Year 11 Term 4 Mathematics Extension 2 Exam, 2008

Question 1. **Marks**

(a) Find the modulus and principal argument of $7-7i$. **2**

(b) Simplify $\dfrac{5+2i}{3+4i}$ in the form $a+ib$ where $a,b\in R$. **2**

(c) Sketch the locus on the Argand diagram : $|z+2-2i|=2$. **3**

(d) A root of $x^2-3x+b=0$ is $2-4i$ where b is a complex number. Find the other root. **1**

(e) (i) Find the three cube roots of -1 in the form $a+ib$ where $a,b\in R$. **3**

(ii) Show accurately on the Argand diagram the points P_1, P_2 and P_3 representing the complex cube roots of -1. **1**

(f) The point P representing z in the Argand diagram satisfies the equation :

$$|z-1-5i|=|z+2-3i|.$$ **2**

(i) Find the Cartesian equation of the locus of the point P.

(ii) Find the minimum value of $|z|$. **1**

Question 2.

(a) $OABC$ is a square where $\overrightarrow{OA}$ is represented by the vector 2+ 3i. **4**
Find the vectors representing $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

(b) (i) Find the square roots of $5-12i$. **2**

(ii) Hence solve $x^2-(8-6i)x+2-12i=0$. **2**

(c) (i) Write in Modulus Argument form : $3\sqrt{3}+3i$. **2**

(ii) Hence show that $3\sqrt{3}+3i$ is a root of $x^4-36x^2+1296=0$. **3**

(iii) Deduce a quadratic factor of x^4-36x^2+1296 with real coefficients giving reasons. **2**

Question 3.

(a) **2**

(i) If $z = r\left(\cos\vartheta + i\sin\vartheta\right)$ and $\left|z\right| = 1$ show $\cos n\vartheta = \dfrac{z^n + \dfrac{1}{z^n}}{2}$

and $\sin n\vartheta = \dfrac{z^n - \dfrac{1}{z^n}}{2i}$ for

n integer.

(ii) Find an expression for $\sin n\vartheta \sin 2\vartheta$ in terms of *z*, and deduce that **2**

$$\sin n\vartheta \sin 2\vartheta = \frac{1}{2}\left\{\cos(n-2)\vartheta - \cos(n+2)\vartheta\right\}.$$

(ii) Evaluate $\displaystyle\int_0^{2\pi} \sin n\vartheta \sin 2\vartheta \, d\vartheta$ if n is an integer. **4**

(b) In the Argand diagram the points *A*, *B* and *C* represent the triangle *ABC.*

The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are represented by z_1 and z_2 respectively with

$Arg\ z_1 = \alpha$, $Arg\ z_2 = \beta$ and $\angle BAC = \vartheta$.

(i) Draw the triangle *ABC* in the Argand diagram with $\alpha > \beta > 0$. **1**

(ii) Write z_1 and z_2 in Mod Arg form. **2**

(iii) Show that the Area *A* of triangle *ABC* is given by **:** **2**

$$A = \left|\frac{1}{2}\mathrm{Im}\left(z_1\,\overline{z_2}\right)\right| .$$

(iv) Hence find the area of ΔABC if the coordinates in the *x*-*y* plane of *A*, *B* and *C* are **2**

(3, 2), (7, 5) and $\left(9, 4\right)$ respectively.

Question 4.

(a) Graph the intersection of the regions defined by **:** **5**

$z\,\overline{z} \geq 9$, $z + \overline{z} \leq 8$ and $0 < Arg\ z < \dfrac{\pi}{4}$.

(b) Let α be the complex root of $z^7 = 1$ with the smallest positive argument.
Let $\vartheta = \alpha + \alpha^2 + \alpha^4$ and $\delta = \alpha^3 + \alpha^5 + \alpha^6$.

(i) Write α in Mod-Arg form. **1**

(ii) Explain why $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = 0$ **1**

(iii) Show $\vartheta + \delta = -1$ and $\vartheta\delta = 2$. **2**

(iv) Write a quadratic equation in the variable t with roots ϑ and δ. **1**

(v) Solve this quadratic equation and deduce that $\vartheta = -\frac{1}{2} + i\frac{\sqrt{7}}{2}$ **3**

and **not** $-\frac{1}{2} - i\frac{\sqrt{7}}{2}$.

(vi) Find the value of $\cos\frac{4\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{\pi}{7}$ **1**

(vii) Deduce the value of $\sin\frac{4\pi}{7} + \sin\frac{2\pi}{7} - \sin\frac{\pi}{7}$. **1**

End of Exam

Y11 – T4 – II answers. 2008

1 (a) $7-7i$ Modulus $|z| = 7\sqrt{2}$ units

Argument $= -\frac{\pi}{4}$

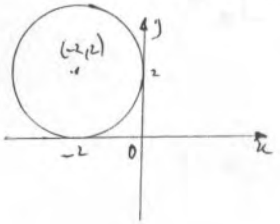
(b) $\frac{5+2i}{3+4i} = \frac{(5+2i)}{(3+4i)}\frac{(3-4i)}{(3-4i)}$

$= \frac{15-20i+6i+8}{3^2+4^2}$

$= \frac{23-14i}{25}$

$= \frac{23}{25} - \frac{14}{25}i$

(c)



(d) $z_1 = 2-4i$

$z_1 + z_2 = 3$

$\therefore$ other root $z_2 = 1+4i$

(e)(i) $z^3 = -1$

$z = \cos\left(\frac{\pi+2\pi k}{3}\right) + i\sin\left(\frac{\pi+2\pi k}{3}\right)$ k integer

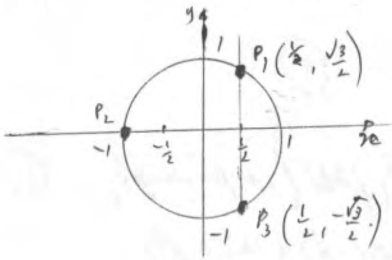
$k=0$ $z_0 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$k=1$ $z_1 = \cos\pi + i\sin\pi = -1 + 0i$

$k=2$ $z_2 = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$\therefore$ Roots of -1 are $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $-1+0i$, $\frac{1}{2} - \frac{\sqrt{3}}{2}i$

(ii)



(f) (i) $(x-1)^2 + (y-5)^2 = (x+2)^2 + (y-3)^2$

$-2x+1 \quad -10y+25 = 4x+4 \quad -6y+9$

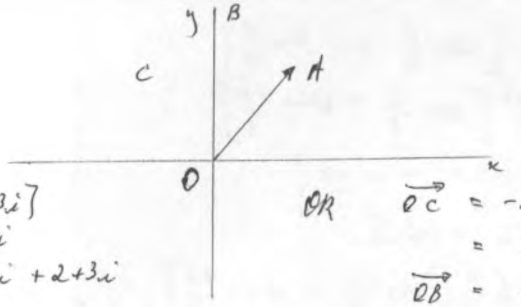
$6x + 4y - 13 = 0$

(ii) $|z|_{min} = \left|\frac{13}{\sqrt{6^2+4^2}}\right|$

$= \frac{13}{\sqrt{52}}$

$= \frac{\sqrt{13}}{2}$ units.

2 (a)



$\vec{OC} = i[2+3i]$

$= -3+2i$

$\vec{OB} = -3+2i+2+3i$

$= -1+5i$

OR $\vec{OC} = -i[2+3i]$

$= 3-2i$

$\vec{OB} = 3-2i+2+3i$

$= 5+i$

(b) $(x+iy)^2 = 5-12i$ $x, y \in R$

$x^2 - y^2 = 5$ ①

$2xy = -12$

$xy = -6$ ②

$y = \frac{-6}{x}$

$\therefore x^2 - \frac{36}{x^2} = 5$

$x^4 - 5x^2 - 36 = 0$

$(x^2-9)(x^2+4) = 0$

$x^2 = 9$ or $x^2 = -4$

$x=3$ $y=-2$ No soln $x \in R$

$x=-3$ $y=2$

$\therefore$ square roots of $5-12i$ are $\pm(3-2i)$

ii) $z^2 - (8-6i)z + 2-12i = 0$

$z = \frac{8-6i \pm \sqrt{(8-6i)^2 - 8 + 48i}}{2}$

$= \frac{8-6i \pm \sqrt{64-96i-36-8+48i}}{2}$

$= \frac{8-6i \pm \sqrt{20-48i}}{2}$

$= 4-3i \pm \sqrt{5-12i}$

$= 4-3i \pm (3-2i)$

$z = 7-5i$ or $1-i$

$(i)\ 3\sqrt{3}+3i = 6\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$

(ii) $(3\sqrt{3}+3i)^2 = 6^2\left[\cos\frac{2\pi}{6} + i\sin\frac{2\pi}{6}\right]$

$= 36\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$

$= 18 + 18\sqrt{3}i$

$(3\sqrt{3}+3i)^4 = 6^4\left[\cos\frac{4\pi}{6} + i\sin\frac{4\pi}{6}\right]$

$x = 3\sqrt{3}+3i \qquad = 1296\left[-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right]$

$= -648 + 648\sqrt{3}i$

$\therefore x^4 - 36x^2 + 1296 = -648 + 648\sqrt{3}i - 36(18+18\sqrt{3}i) + 1296$

$= -648 + 648\sqrt{3}i - 648 - 648\sqrt{3}i + 1296$

$= 0$

$\therefore x = 3\sqrt{3}+3i$ is a root of $x^4 - 36x^2 + 1296 = 0$

(iii) Another root is $x = 3\sqrt{3} - 3i$ [conjugate root with real coefficients]

$\therefore$ Quadratic factor $(x - 3\sqrt{3} - 3i)(x - 3\sqrt{3} + 3i)$

$= (x - 3\sqrt{3})^2 + 9$

$= x^2 - 6\sqrt{3}x + 36$ OR $x^2 + 6\sqrt{3}x + 36$.

Q3 (i) $|z| = 1 \quad \therefore z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n$

$z^n = \cos n\theta + i\sin n\theta \qquad -①$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$\frac{1}{z^n} = \cos n\theta - i\sin n\theta \qquad -②$

$\therefore z^n + \frac{1}{z^n} = 2\cos n\theta \quad \& \quad z^n - \frac{1}{z^n} = 2i\sin n\theta$

$\cos n\theta = \dfrac{z^n + \frac{1}{z^n}}{2} \qquad \sin n\theta = \dfrac{z^n - \frac{1}{z^n}}{2i}$

) $\sin n\theta \sin 2\theta = \dfrac{z^n - \frac{1}{z^n}}{2i} \cdot \dfrac{z^2 - \frac{1}{z^2}}{2i}$

$= -\frac{1}{4}\left[z^{n+2} - z^{n-2} - \frac{1}{z^{n-2}} + \frac{1}{z^{n+2}}\right]$

$= -\frac{1}{4}\left[z^{n+2} + \frac{1}{z^{n+2}} - \left(z^{n-2} + \frac{1}{z^{n-2}}\right)\right]$

$\therefore \sin n\theta \sin 2\theta = -\frac{1}{2}\left[\dfrac{z^{n+2} + \frac{1}{z^{n+2}}}{2} - \left(\dfrac{z^{n-2} + \frac{1}{z^{n-2}}}{2}\right)\right]$

$= \frac{1}{2}\left[\cos(n-2)\theta - \cos(n+2)\theta\right]. \qquad (1)$

$\therefore \int_0^{2\pi} \sin n\theta \sin 2\theta = \frac{1}{2}\int_0^{2\pi} \cos(n-2)\theta - \cos(n+2)\theta \, d\theta$

$= \frac{1}{2}\left[\frac{\sin(n-2)\theta}{n-2} - \frac{\sin(n+2)\theta}{n+2}\right]_0^{2\pi}$

$= 0 \qquad n \neq \pm 2 \qquad (2)$

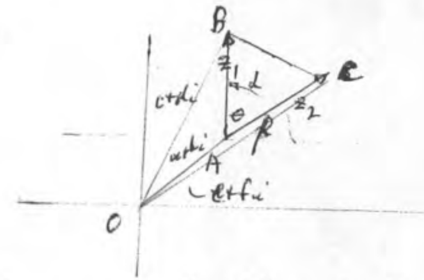
OR $\quad = \frac{1}{2}\int_0^{2\pi} 1 - \cos 4\theta \, d\theta \qquad n = 2$

$= \frac{1}{2}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi}$

$= \pi \qquad n = 2 \qquad (1)$

OR $= -\pi \qquad n = -2$.

(b)



(1)

(ii) $z_1 = |AB|\{\cos\alpha + i\sin\alpha\} \qquad (1)$

$z_2 = |AC|\{\cos\beta + i\sin\beta\}$

(iii) $RHS = \frac{1}{2}\,\mathrm{Im}\left\{|AB|\{\cos\alpha + i\sin\alpha\}\,|AC|\{\cos\beta - i\sin\beta\}\right\} \qquad (1)$

$= \frac{1}{2}\,\mathrm{Im}\,|AB||AC|\{\cos\alpha + i\sin\alpha\}\{\cos(-\beta) + i\sin(-\beta)\}$

$= \frac{1}{2}\,\mathrm{Im}\,|AB||AC|\{\cos(\alpha-\beta) + i\sin(\alpha-\beta)\}$

$= \frac{1}{2}|AB||AC|\sin(\alpha-\beta)$

$= \frac{1}{2}|AB||AC|\sin\theta$ where $\alpha-\beta=\theta$

$=$ Area ΔABC.

(iv) $z_1 = 4+3i$

$z_2 = 6+2i$

Area $= \frac{1}{2}\,\mathrm{Im}(z_1\bar{z}_2)$

$= \frac{1}{2}\,\mathrm{Im}\left[(4+3i)(6-2i)\right]$

$= \frac{1}{2}\left[-8+18\right]$

$= 5$ unit^2.

Q4 $z\bar{z} \geq 9$

or $x^2+y^2 \geq 9$ (1)

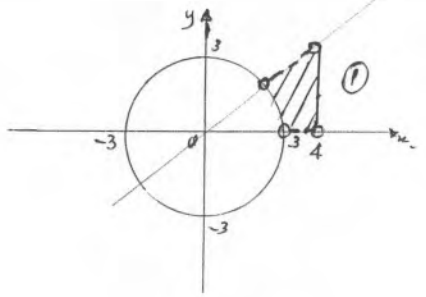
$z+\bar{z} \leq 8$

$2x \leq 8$

$x \leq 4$ (1)

$0 < \mathrm{Arg}\, z < \frac{\pi}{4}$

(1) (1)



(b) (i) $z^7=1 \implies z=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$ smallest + argument.

$\therefore z^7-1=0$

$(z-1)(z^6+z^5+z^4+z^3+z^2+z+1)=0$

or $(z-1)(1+z+z^2+z^3+z^4+z^5+z^6)=0$

$\therefore$ If α complex then $\alpha\neq 1$

hence $1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6=0$

(iii) $\theta=\alpha+\alpha^2+\alpha^4$ $\quad\delta=\alpha^3+\alpha^5+\alpha^6$

$\theta+\delta=\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6$

$=-1$

And $\theta\delta = (\alpha+\alpha^2+\alpha^4)(\alpha^3+\alpha^5+\alpha^6)$

$= \alpha(1+\alpha+\alpha^3)(\alpha^3)(1+\alpha^2+\alpha^3)$

$= \alpha^4(1+\alpha^2+\alpha^3+\alpha+\alpha^3+\alpha^4+\alpha^3+\alpha^5+\alpha^6)$

$= \alpha^4(2\alpha^3+1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6)$

$= \alpha^4(2\alpha^3+0)$

$= 2\alpha^7$

$= 2\times 1$

$= 2$

(iv) $\theta+\alpha=-\frac{b}{a}$ $\quad\theta\alpha=\frac{c}{a}$

$\therefore \frac{b}{a}=1$ $\quad\therefore \frac{c}{a}=2$

$t^2+\left(\frac{b}{a}\right)t+\frac{c}{a}=0$

$t^2+t+2=0$

(v) $t=\frac{-1\pm\sqrt{1-8}}{2}$

$= -\frac{1}{2}\pm\frac{\sqrt{7}}{2}i$

Now $\theta=\alpha+\alpha^2+\alpha^4$

$= \cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}+\left(\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}\right)^2+\left(\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}\right)^4$

$= \cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}+\cos\frac{4\pi}{7}+i\sin\frac{4\pi}{7}+\cos\frac{8\pi}{7}+i\sin\frac{8\pi}{7}$

$= \cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}+\cos\frac{4\pi}{7}+i\sin\frac{4\pi}{7}-\cos\frac{\pi}{7}-i\sin\frac{\pi}{7}$

$= \cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}-\cos\frac{\pi}{7}+i\left[\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}-\sin\frac{\pi}{7}\right]$

But $\sin\frac{\pi}{2} > \sin\frac{2\pi}{7} > \sin\frac{\pi}{7} > 0$

$\therefore \quad \sin\frac{2\pi}{7} - \sin\frac{\pi}{7} > 0$

$\sin\frac{4\pi}{7} > 0$

$\therefore \quad \sin\frac{4\pi}{7} + \sin\frac{2\pi}{7} - \sin\frac{\pi}{7} > 0$

$\operatorname{Im}(t) > 0.$

$\therefore \quad \theta = -\frac{1}{2} + \frac{\sqrt{7}}{2}i$

(vi) $\cos\frac{4\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2}$

(vii) $\sin\frac{4\pi}{7} + \sin\frac{2\pi}{7} - \sin\frac{\pi}{7} = \frac{\sqrt{7}}{2}.$