Year 11 Term 4 Extension II   2007

Question 1. **Marks**

(a) Let $z_1 = 2 - i$ and $z_2 = iz_1$, find $z_1 - z_2$. **2**

(b) Let $z = \dfrac{-9+15i}{4-i}$,

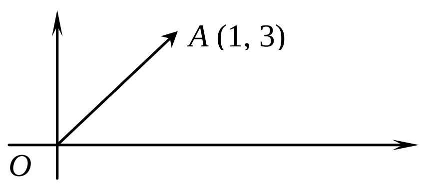
(i) By simplifying $\dfrac{-9+15i}{4-i}$, express z in the form $a+bi$, where a and b are real numbers. **3**

(ii) Hence, or otherwise find $|z|$ and $\arg z$. **2**

(c) Solve $z^2 = 35 - 12i$, expressing your answer in the form $a+bi$. **3**

(d) Sketch the region on an Argand diagram neatly, where the inequalities $|z-2| \le \text{Re}(z)$ and $\text{Im}(z) \ge 0$ hold simultaneously. **4**

(e) Let point A, below, represent the complex number $1+3i$ in the Argand diagram. **2**



Not to scale

OA is rotated 45^0 anticlockwise about O and enlarged by a factor of $\sqrt{2}$.
Find this new complex number in the form $a+bi$.

Question 2. **[Start a New Page]**

(a) Given $1-2i$ is a root of $x^2 + bx + c = 0$, where b and c are real numbers, Find the value of bc. **3**

(b) (i) Sketch neatly on the same diagram the loci of z when: **2**

(α) $|z-(3+2i)| = 2$. (β) $|z+3| = |z-5|$.

[clearly label each sketch]

(ii) Hence write down all the values of z which satisfy simultaneously: **2**

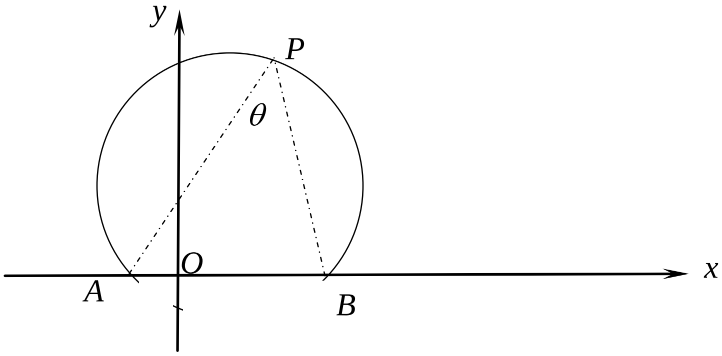
$|z-(3+2i)| = 2$ and $|z+3| = |z-5|$.

(iii) Determine the values of k for which the simultaneous equations: **2**
$|z-(3+2i)| = 2$ and $|z-2i| = k$ have exactly one solution for z.

(iv) For the locus of z when $|z-(3+2i)| = 2$, **2**
find the maximum value of $\arg z$.

(c) The diagram below shows the locus for all complex numbers z in the Argand diagram, such that $\arg\left(\dfrac{z-3}{z+1}\right)=\dfrac{\pi}{3}$.

The angle formed by the intervals AP and BP is θ, where P represents the complex number z and A is $(-1,0)$ and B is $(3,0)$.



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(i) Copy this diagram neatly and find θ, giving reasons. **2**

(ii) Find the coordinates for the centre of this circle. **2**

Question 3. **[Start a New Page]**

(a) Let P, Q and R represent the complex numbers z_1, z_2 and z_3 respectively. **3**
By drawing a diagram or otherwise, determine the geometric properties that characterize ΔPQR if $z_2 - z_1 = i(z_3 - z_1)$?
Give clear reasons for your answer.

(b) Given $z = \cos\theta + i\sin\theta$ and that m and n are integers.

(i) Show that $\dfrac{z^n + z^{-n}}{2} = \cos n\theta$. **2**

(ii) Hence, using part (i), show that **2**

$$\cos m\theta \times \cos n\theta = \frac{1}{2}\left[\cos(m+n)\theta + \cos(m-n)\theta\right].$$

(c) Let $P(z) = z^8 - \dfrac{5}{2}z^4 + 1$. The complex number α is a root of $P(z)=0$.

(i) Show that $i\alpha$ and $\dfrac{1}{\alpha}$ are also roots of $P(z)=0$. **3**

(ii) Find one of the roots of $P(z)=0$ in exact form. **3**

(iii) Hence find all the roots of $P(z)=0$. **2**

Question 4. **[Start a New Page}** **Marks**

(a) The point P on the Argand diagram represents the complex number z, **3**

where z satisfies $\frac{1}{z}+\frac{1}{\overline{z}}=1.$

Find the locus of P as z varies

(b) (i) State the triangle inequality for the two complex numbers z_1 and z_2. **1**

(ii) If $z_1 = 3+4i$ and $|z_2| = 13,$ find the greatest value for $|z_1 + z_2|$. **1**

(iii) Further, if $0 < Arg\, z_2 < \frac{\pi}{2}$ and $|z_1 + z_2|$ has this greatest value, **2**

find z_2.

(c) Given $w = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5},$

(i) Show that w^k is a solution of $z^5 - 1 = 0,$ where k is an integer. **2**

(ii) Explain why $1 + w + w^2 + w^3 + w^4 = 0.$ **2**

(iii) Hence, or otherwise find the value of: **2**

$$(w-1)(1+2w+3w^2+4w^3+5w^4).$$

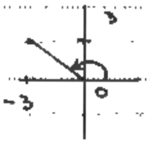
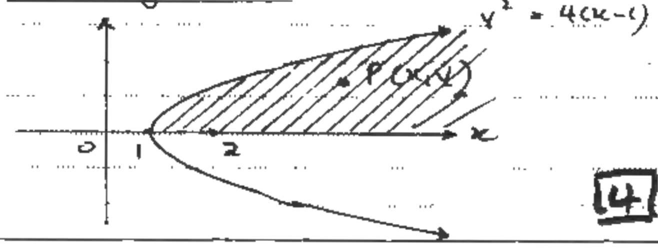
(iv) Given that $S = \frac{w}{1-w^2} + \frac{w^2}{1-w^4} + \frac{w^3}{1-w} + \frac{w^4}{1-w^3},$ **2**

Show that $S = 0.$

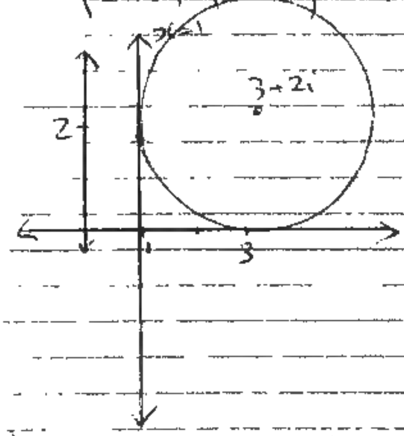
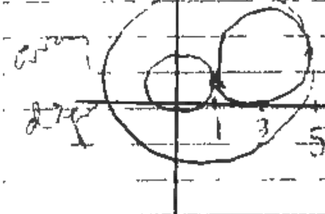
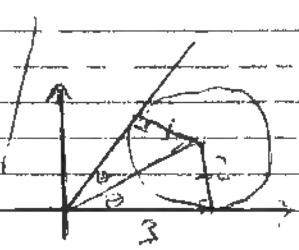
THE END

MATH. EXT 2 SOLUTIONS, TERM 4 2007

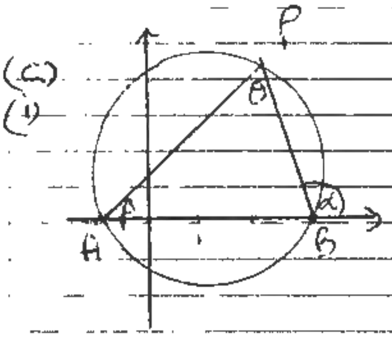
MATHEMATICS Extension 2: Question 1

Suggested Solutions	Marks	Marker's Comments
(a) $z_1 - z_2 = 2 - i - i(2-i)$ $= 2 - i - 2i - 1$ $= \underline{1 - 3i}$ $\boxed{2}$		
(b) (i) $z = \frac{(-9+15i)}{(4-i)} \times \frac{(4+i)}{(4+i)} = \frac{-36 - 9i + 60i - 15}{17}$ $\boxed{3}$ $\therefore z = \frac{-51+51i}{17} = \underline{-3+3i}$		$\frac{1}{2}$ for $\frac{4+i}{4+i}$, $\frac{1}{2}$ for 17 1 for expanding $\frac{1}{2}$ for $a = -3$ $\frac{1}{2}$ for $b = 3$
(ii) $\lvert z\rvert = \sqrt{9+9} = \underline{\sqrt{18}} = 3\sqrt{2}$  $\arg z = \underline{\frac{3\pi}{4}}$ $\boxed{2}$		1 for $\lvert z\rvert = \sqrt{18}$ = 1 for $\arg z = \frac{3\pi}{4}$
(c) Let $z = x + iy$, $x, y \in \mathbb{R}$ $z^2 = x^2 - y^2 + 2xyi = 35 - 12i$ METHOD 1 $\boxed{3}$ $\therefore x^2 - y^2 = 35$ — (1) $xy = -6$ — (2) so $x^2y^2 = 36$ and $x^4 - x^2y^2 = 35x^2$ i.e. $x^4 - 36 = 35x^2$ $x^4 - 35x^2 - 36 = 0$ $(x^2 - 36)(x^2 + 1) = 0$ $x = \pm 6$ or no real soln $\therefore y = \mp 1$ $\therefore z = \underline{6 - i}$ or $\underline{-6 + i}$		1 for a method 1 for $6 - i$ 1 for $-6 + i$
(d) Let $z = x + iy$ $Re(z) = x$ | $Im(z) = y$ $\lvert z - 2\rvert = \sqrt{(x-2)^2 + y^2}$ $\therefore \sqrt{(x-2)^2 + y^2} \le x$ $x^2 - 4x + 4 + y^2 \le x^2$ $\underline{y^2 \le 4(x-1)}$ $\underline{\text{and } y \geq 0}$  $\boxed{4}$		$\frac{1}{2}$ each $Re(z) = x$ $Im(z) = y$ $\frac{1}{2}$ for $\sqrt{\ldots} \le x$ 1 for $y^2 \le 4(x-1)$ $\frac{1}{2}$ for sketch of 1 for shaded region
(e) New number $= (1+3i) \times \sqrt{2} \times \text{cis}\frac{\pi}{4}$ $= (1+3i) \times \sqrt{2} \times \frac{1}{\sqrt{2}}(1+i)$ $\boxed{2}$ $= \underline{-2+4i}$		

MATHEMATICS Extension 2: Question 2

Suggested Solutions	Marks	Marker's Comments
(a) $(1-2i)$ is given $\therefore (1+2i)$ is also a root as all the coefficients are real numbers.		1 mk
~~$\alpha\beta = \frac{c}{a} = c = (1+2i)(1-2i)$~~ $= 5$		½
$\alpha+\beta = -b/a = -b = 1+2i+1-2i$ $-b = 2$ $b = -2$		½
$\therefore bc = -2 \times 5$ $= -10$		1 mk
(b)(i) $\lvert z-(3+2i)\rvert = 2$ is a circle centre $(3+2i)$ radius 2 $\lvert z+3\rvert = \lvert z-5\rvert$ is the perp. bisector of interval joining -3 and 5, ie $x=1$	½ ½ for $x=1$ ~~1 for~~ circle ~~incl.~~ centre	
		
(ii) since $x=1$ is a tangent to the circle there's only one point of intersection $z = 1+2i$		1 for answer 1 for explanation
(iii) $\lvert z-2i\rvert = k$ represents a circle, centre $2i$, radius k.  $\therefore$ only one solution for z when $k=1$ or $k=5$		1 mk for each answer
(iv) max $\arg z = 2\theta$ $= 2\tan^{-1}(2/3)$ ~~$= 1.1760^c$~~ $(67°23')$ 		½ for diagram/explanation 1 for $\theta = 2\tan^{-1}(2/3)$ ½ for answer

MATHEMATICS Extension 2: Question 2

Suggested Solutions	Marks	Marker's Comments
(c) (i)  $\arg(z-3)=\alpha$, $\arg(z+1)=\beta$, $\alpha-\beta=\theta$ (exterior angle of a triangle equals the sum of 2 interior opposite angles)		1st mark
but $\alpha-\beta=\frac{\pi}{3}$		
$\therefore \theta=\frac{\pi}{3}$		2nd mark
(ii) centre will lie on $x=1$, since the perpendicular bisector of the chord AB passes through the centre		
 $\angle AOB=2\theta$ (angle at centre is double the angle at the circumference), $\angle AOB=\frac{2\pi}{3}$		1st mark
$\frac{y}{2}=\tan\frac{\pi}{6}$		
$y=\frac{2}{\sqrt{3}}$		
$\therefore$ centre is $\left(1,\frac{2}{\sqrt{3}}\right)$		2nd mark

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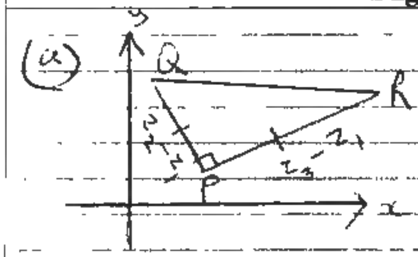
MATHEMATICS Extension 2: Question 3

Suggested Solutions	Marks	Marker's Comments
(a) 		
ΔPQR is right angled at $P(z_1)$ and is isosceles $PQ = PR$.		1 for answer/diagram
This is because the multiplication of $(z_3 - z_1)$ by i rotates this length $90°$ anticlockwise, it doesn't change the length as $\lvert z_2 - z_1\rvert = \lvert i(z_3 - z_1)\rvert$		1
$= \lvert i\rvert \times \lvert z_3 - z_1\rvert$		1
$= \lvert z_3 - z_1\rvert$		
(b)(i) $\dfrac{z^n + z^{-n}}{2} = \dfrac{(\text{cis}\,\theta)^n + (\text{cis}\,\theta)^{-n}}{2}$		
$= \dfrac{\cos n\theta + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta)}{2}$		1 mk
$= \dfrac{\cos n\theta + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)}{2}$		
$= \dfrac{2\cos(n\theta)}{2}$		1 mk
$= \cos(n\theta)$		
$\therefore$ QED		
(ii) $\cos(m\theta) \times \cos(n\theta) = \dfrac{(z^m + z^{-m})}{2}\dfrac{(z^n + z^{-n})}{2}$		½
$= \dfrac{z^{m+n} + z^{-m-n} + z^{m-n} + z^{-m+n}}{4}$		½
$= \dfrac{z^{m+n} + z^{-(m+n)} + z^{m-n} + z^{-(m-n)}}{4}$		½
$= \dfrac{2\cos(m+n)\theta + 2\cos(m-n)\theta}{4}$		½
$= \frac{1}{2}\left[\cos(m+n)\theta + \cos(m-n)\theta\right]$		
QED.		

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MATHEMATICS Extension 2: Question 3

Suggested Solutions	Marks	Marker's Comments
(c) (i) $P(z) = z^8 - \frac{5}{2}z^4 + 1$		
α is a root of $P(z)=0$		
$\therefore \alpha^8 - \frac{5}{2}\alpha^4 + 1 = 0$		1
$P(i\alpha) = (i\alpha)^8 - \frac{5}{2}(i\alpha)^4 + 1$		
$= i^8\alpha^8 - \frac{5}{2}(i\alpha)^4 + 1$		
$= \alpha^8 - \frac{5}{2}\alpha^4 + 1$		
$= 0$		1
$P\left(\frac{1}{\alpha}\right) = \left(\frac{1}{\alpha}\right)^8 - \frac{5}{2}\left(\frac{1}{\alpha}\right)^4 + 1$		
$= \frac{1}{\alpha^8}\left(1 - \frac{5}{2}\alpha^4 + \alpha^8\right)$		
$= \frac{1}{\alpha^8} \times 0$		1
$= 0$		
$\therefore$ $i\alpha$ and $\frac{1}{\alpha}$ are also roots of $P(z)$		
(ii) $z^8 - \frac{5}{2}z^4 + 1 = 0$		
$2z^8 - 5z^4 + 2 = 0$		1
$(2z^4 - 1)(z^4 - 2) = 0$		
$\therefore z^4 = \frac{1}{2}$ or $z^4 = 2$		1
$\therefore$ one root is $z = \sqrt[4]{2}$		1
(iii) the roots are $\pm\sqrt[4]{2}$, $\pm\sqrt[4]{2}\,i$, $\pm\frac{1}{\sqrt[4]{2}}$, $\pm\frac{i}{\sqrt[4]{2}}$		2 marks

MATHEMATICS Extension 2: Question 4

Suggested Solutions	Marks	Marker's Comments

(a) Let $z = x + iy$ so $\frac{1}{z} + \frac{1}{\bar{z}} = 1$ $\boxed{3}$

$\bar{z} = x - iy$

$$\frac{\bar{z} + z}{z\bar{z}} = 1$$

$$\frac{2x}{x^2+y^2} = 1$$

$\therefore\ x^2 + y^2 = 2x$

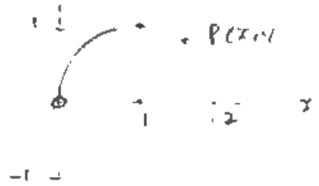
i.e. $x^2 - 2x + y^2 = 0$

i.e. $(x-1)^2 + y^2 = 1$

$\therefore$ locus of z is $\underline{(x-1)^2 + y^2 = 1}$ except $(0,0)$

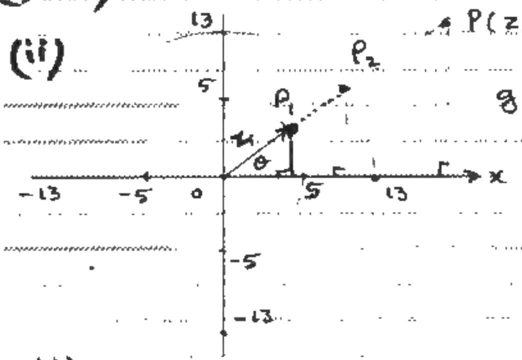
a circle $c(1,0)$

$r = 1$



(b) (i) $|z_1 + z_2| \le |z_1| + |z_2|$ ✓ equality iff collinear ($z_2 = kz_1$)

(ii)



greatest $|z_1 + z_2|$

$= |z_1| + |z_2|$

$= \underline{5} + 13$

$OP = \underline{18}$ ✓

(iii) As z_1 and z_2 are collinear $\boxed{2}$

$OP = 18$

$\therefore\ \overrightarrow{OP_2} = z_2 = \frac{13}{5}\cdot\overrightarrow{OP_1} = \frac{13}{5} z_1 = \frac{13}{5}(3 + 4i)$

$z_2 = 13 \operatorname{cis}\theta$ where $\tan\theta = \frac{4}{3}$

(c) (i) $w = \operatorname{cis}\frac{2\pi}{5}$

$\left[w^k = \operatorname{cis}\frac{2\pi k}{5} \quad \text{de Moivres Thm} \right]$

$\therefore\ z^5 - 1 = (w^k)^5 - 1 = (w^5)^k - 1$ $\boxed{2}$

$= \left(\operatorname{cis}\frac{2\pi}{5}\times 5\right)^k - 1$

$= (\operatorname{cis} 2\pi)^k - 1$

$= 1 - 1$

i.e. $z^5 - 1 = 0$

$\therefore\ w^k$ is a solution of $z^5 - 1 = 0$ $k \in \mathbb{Z}$

(ii) As $z^5 - 1 = 0$ $\therefore$ 5 roots $= w^0, w, w^2, w^3, w^4$

$= 1, w, w^2, w^3, w^4$

and $\Sigma\alpha = 1 + w + w^2 + w^3 + w^4 = -\frac{b}{a} = 0$

$\boxed{2}$

MATHEMATICS Extension 2: Question 4.

Suggested Solutions	Marks	Marker's Comments
(c) (iii) $\boxed{2}$		
$(\omega-1)(1+2\omega+3\omega^2+4\omega^3+5\omega^4)$		
$= \omega+2\omega^2+3\omega^3+4\omega^4+5\times1-1-2\omega-3\omega^2-4\omega^3-5\omega^4$		
$= -\omega-\omega^2-\omega^3-\omega^4+5-1$	✓	
$= -(\omega+\omega^2+\omega^3+\omega^4)+4$		
$= -(-1)+4$		
$= 5$	✓	
(iv) METHOD 1: $S = \dfrac{\omega}{1-\omega^2}+\dfrac{\omega^2}{1-\omega^4}+\dfrac{\omega^3}{1-\omega}+\dfrac{\omega^4}{1-\omega^3}$		
$\therefore \omega S = \dfrac{\omega^2}{1-\omega^2}+\dfrac{\omega^3}{1-\omega^4}+\dfrac{\omega^4}{1-\omega}+\dfrac{1}{1-\omega^3}$		
$= \dfrac{\omega^2}{1-\omega^2}+\dfrac{\omega^4}{\omega-1}+\dfrac{\omega^4}{1-\omega}+\dfrac{\omega^2}{\omega^2-1}$	✓	
$\boxed{2}$ $= \dfrac{\omega^2}{1-\omega^2}+\dfrac{\omega^2}{\omega^2-1}+\dfrac{\omega^4}{\omega-1}+\dfrac{\omega^4}{1-\omega}$	✓	
$= \dfrac{\omega^2}{1-\omega^2}-\dfrac{\omega^2}{1-\omega^2}+\dfrac{\omega^4}{\omega-1}-\dfrac{\omega^4}{\omega-1}$		
$= 0$		
$\therefore S = 0$ as $\omega \neq 0$.		
METHOD 2		
$S = \dfrac{\omega}{1-\omega^2}+\dfrac{\omega^2}{1-\omega^4}+\dfrac{\omega^3}{1-\omega}+\dfrac{\omega^4}{1-\omega^3}$		

$$= \frac{\omega}{1-\omega^2}+\frac{\omega^4}{\omega^2-\omega}+\frac{\omega^4}{\omega-\omega^2}+\frac{\omega}{\omega^2-1}$$

$$= \frac{\omega}{1-\omega^2}-\frac{\omega}{1-\omega^2}+\frac{\omega^4}{\omega^2-\omega}-\frac{\omega^4}{\omega^2-\omega}$$

$$= 0$$