Question 1. **(15 Marks)** **[START A NEW PAGE]** **Marks**

a) Given that $z = 3 - 4i$ and $w = 1 + 2i$, write down the values of each of the following, expressing your answers in simplest form. **5**

i) $\overline{z}$ ii) z^2 iii) $|z|$ iv) $\dfrac{z}{w}$ v) $\arg(w)$

b) (Detach and use the sheet provided)
On the Argand diagram provided, the points A and B represent the complex numbers α and β respectively. On the same sheet construct, with the aid of geometrical instruments, points P,Q, R, S and T which respectively represent: **5**

i) $\alpha + \beta$ ii) $-\beta$ iii) $\alpha - \beta$ iv) $2i\beta$ v) $\overline{\alpha}$

Clearly indicate any significant lengths and angles.

(Ensure that the Argand diagram is handed in with the rest of question 1.)

c) i) Find all real numbers x and y that satisfy $(x + iy)^2 = -3 + 4i$. **3**

ii) Hence solve the equation $z^2 - 3z + (3 - i) = 0$ **2**

Question 2. **(15 Marks)** **[START A NEW PAGE]** **Marks**

a) A complex root of $z^3 + z^2 + z - 39 = 0$ is known to be $z = 3i - 2$. Deduce the values of the other two roots, giving reasons for your answers. **3**

b) i) If z = $(\cos\theta + i\sin\theta)$, use de Moivre's Theorem to prove that **3**

$$\left(z^n + \frac{1}{z^n}\right) = 2\cos n\theta.$$

ii) Hence, or otherwise, prove that $8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$. **2**

c) z satisfies $|z - 4 + 2i| = 3$ and w satisfies $\arg(w + 1 - i) = \frac{\pi}{4}$. The point P represents z on an Argand diagram and Q represents w.

i) Sketch the loci of P and Q on an Argand diagram. **4**

ii) If $w = x + iy$, write down the equation of the locus of Q in terms of x and y. **2**

iii) Hence deduce the minimum value of $|z - w|$. **1**

Question 3. **(15 Marks)** **[START A NEW PAGE]** **Marks**

a) i) Express $z = \dfrac{\sqrt{3}+i}{1+i}$ in mod-arg form. **3**

ii) Hence find the smallest positive integer *m* for which z^m is real. **2**

b) i) On an Argand diagram, sketch the locus of points which satisfy the relationship $|z-2| = |z+1-3i|$, clearly showing its position and orientation. **2**

ii) On the same Argand diagram, sketch the locus of $z + \bar{z} = 4$. **2**

iii) Shade the area where $|z-2| \le |z+1-3i|$ and $z + \bar{z} \ge 4$ simultaneously. **1**

c) The points A, B and C represent z_1, z_2 and $z_1 + z_2$ in an Argand diagram where z_1 and z_2 satisfy the relationship

$\dfrac{(z_2 - z_1)}{(z_2 + z_1)} = ki$ where k is a positive number (and $i = \sqrt{-1}$).

i) Illustrate such an arrangement on an Argand diagram, denoting clearly properties of the quadrilateral *OACB* and its diagonals (*O* is the origin). **2**

ii) What is the shape of the quadrilateral OACB if α) $k = 1$ β) $k \ne 1$. **2**

iii) Write down the area of the quadrilateral *OACB* in terms of z_1 and z_2. 1

Question 4. **(15 Marks)** **[START A NEW PAGE]** **Marks**

a) i) Solve the equation $z^6 + 8 = 0$, writing the roots in mod-arg form. **3**

ii) Illustrate these roots on a clear Argand diagram. **2**

iv) Write $z^6 + 8$ in factorized form over the field of Real numbers. **3**

b) The diagram shows an isosceles triangle *ABP* in the Argand diagram, with base *AB* and $\angle APB = \alpha$. *PM* is the perpendicular bisector of *AB* and so bisects $\angle APB$. Given that *A* and *B* represent the complex numbers z_1 and z_2 respectively, find, in terms of z_1, z_2 and α, the complex numbers represented by:

i) the vector *AM*. **2**

ii) the vector *MP*. **3**

Hence show that *P* represents the complex number

$$\tfrac{1}{2}(1 - i\cot\frac{\alpha}{2})z_1 + \tfrac{1}{2}(1 + i\cot\frac{\alpha}{2})z_2 .$$ **2**

(END OF EXAMINATION)

Student Number : ___________________________

Use this diagram for **Question 1(b**) and hand it in with the rest of your answers to question 1.

TERM 4 2006 EXT 2

1 a) $z = 3-4i \quad w = 1+2i$

i) $\bar{z} = \underline{3+4i}$

ii) $z^2 = 9-16-24i = \underline{-7-24i}$

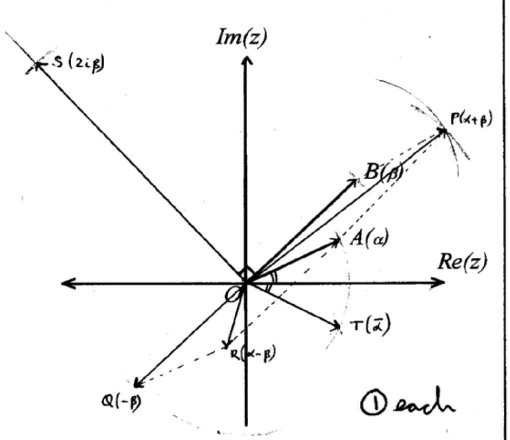
iii) $|z| = \sqrt{3^2+(-4)^2} = \underline{5}$

iv) $\dfrac{z}{w} = \dfrac{(3-4i)(1-2i)}{(1+2i)(1-2i)}$ ① each

$= \dfrac{3-8-4i-6i}{5} = \underline{-1-2i}$

v) $\text{Arg}(w) = \tan^{-1}\left(\frac{2}{1}\right) = \underline{\tan^{-1}2}$

b)



c) $x^2-y^2 = -3$ (Real)

$2xy = 4$ (Imaginary)

$\therefore y = 2/x$

$x^2 - \dfrac{4}{x^2} = -3$

$x^4+3x^2-4 = 0$

$(x^2-1)(x^2+4) = 0 \quad (x^2 \neq 4)$

$\therefore x = \pm 1 \quad \therefore y = \pm 2$

$\underline{(x,y) = (1,2) \text{ or } (-1,-2)}$

Quadratic formula

$z = \dfrac{3 \pm \sqrt{9-4(3-i)}}{2} = \dfrac{3 \pm \sqrt{-3+4i}}{2}$

$= \dfrac{3 \pm (1+2i)}{2} = \underline{2+i} \text{ or } \underline{1-i}$

2 a) As all coefficients are real, ①
complex roots come in conjugate pa[irs]

$\therefore$ Another root is $\underline{-2-3i}$ ①

Sum of roots = "$-b/a$" = -1

$\therefore$ Third root is $\underline{3}$ ①

b) i) $z^n = (\cos\theta + i\sin\theta)^n$ †

$= (\cos n\theta + i\sin n\theta)$ * (de Moivre) ①

$z^{-n} = (\cos(-n\theta) + i\sin(-n\theta))$ (de Moi[vre])

$= \cos n\theta - i\sin n\theta$ † ①

$\therefore z^n + \dfrac{1}{z^n} = 2\cos n\theta$ (adding *+[†]) ①

ii) $\left(z + \dfrac{1}{z}\right)^4 = 16\cos^4\theta$

But $\left(z+\dfrac{1}{z}\right)^4 = z^4 + 4z^2 + 6 + \dfrac{4}{z^2} + \dfrac{1}{z^4}$

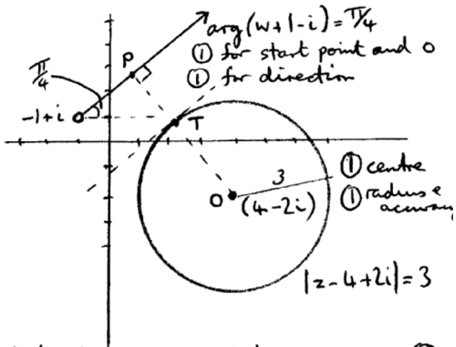
$= \left(z^4 + \dfrac{1}{z^4}\right) + 4\left(z^2 + \dfrac{1}{z^2}\right) + 6$

$= 2\cos 4\theta + 8\cos 2\theta + 6$

Combining & divide thro' by 2

$\underline{8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3}$

c)



ii) Gradient is 1, Intercept 2 ①

Eqn is $\underline{y = x+2, \; x > -1}$ ①

iii) $OP = \left|\dfrac{4-(-2)+2}{\sqrt{1+1}}\right| = \dfrac{8}{\sqrt{2}} = 4\sqrt{2}$

$\therefore PT = \min|z-w| = \underline{4\sqrt{2}-3}$ ①

3 a) i) $z = \dfrac{2\,\text{cis}(\tan^{-1} 1/\sqrt{3})}{\sqrt{2}\,\text{cis}(\tan^{-1} 1)} = \dfrac{2\,\text{cis}(\pi/6)}{\sqrt{2}\,\text{cis}(\pi/4)}$ ① ①

$= \sqrt{2}\,\text{cis}\left(-\frac{\pi}{12}\right)$ ①

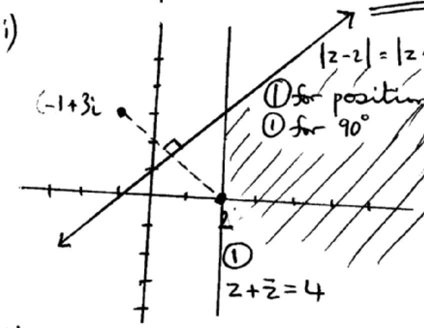
ii) $z^m = (\sqrt{2})^m\left(\cos\left(-\frac{m\pi}{12}\right) + i\sin\left(-\frac{m\pi}{12}\right)\right)$

$= (\sqrt{2})^m\left(\cos\left(\frac{m\pi}{12}\right) - i\sin\left(\frac{m\pi}{12}\right)\right)$ ①

Which is real when $\sin\left(\frac{m\pi}{12}\right) = 0$

Smallest positive m is 12 ①

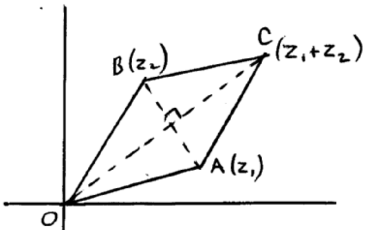
b) i)



ii) $z + \bar{z} = 4 \Rightarrow 2\,\text{Re}(z) = 4 \Rightarrow \text{Re}(z) = 2$ ①

iii) ① for correct area shaded

c) i) $z_2 - z_1$ and $z_2 + z_1$ are represented by the diagonals of the quadrilateral OACB. These diagonals are at 90° as their ratio is ki.



As BC // OA and OB // AC, this is a parallelogram so OB = AC, and OA = BC. A parallelogram with diagonals at 90° ① is a rhombus. Thus $|z_1| = |z_2|$ ①

ii) α) $k = 1 \Rightarrow$ Square ① β) $k \neq 1 \Rightarrow$ Rhombus ①

iii) Area $= \frac{1}{2}|z_1 + z_2||z_2 - z_1|$ ①

4 a) i) $z^6 = -8 = 8\,\text{cis}(\pi + 2k\pi)$ ① $(k \in \mathbb{Z})$

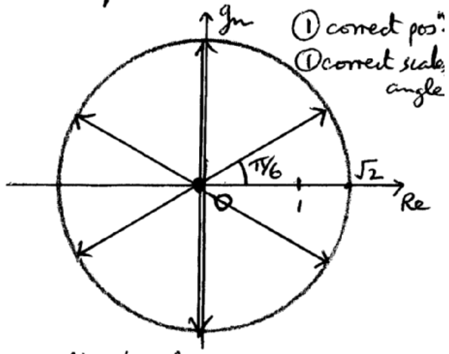
$\therefore z = \sqrt{2}\,\text{cis}\left(\frac{\pi}{6} + \frac{k\pi}{3}\right)$ ① (de Moivre)

$z = \sqrt{2}\,\text{cis}\left(-\frac{5\pi}{6}\right), \sqrt{2}\,\text{cis}\left(-\frac{\pi}{2}\right), \sqrt{2}\,\text{cis}\left(-\frac{\pi}{6}\right)$

$\sqrt{2}\,\text{cis}\left(\frac{\pi}{6}\right), \sqrt{2}\,\text{cis}\left(\frac{\pi}{2}\right), \sqrt{2}\,\text{cis}\left(\frac{5\pi}{6}\right)$ ①

(or equivalent).

ii)



iii) Pair off the linear roots

e.g. $\left(z - \sqrt{2}\,\text{cis}\left(-\frac{5\pi}{6}\right)\right)\left(z - \sqrt{2}\,\text{cis}\left(\frac{5\pi}{6}\right)\right)$

$= \left(z - \sqrt{2}\cos\frac{5\pi}{6} + i\sqrt{2}\sin\frac{5\pi}{6}\right)\left(z - \sqrt{2}\cos\frac{5\pi}{6} - i\sqrt{2}\sin\frac{5\pi}{6}\right)$

$= \left(z^2 - 2\sqrt{2}\cos\frac{5\pi}{6}z + 2\right) = (z^2 - z\sqrt{6} + 2)$

Similarly $\pi/6 \rightarrow (z^2 + z\sqrt{6} + 2)$

Also $\pi/2 \rightarrow (z^2 + 2)$ ①

$\therefore z^6 + 8 = (z^2+2)(z^2+z\sqrt{6}+2)(z^2-z\sqrt{6}+2)$

b) i) M represents $\dfrac{z_1 + z_2}{2}$ ①

$\therefore$ AM represents $\dfrac{z_1 + z_2}{2} - z_1$

$= \dfrac{z_2 - z_1}{2}$ ①

ii) MP is AM rotated by 90° (a-c) and enlarged in ratio $\frac{|PM|}{|AM|} = \cot\frac{\alpha}{2}$

$\therefore$ MP represents $\left(\dfrac{z_2 - z_1}{2}\right) i \cot\dfrac{\alpha}{2}$ ① ① ①

P represents OM + MP ①

$= \dfrac{z_1 + z_2}{2} + \left(\dfrac{z_2 - z_1}{2}\right) i \cot\dfrac{\alpha}{2}$

$= \frac{1}{2}z_1(1 - \cot\alpha/2) + \frac{1}{2}(1 + \cot\alpha/2)z_2$