# Mathematics Extension 2 Term 4 Assessment 2005

**Question 1 (15 Marks)** **Marks**

(a) Given that $\alpha = 3 + 2i$ and $\mu = 2 - 5i$, find

(i) $\alpha\mu$ 2
(ii) $\text{Re}(\alpha\mu)$ 1
(iii) $\text{Im}(\alpha\mu)$ 1

(b) Describe geometrically, on an Argand diagram

(i) $\text{Re}(z) = 2$. 1
(ii) $Im(z) < 2$ 1

(c) Expand and simplify $(2 - 3i)^4$ 3

(d) Calculate the modulus of the product of the roots of the equation 3
$(2 + i)x^2 + 3x - (1 - i) = 0$.

(e) Show that the point representing $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ is on a circle with radius 1 3
and centre at (1, 0).

**Question 2 (15 Marks)**

(a) The complex number $z = \sqrt{3} + i$ is represented on an Argand diagram by the point $A$.

(i) The points $B$, $C$, $D$ and $E$ are the points representing $-z$, $iz$, $1 - z$ and $\bar{z}$ respectively. Mark clearly on an Argand diagram the points $A$, $B$, $C$, $D$, and $E$. Clearly indicate important geometrical relationships between these points. 4

(ii) $F$ is the point in the second quadrant such that $\Delta ABF$ is equilateral. What complex number is represented by the point $F$? 2

(b) (i) Find $\sqrt{6i - 8}$, in the form $a + ib$. 3

(ii) Hence, solve the equation $2z^2 - (3 + i)z + 2 = 0$, expressing the values of $z$ in the form $a + ib$. 2

(c) A point $z$ on the Argand diagram is given by $z = w^2 + 2iw$, where $w = u + iv$ and $u$ and $v$ are real. Find the locus of $z$ when

(i) $u = 0$ and $v$ varies. 1
(ii) $v = 1$ and $u$ varies. 1
(iii) Sketch the two loci, showing any important features. 2

**Question 3 (15 Marks)** **Marks**

(a) Sketch the locus $|z + 2 - 3i| \leq 5$. 2

(b) Show that $w = 2\sqrt{3}\,i - 2$ is a root of the equation $z^3 = 64$. 2

(c) If $z \neq 0$, show that $u = z + \dfrac{|z|^2}{z}$ is always real, where $z = x + iy$ and $x \in \Re$, $y \in \Re$. 3

(d) If $z = \cos\theta + i\sin\theta$, prove that $\dfrac{2}{1+z} = 1 - i\tan\dfrac{\theta}{2}$. 3

(e) Sketch the region, in an Argand diagram where points satisfy the set of inequalities: 2
$2 \leq |z| \leq 4$ and $-\dfrac{\pi}{3} \leq \arg z \leq \dfrac{\pi}{3}$.

(f) Prove that if $Z_1$, $Z_2$ are complex numbers then $|Z_1 - Z_2|^2 + |Z_1 + Z_2|^2 = 2\{|Z_1|^2 + |Z_2|^2\}$. 3

**Question 4 (15 Marks)**

(a) $2 - i$ is one root of $x^2 - (3 - i)x + k = 0$. Find $k$ and the other root of the equation. 2

(b) Sketch the region in the Argand diagram defined by $\left|z^2 - \bar{z}^2\right| \geq 8$ 3

(c) (i) Show that $(1 + i)$ is a root of the polynomial $P(x) = x^3 + x^2 - 4x + 6$. 2
(ii) Hence resolve $P(x)$ into irreducible factors over the complex field. 3

(d) Find the fourth roots of $2\sqrt{3} + 2i$. 5

✎ END OF EXAM ☺

## Mathematics Extension 2 Term 4 Assessment 2005 - solutions

**Question 1 (15 Marks)** **Marks**

(a) Given that $\alpha = 3 + 2i$ and $\mu = 2 - 5i$, find

(i) $\alpha\mu = (3 + 2i)(2 - 5i)$ **1mk**

$= 11 - 4i$ **1mk** 2

(ii) $\text{Re}(\alpha\mu) = 11$ **1mk** 1

(iii) $\text{Im}(\alpha\mu) = -4$ **1mk** 1

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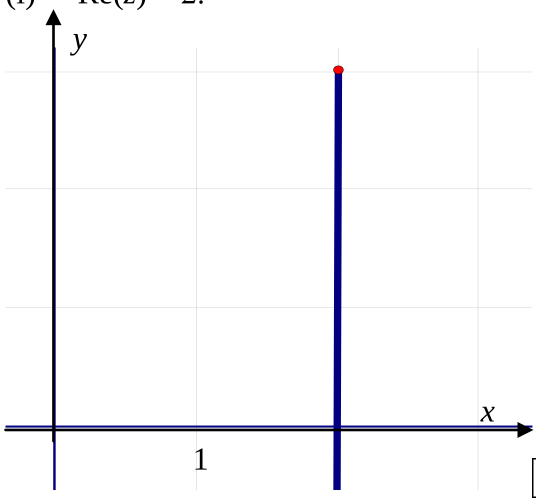
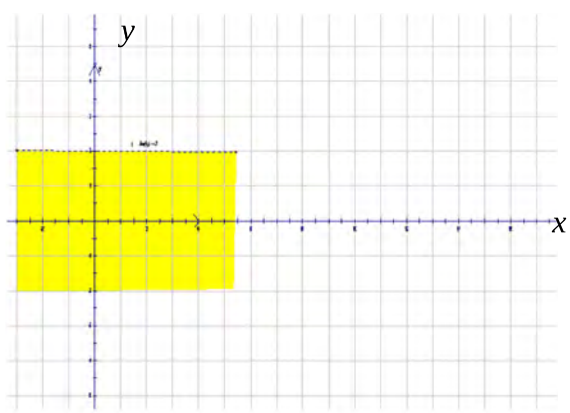
(b) Describe geometrically, on an Argand diagram

(i) Re(z) = 2. 1




**1mk**

(ii) $Im(z) < 2$ 1




**1mk**

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(c) Expand and simplify $(2-3i)^4$ 3

$(2-3i)^4 = 2^4 + 4(2)^3(-3i) + 6(2)^2(-3i)^2 + 4(2)(-3i)^3 + (-3i)^4$ **1mk**

$= 16 - 96i + 216i^2 - 216i^3 + 81i^4$ **1mk**

$= -119 + 120i$ **1mk**

(d) Calculate the modulus of the product of the roots of the equation $(2 + i)x^2 + 3x - (1 - i) = 0$. 3

Let the roots be $\alpha, \beta$. Then $\alpha\beta = \frac{-1+i}{2+i}$ **1mk**

$\therefore\ \alpha\beta = \frac{-1+i}{2+i}\times\frac{2-i}{2-i}$

$= \frac{-3+3i}{5}$ **1mk**

$|\alpha\beta| = \sqrt{\left(\frac{-3}{5}\right)^2+\left(\frac{3}{5}\right)^2} = \frac{3\sqrt{2}}{5}$ **1mk**

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(e) Show that the point representing $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ is on a circle with radius 1 and centre at (1, 0). 3

Let $z = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$\therefore\ |z| = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$ ➔ equal to radius of circle **1mk**

Now equation of the circle with centre (1,0) and radius 1 is $(x - 1)^2 + y^2 = 1$ **1mk**

Since $x = \frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$ and substitution into LHS $= \left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1 = RHS$ **1mk**

$\therefore\ \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ lies on circle centred (1, 0) and radius 1.

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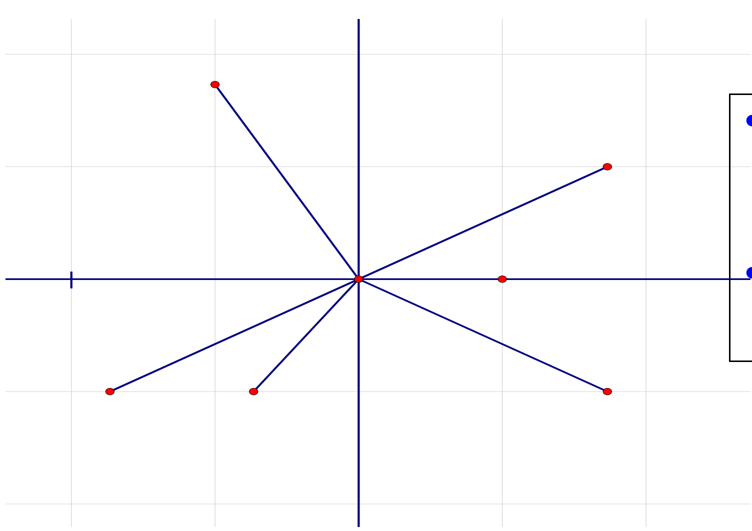
**Question 2 (15 Marks)**

(a) The complex number $z = \sqrt{3} + i$ is represented on an Argand diagram by the point $A$.

(i) The points $B$, $C$, $D$ and $E$ are the points representing $-z$, $iz$, $1 - z$ and $\bar{z}$ respectively. Mark clearly on an Argand diagram the points $A$, $B$, $C$, $D$, and $E$. Clearly indicate important geometrical relationships between these points. 4



- **Points B, C, D, E with relationship to point A, award 1mk each**
- **Just points award 2 marks only**

(ii) $F$ is the point in the second quadrant such that $\Delta ABF$ is equilateral. What complex number is represented by the point $F$? 2

$F$ must be the same distance from A as B is so must have point $(-\sqrt{3}, 2)$

So has the complex number $z_2 = -\sqrt{3} + 2i$ **1mk**

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(b) (i) Find $\sqrt{6i - 8}$, in the form $a + ib$. 3

Let $\Delta = 6i - 8$ then let $\Delta = (a + ib)^2$ where $a$, $b$ are Real

$\therefore a^2 - b^2 = -8$ and $2ab = 6 \therefore ab = 3$ **1mk**

$\therefore a^2 - \dfrac{9}{a^2} = -8$ ➔ $a^4 + 8a^2 - 9 = 0$

$(a^2 + 9)(a^2 - 1) = 0$ since $a$ is real ➔ $a = 1$ or $a = -1$ so $b = 3$ or $b = -3$ respectively. **1mk**

$\therefore \sqrt{6i - 8} = 1 + 3i$ or $-1 - 3i$ **1mk**

(ii) Hence, solve the equation $2z^2 - (3 + i)z + 2 = 0$, expressing the values of $z$ in the form $a + ib$. 2

$\therefore z = \dfrac{(3+i) \pm (1+3i)}{4}$ by quadratic formula **1mk**

$\therefore z = 1 + i$ or $z = \dfrac{1}{2} - \dfrac{i}{2}$ **1mk**

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(c) A point $z$ on the Argand diagram is given by $z = w^2 + 2iw$, where $w = u + iv$ and $u$ and $v$ are real. Find the locus of $z$ when

(i) $u = 0$ and $v$ varies. 1

when $u = 0$ then $w = iv$ and $z = -v^2 - 2v$

if $z = x + iy$ where $x, y$ are real $\therefore x = -v^2 - 2v$ and $y = 0$

$\therefore$ locus is $y = 0$ **1mk**

(ii) $v = 1$ and $u$ varies. 1

when $v = 1$ then $w = u + i$ and $z = (u + i)^2 + 2i(u + i)$

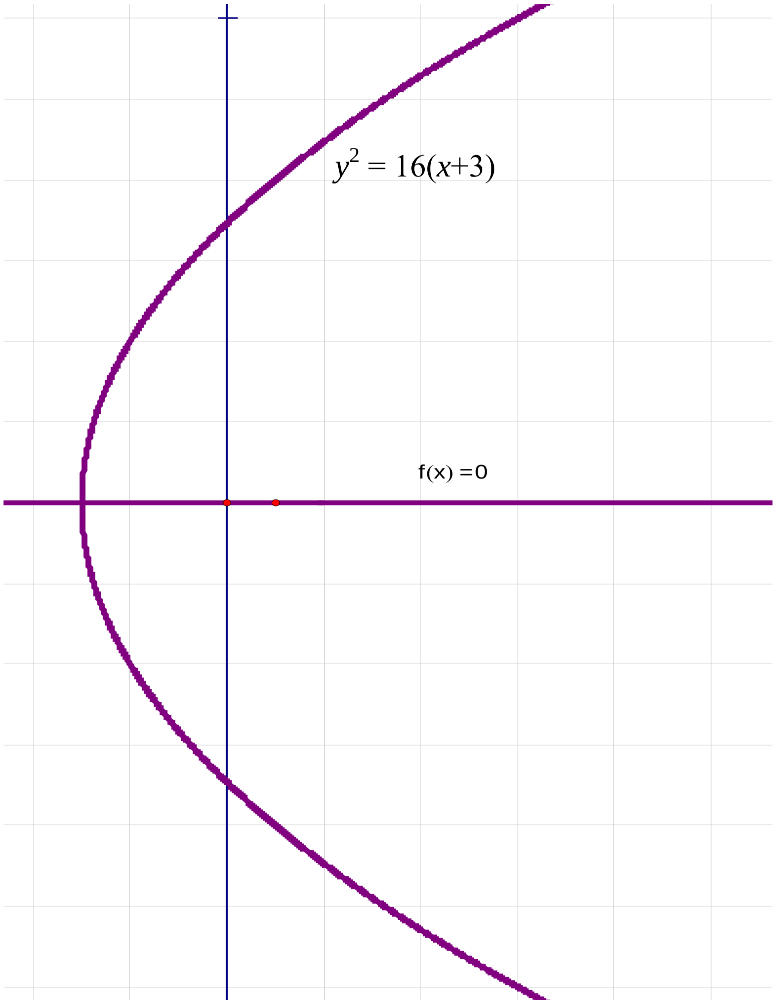
$\therefore z = (u^2 - 3) + 4ui$

$\therefore$ if $z = x + iy$ where $x, y$ are real then

$x = u^2 - 3$ and $y = 4u \therefore y^2 = 16(x + 3)$ **1mk**

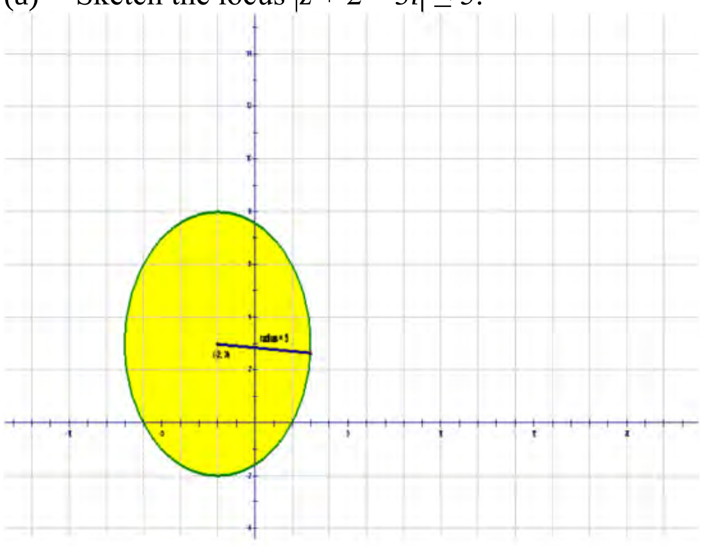
(iii) Sketch the two loci, showing any important features. 2

See next page **1mk each**




**Question 3 (15 Marks)** **Marks**

(a) Sketch the locus $|z + 2 - 3i| \le 5$. 2



1mk for circle centre $(-2,3)$ and radius= 5

1mk for correct region

(b) Show that $w = 2\sqrt{3}\,i - 2$ is a root of the equation $z^3 = 64$. 2

$w = 2\sqrt{3}\,i - 2 = 4(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$ **1mk**

$\therefore\ w^3 = 4^3(\cos 2\pi + i\sin 2\pi)$ by De Moivre's Theorem

$= 64$ as required **1mk**

(c) If $z \neq 0$, show that $u = z + \frac{|z|^2}{z}$ is always real, where $z = x + iy$ and $x \in \Re$, $y \in \Re$. 3

$z = x + iy$ then $|z|^2 = x^2 + y^2$ & $z^2 = (x^2 - y^2) + 2ixy$ **1mk**

Now $u = z + \frac{|z|^2}{z} = \frac{x^2 - y^2 + 2ixy + x^2 + y^2}{x + iy}$ **1mk**

$= \frac{2x(x + iy)}{x + iy}$ **1mk**

$= 2x$ which is real as $x$ is real

**Alternatively:**

$u = z + \frac{|z|^2}{z} = z + \frac{|z|^2 \times \bar{z}}{z \times \bar{z}}$ **1mk**

$= z + \frac{|z|^2 \times \bar{z}}{(\bar{z})^2}$ **1mk**

$= z + \bar{z}$ **1mk**

$= x + iy + x - iy$

$= 2x$ which is real as $x$ is real.

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(d) If $z = \cos\theta + i\sin\theta$, prove that $\frac{2}{1+z} = 1 - i\tan\frac{\theta}{2}$. 3

LHS $= \frac{2}{1 + \cos\theta + i\sin\theta} \times \frac{(1 + \cos\theta) - i\sin\theta}{(1 + \cos\theta) - i\sin\theta}$ **1mk**

$= \frac{2 + 2\cos\theta - 2i\sin\theta}{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}$

$= \frac{2 + 2\cos\theta - 2i\sin\theta}{2 + 2\cos\theta}$

$= 1 - \frac{i\sin\theta}{1 + \cos\theta}$ **1mk**

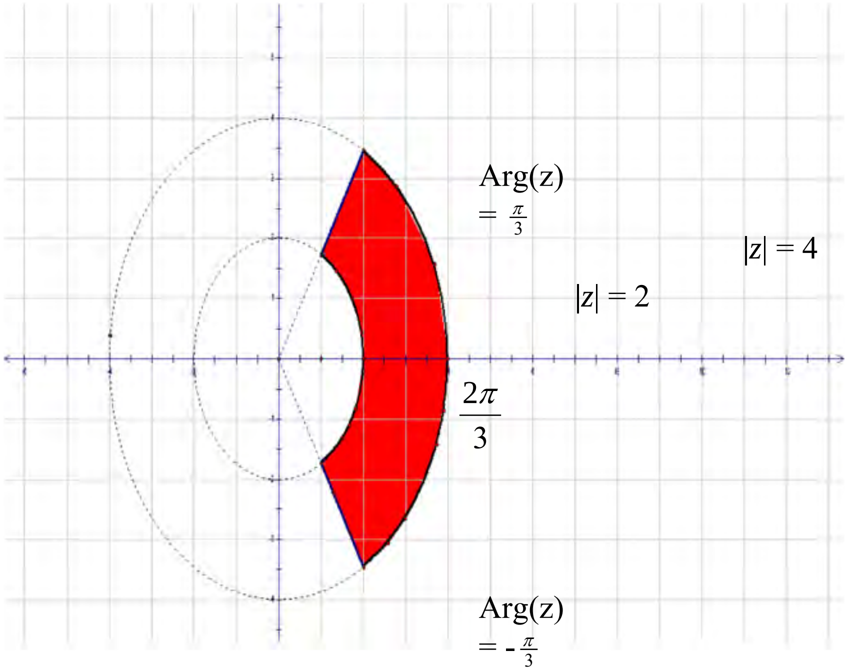
$= 1 - \frac{2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1 + 2\cos^2\frac{\theta}{2} - 1}$ **1mk**

$= 1 - \frac{i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$

$= 1 - i\tan\frac{\theta}{2}$ = RHS [Students could also let $t = \tan\frac{\theta}{2}$]

(e) Sketch the region, in an Argand diagram where points satisfy the set of inequalities: 2

$2 \le |z| \le 4$ and $-\dfrac{\pi}{3} \le \arg z \le \dfrac{\pi}{3}$.




**1mk** for the boundary

**1mk** for the region

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(f) Prove that if $Z_1$, $Z_2$ are complex numbers then $|Z_1 - Z_2|^2 + |Z_1 + Z_2|^2 = 2\{|Z_1|^2 + |Z_2|^2\}$. 3

$LHS = |Z_1 - Z_2|^2 + |Z_1 + Z_2|$

$= (Z_1 + Z_2)(\overline{(Z_1 + Z_2)}) + (Z_1 - Z_2)(\overline{(Z_1 - Z_2)})$ **1mk**

$= (Z_1 + Z_2)(\overline{Z}_1 + \overline{Z}_2) + (Z_1 - Z_2)(\overline{Z}_1 - \overline{Z}_2)$

$= 2(Z_1\overline{Z}_1 + Z_2\overline{Z}_2)$ **1mk**

$= 2\left(|Z_1|^2 + |Z_2|^2\right) = \text{RHS}$ **1mk**

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**Question 4 (15 Marks)**

(a) $2 - i$ is one root of $x^2 - (3 - i)x + k = 0$. Find $k$ and the other root of the equation. 2

Let $\alpha$ and $\beta$ be the roots of the equation, so let $\beta = 2 - i$

$\therefore \alpha + 2 - i = (3 - i) \Rightarrow \alpha = 1$ **1mk**

$\therefore \alpha\beta = k \Rightarrow k = 2 - i$ **1mk**

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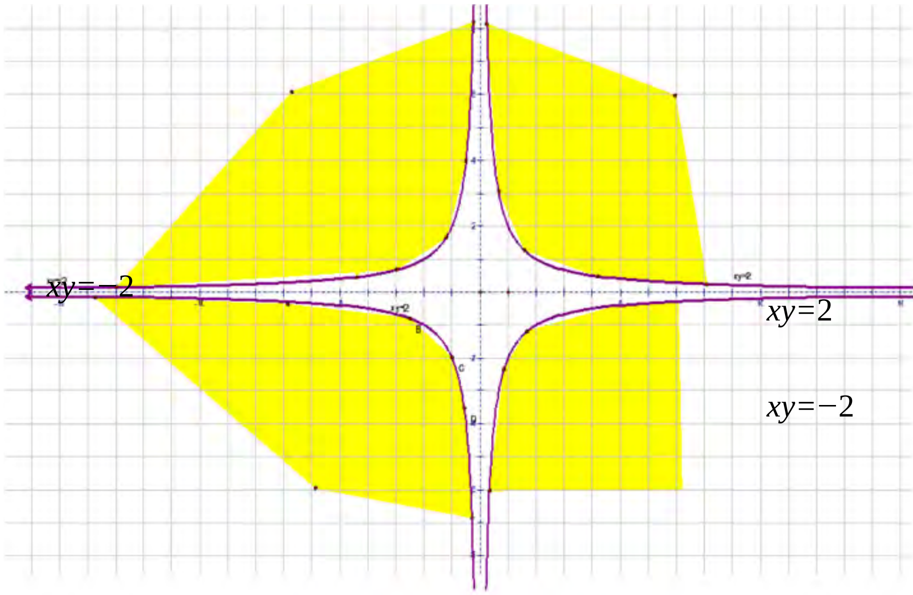
(b) Sketch the region in the Argand diagram defined by $\left|z^2 - \overline{z}^2\right| \ge 8$ 3

$z^2 - \overline{z}^2 = (z - \overline{z})(z + \overline{z})$

$= (2iy)(2x) = 4ixy$

$\therefore \left|z^2 - \overline{z}^2\right| \ge 4|xy|$

$\therefore 4|xy| \ge 8 \Rightarrow |xy| \ge 2$ **1mk**




1 mk for the graphs and 1 mk for the correct region.

(c) (i) Show that $(1 + i)$ is a root of the polynomial $P(x) = x^3 + x^2 - 4x + 6$. 2

If $(1 + i)$ is a root then $P(1 + i) = 0$

$P(1 + i) = (1 + i)^3 + (1 + i)^2 - 4(1 + i) + 6$

$= 1 + 3i + 3i^2 + i^3 + 1 + 2i + i^2 - 4 - 4i + 6$ **1mk**

$= 4 + i - 3 - 1 - i = 0$ **1mk**

$\therefore (1 + i)$ is a root of $P(x)$.

(ii) Hence resolve $P(x)$ into irreducible factors over the complex field. 3

Since $P(x)$ has real coefficients ➔ the complex roots occur in conjugate pairs.

$\therefore (1 - i)$ is also a root. **1mk**

$\therefore$ Let $P(x)$ have roots $(1 + i)$, $(1 - i)$ and $\beta$

Sum of the roots $= (1 + i) + (1 - i) + \beta = -1$ ➔ $\beta = -3$ **1mk**

$\therefore P(x) = (x + 3)\{x - (1 + i)\}\{x - (1 - i)\}$ **1mk**

(d) Find the fourth roots of $2\sqrt{3} + 2i$. 5

Let $z = x + iy$ be one of the fourth roots. Then $z^4 = 2\sqrt{3} + 2i$

$\therefore z^4 = 16(\cos 4\theta + i\sin 4\theta)$ by De Moivre's Theorem.

Equating the real parts ➔ $16\cos 4\theta = 2\sqrt{3}$ **1mk**

Equating imaginary parts ➔ $16\sin 4\theta = 2$ **1mk**

$\therefore \tan 4\theta = \dfrac{1}{\sqrt{3}}$ **1mk** ➔ $4\theta = \dfrac{\pi}{6} + 2k\pi$, where $k$ are integers.

$\therefore \theta = \dfrac{(12k + 1)\pi}{24}$, where $k = 0, 1, -1, -2$ **1mk**

$\therefore$ Four roots $\sqrt{3} + i$ are: $2cis\dfrac{\pi}{24}$, $2cis\dfrac{13\pi}{24}$, $2cis\dfrac{-11\pi}{24}$, $2cis\dfrac{-23\pi}{24}$ **1mk**

**✎ END OF EXAM ☺**