

$= ax - by + i(bx+ay)$

$- [ax - by - i(bx+ay)]$

$= ax - by + i(bx+ay) - ax + by + i(bx+ay)$

$z\alpha - \bar{z}\bar{\alpha} = 2i(bx+ay)$

so $2i(bx+ay) = 0$

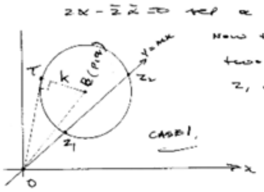
$\Rightarrow bx + ay = 0$ or $y = -\frac{b}{a}x$

this represents a line thru O (2)

as it is of the form $Ax+By+C=0$, $C=0$.

(ii) $|z-\beta| = k$ rep. a circle $B=(p,q)$ $r=k$

$z\alpha - \bar{z}\bar{\alpha} = 0$ rep a line $y = mx$.



Now the line is a secant two points of intersection z_1 and z_2

CASE 1.

Now $|z_1||z_2| = OP_1 . OP_2 = OT^2$ (sq. on tangent to circle equals product of intercepts on secant)

so $|z_1|.|z_2| = OB^2 - BT^2$ (Pyth. Thm)

$\therefore |z_1||z_2| = p^2+q^2-k^2$ (3)

or Solve $(x-p)^2 + (y-q)^2 = k^2$ — (1)

$y = -\frac{b}{a}x$ — (2)

to get $\left(\frac{a^2+b^2}{a^2}\right)x^2 + 2\left(\frac{bq}{a} - 2p\right)x + p^2+q^2-k^2 = 0$

$\Rightarrow x_1x_2 = \frac{a^2}{a^2+b^2}(p^2+q^2-k^2)$

then!!! get to $|z_1|^2|z_2|^2 = (x_1^2+y_1^2)(x_2^2+y_2^2)$

Q4 $P(x) = x^3 + 2x^2 + 19x + 18$

(a) (i) $\alpha+\beta+\gamma = -\frac{b}{a} = \underline{-2}$ (1)

(ii) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= (-2)^2 - 2\times 19$ (2)

$= \underline{-34}$

(iii) 1 real zero because:

- If all the zeros were real $\Sigma\alpha^2 \geq 0$

but $\Sigma\alpha^2 < 0$

$\therefore$ there is at least one complex zero

HENCE and as the coeffts of P(x) are all real

$\therefore$ complex zeros are in conjugate pairs

and as deg $P(x) = 3$, by F.T.A

there are $3-2=1$ real zero

- Solving $P(x) = 0$ (2)

get $P(-1) = 0$

OTHERWISE $\therefore (x+1)(x^2+x+18) = 0$

for x^2+x+18 $\Delta = -71 < 0$

$\therefore$ 2 unreal roots

$\therefore$ only one real zero $(x=-1)$.

(b) (i) $z^5 - i = i^5 - i$

$= i^4 \times i - i$

$= 1\times i - i$ (2)

$= 0$

$\therefore z = i$ is a solution

to show $1 - iz - z^2 + iz^3 + z^4 = 0$

use long division $(z-i)\overline{)z^5 \ldots\ldots -i}$

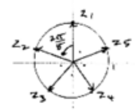
or geometric series "$r = \frac{-iz}{1}$, $n=5$"

or show $(z-i)(1-iz-z^2+iz^3+z^4) = z^5 - i = 0$ by expanding.

$\therefore 1 - iz - z^2 + iz^3 + z^4 = 0$ as $z - i \neq 0$.

Q4 (b) (ii) $z^5 - i = 0$

Since one root is $i = \text{cis}\frac{\pi}{2}$
and the roots are equally spaced
around the circle by $\frac{2\pi}{5}$, $n = 5$.



$z_1 = i = \text{cis}\frac{\pi}{2}$

next $z_2 = \text{cis}\left(\frac{\pi}{2} + \frac{2\pi}{5}\right) = \text{cis}\frac{9\pi}{10} = -\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}$

✓ $z_3 = \text{cis}\left(\frac{\pi}{2} + \frac{4\pi}{5}\right) = \text{cis}\frac{13\pi}{10} = -\cos\frac{3\pi}{10} - i\sin\frac{3\pi}{10}$

✓ $z_4 = \text{cis}\left(\frac{\pi}{2} + \frac{6\pi}{5}\right) = \text{cis}\frac{17\pi}{10} = \cos\frac{3\pi}{10} - i\sin\frac{3\pi}{10}$

✓ $z_5 = \text{cis}\left(\frac{\pi}{2} + \frac{8\pi}{5}\right) = \text{cis}\frac{21\pi}{10} = \cos\frac{\pi}{10} + i\sin\frac{\pi}{10}$

(2)

or Let $z = r\,\text{cis}\,\theta$

$\therefore z^5 = r^5\,\text{cis}\,5\theta = i = 1\,\text{cis}\left(\frac{\pi}{2} + 2k\pi\right)$, $k = 0, 1, 2, 3, 4$.

generates above roots.

---

iii) As $z^5 - i = (z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5) = 0$

i.e. $= (z - i)(z - z_2)(z - z_3)(z - z_4)(z - z_5)$

and noting:

"$(z - \alpha)(z - \beta) = z^2 - (\alpha + \beta)z + \alpha\beta$"

Pair z_2 and z_5, z_3 and z_5

and eg: $\alpha\beta = \left(-\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$

$= -\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$

$= -1\left(\cos^2\frac{\pi}{10} + \sin^2\frac{\pi}{10}\right)$

$= -1$

(4)

z_2 and z_5 / z_3 and z_4

so $z^5 - i = (z - i)\left[z^2 - \left(2i\sin\frac{\pi}{10}\right)z - 1\right]\left[z^2 - \left(-2i\sin\frac{3\pi}{10}\right)z - 1\right] = 0.$

i.e. $(z - i)\left(z^2 - 2i\sin\frac{\pi}{10}z - 1\right)\left(z^2 + 2i\sin\frac{3\pi}{10}z - 1\right) = 0.$ q.e.d.

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iv) As $(z - i)\left[z^2 - 2i\sin\frac{\pi}{10}z - 1\right]\left[z^2 + 2i\sin\frac{3\pi}{10}z - 1\right] = z^5 - i$

i.e. $(z - i)(1 - iz - z^2 + iz^3 - z^4) = z^5 - i$

the COEFFT. OF z^2 in $[\ldots][\ldots]$

is $-1 - 4i^2\sin\frac{\pi}{10}\sin\frac{3\pi}{10} - 1$

i.e. $-2 + 4\sin\frac{\pi}{10}\sin\frac{3\pi}{10}$

(2)

but coefft of z^2 in $1 - iz - z^2 + iz^3 - z^4$ is -1

so $-2 + 4\sin\frac{\pi}{10}\sin\frac{3\pi}{10} = -1$

$\Rightarrow \sin\frac{\pi}{10} \times \sin\frac{3\pi}{10} = \frac{1}{4}$ qed