Year 11 Mathematics Extension II Term 4 Assessment 2009

Question 1: (15 Marks) (START A NEW PAGE) **Marks**

(a) Given that $z = 2 + i$ and $w = 1 + 3i$, express the following in the form $z = x + iy$ where x and y are real numbers.

(i) $z + \overline{w}$ **2**

(ii) z^2 **2**

(iii) $\dfrac{z}{w}$ **2**

(b) Given that $z = 2 + 3i$ is a root of the equation $z^3 - 2z^2 + pz + q = 0$, where p and q are real.

(i) Find the other two roots of the equation. **2**

(ii) Find the values of p and q. **2**

(c) (i) If $z = \cos\theta + i\sin\theta$, use De Moivre's Theorem to prove that $z^n + z^{-n} = 2\cos n\theta$. **2**

(ii) Hence deduce that $\cos\theta\cos 2\theta = \dfrac{1}{2}\left(\cos 3\theta + \cos\theta\right)$. **3**

Question 2: (15 Marks) (START A NEW PAGE) **Marks**

(a) (i) Solve $\alpha^2 = 5 - 12i$, expressing both answers in the form $a + ib$ where a an b are real numbers. **3**

(ii) Hence solve $z^2 - (1 - 4i)z - (5 - i) = 0$, expressing both answers in the form $x + iy$ where x and y are real numbers. **3**

(b) Sketch, on an Argand Diagram, the intersection of the regions $|z| \geq 4$ and $0 \leq \arg(z) \leq \dfrac{\pi}{4}$. **3**

(c) If $z = \dfrac{1+i}{1-i}$ and $w = \dfrac{\sqrt{2}}{1-i}$ are two complex numbers.

(i) Find the modulus and arguments of z and w. **2**

(ii) Plot the points representing z, w and $z + w$ on an Argand Diagram. **2**

(iii) Hence show that $\tan\left(\dfrac{3\pi}{8}\right) = 1 + \sqrt{2}$ **2**

Question 3: (15 Marks) (START A NEW PAGE) **Marks**

(a) (i) Express $1+i\sqrt{3}$ in mod-arg form. **2**

(ii) Hence evaluate $\left(1+i\sqrt{3}\right)^8$, expressing your answer in the form $x+iy$ where x and y are real numbers and not trigonometric expressions. **2**

(b) Solve $\bar{z}-\dfrac{12}{z}=1-i$, expressing your answer in the form $a+ib$ where a an b are real numbers. **4**

(c) The point Z represents the complex number z in the complex plane. If $\text{Re}\left(\dfrac{z-ib}{z-a}\right)=0$, where a and b are real, prove that the locus of Z is a circle and give its centre and radius. **4**

(d) If $z=r(\cos\theta+i\sin\theta)$, prove that $\dfrac{z}{z^2+r^2}$ is real and express its value in terms of θ. **3**

Question 4: (15 Marks) (START A NEW PAGE) **Marks**

(a) If α and β are two complex numbers $(\alpha\neq\beta)$ and $|\alpha|=1$, show that $\left|\dfrac{\alpha\bar{\beta}-1}{\alpha-\beta}\right|=1$. **3**

(b) (i) The points P, Q and R represent the complex numbers z, w and $z+w$ respectively on an Argand diagram, where $\arg(w)>\arg(z)$ and $0<\arg(z)<\dfrac{\pi}{2}$. Plot the positions of points P, Q and R on an Argand diagram. **1**

(ii) If O is the origin, what type of quadrilateral is $OPRQ$ given that $\left|\dfrac{z-w}{z+w}\right|=1$? **1**

(iii) Hence, what can be stated about the expression $\dfrac{w}{z}$? **2**

(c) (i) Write -1 in mod-arg form and hence express both complex roots of $z^3=-1$ in mod-arg form. **3**

(ii) If $z=\lambda$ is either of the two complex roots of $z^3=-1$, prove that $\lambda^2-\lambda+1=0$. **2**

(iii) Evaluate $\left(\lambda^2+\lambda+1\right)^8+\left(\lambda^2+\lambda-1\right)^8$. **3**

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Alternative

Question 4: (15 Marks) (START A NEW PAGE) **Marks**

(a) Find the greatest value of $|z|$ if the complex number z satisfies the equation $\left|z-\frac{4}{z}\right|=8$. **4**

(b) (i) Find the two complex cube roots of unity expressing them in mod-arg form. **3**

(ii) If ω is one of the complex cube roots of unity prove that $1+\omega+\omega^2=0$. **2**

(iii) Evaluate $(1+\omega-\omega^2)^8+(1-\omega+\omega^2)^8$. **3**

(iv) Prove that if n is an integer, then $1+\omega^n+\omega^{2n}=3$ or 0, depending on whether n is or is not a multiple of 3. **3**

Extra Questions

(a) On the Argand Diagram provided and with the aid of ruler, protractor and a pair of compasses, show the position of the complex numbers represented by

(i) $\bar{z}_1$ **1**

(ii) $-z_2$ **1**

(iii) z_1-z_2 **1**

(iv) iz_1 **1**

(v) iz_1+z_2 **1**

Clearly show all relevant lengths and angles.

If ω is a complex root of $\omega^8=1$, prove that $\omega+\omega^7$ is real.

Year 11 Mathematics Extension II Term 4 Assessment 2009
SOLUTIONS

Question 1: (15 Marks) (START A NEW PAGE) **Marks**

(a) Given that $z = 2 + i$ and $w = 1 + 3i$, express the following in the form $z = x + iy$ where x and y are real numbers.

(i) $z + \overline{w} = (2 + i) + (1 - 3i)$ **2**

$= 3 - 2i$

(ii) $z^2 = (2 + i)^2$ **2**

$= 4 + 4i - 1$

$= 3 + 4i$

(iii) $\frac{z}{w} = \frac{2+i}{1+3i} \times \frac{1-3i}{1-3i}$ **2**

$= \frac{(2+i)(1-3i)}{1+9}$

$= \frac{2 - 6i + i + 3}{10}$

$= \frac{1}{2} - \frac{1}{2}i$

(b) Given that $z = 2 + 3i$ is a root of the equation $z^3 - 2z^2 + pz + q = 0$, where p and q are real.

(i) Find the other two roots of the equation. **2**

Since $z = 2 + 3i$ is a root then so is $z = 2 - 3i$ (since coefficients are real)

Let roots be $z(= \alpha) = 2 + 3i$, $z(= \beta) = 2 - 3i$ and $z = \gamma$

$\therefore \alpha + \beta + \gamma = 2 \quad (\text{sum of roots})$

$(2 + 3i) + (2 - 3i) + \gamma = 2$

$\gamma = -2$

(ii) Find the values of p and q. **2**

$\alpha\beta\gamma = -q \quad \ldots\ldots\ldots \quad (1)$

$\alpha\beta + \alpha\gamma + \beta\gamma = p \quad \ldots\ldots\ldots \quad (2)$

from (1) $(2+3i)(2-3i)(-2) = -q$

$\Rightarrow \quad q = 26$

from (2) $(2+3i)(2-3i) + (2+3i)(-2) + (2-3i)(-2) = p$

$\Rightarrow \quad p = 5$

$\therefore \quad p = 5$ and $q = 26$

(c) (i) If $z = \cos\theta + i\sin\theta$, use De Moivre's Theorem to prove that $z^n + z^{-n} = 2\cos n\theta$. **2**

$$
\begin{aligned}
z^n + z^{-n} &= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n} \\
&= (\cos n\theta + i\sin n\theta) + (\cos(-n\theta) + i\sin(-n\theta)) \quad \text{(by De Moivre's Theorem)} \\
&= (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta) \\
&= 2\cos n\theta
\end{aligned}
$$

(ii) Hence deduce that $\cos\theta\cos 2\theta = \frac{1}{2}(\cos 3\theta + \cos\theta)$. **3**

Now $\cos n\theta = \dfrac{z^n + z^{-n}}{2}$

$$
\begin{aligned}
\cos\theta\cos 2\theta &= \left(\frac{z + z^{-1}}{2}\right)\left(\frac{z^2 + z^{-2}}{2}\right) \\
&= \frac{z^3 + z^{-1} + z + z^{-3}}{4} \\
&= \frac{1}{4}\left(\left(z^3 + z^{-3}\right) + \left(z + z^{-1}\right)\right) \\
&= \frac{1}{4}(2\cos 3\theta + 2\cos\theta)
\end{aligned}
$$

$\therefore \quad \cos\theta\cos 2\theta = \frac{1}{2}(\cos 3\theta + \cos\theta)$

Question 2: (15 Marks) (START A NEW PAGE) **Marks**

(a) (i) Solve $\alpha^2 = 5-12i$, expressing both answers in the form $a+ib$ where a an b are real numbers. **3**

Let $\alpha = a+ib$ where a and b are real

$(a+ib)^2 = 5-12i$

$a^2 - b^2 + 2iab = 5-12i$

$a^2 - b^2 = 5$ ....(1)

$ab = -6$ ....(2)

By inspection $a=3$ and $b=-2$ or $a=-2$ and $b=3$

$\therefore \quad \alpha = 3-2i$ or $-3+2i$

(ii) Hence solve $z^2 - (1-4i)z - (5-i) = 0$, expressing both answers in the form $x+iy$ where x and y are real numbers. **3**

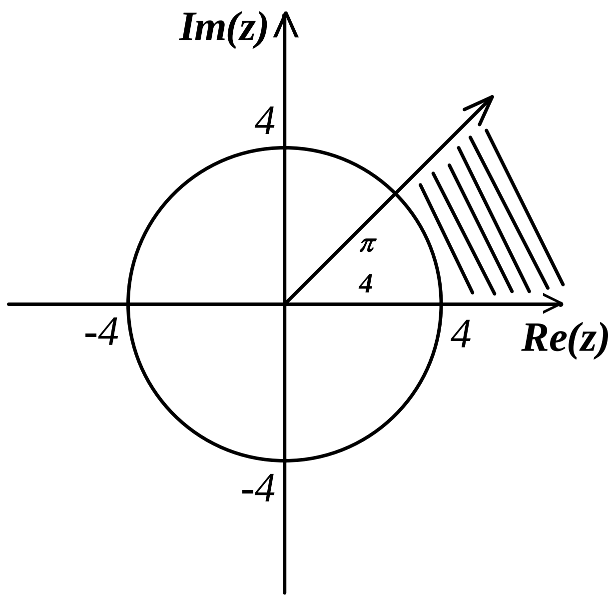
$$z = \frac{(1-4i) \pm \sqrt{(1-4i)^2 - 4(1)(-(5-i))}}{2}$$

$$= \frac{(1-4i) \pm \sqrt{5-12i}}{2}$$

$$= \frac{(1-4i) \pm (3-2i)}{2}$$

$z = 3-2i$ or $-1-i$

(b) Sketch, on an Argand Diagram, the intersection of the regions $|z| \geq 4$ and $0 \leq \arg(z) \leq \frac{\pi}{4}$. **3**



(c) If $z=\dfrac{1+i}{1-i}$ and $w=\dfrac{\sqrt{2}}{1-i}$ are two complex numbers.

(i) Find the modulus and arguments of z and w. **2**

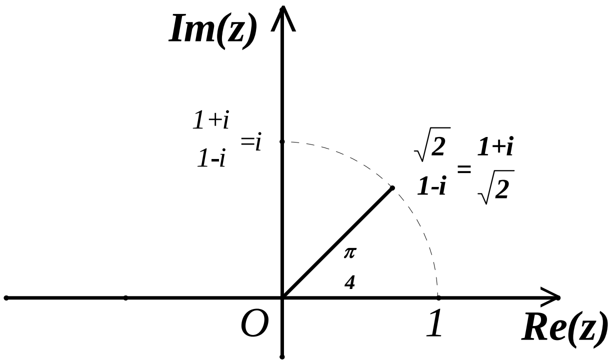
$$|z|=\frac{|1+i|}{|1-i|}=\frac{\sqrt{2}}{\sqrt{2}}=1$$

$$\arg z=\arg\left(\frac{1+i}{1-i}\right)=\arg(1+i)-\arg(1-i)=\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)=\frac{\pi}{2}$$

$$|w|=\frac{|\sqrt{2}|}{|1-i|}=\frac{\sqrt{2}}{\sqrt{2}}=1$$

$$\arg w=\arg\left(\frac{\sqrt{2}}{1-i}\right)=\arg(\sqrt{2})-\arg(1-i)=0-\left(-\frac{\pi}{4}\right)=\frac{\pi}{4}$$

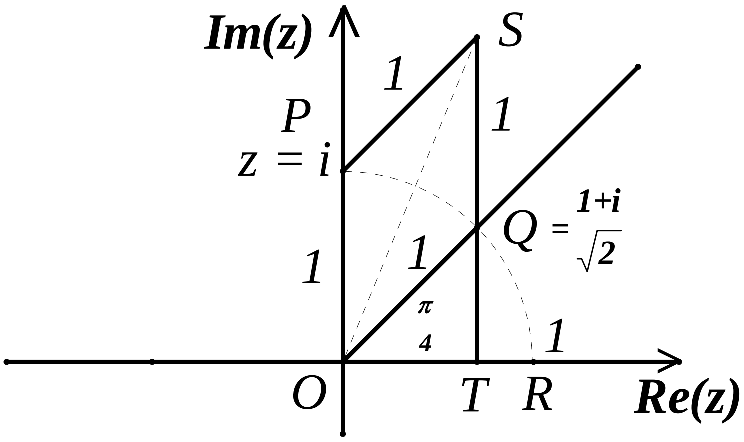
(ii) Plot the points representing z, w and $z+w$ on an Argand Diagram. **2**



(iii) Hence show that $\tan\left(\dfrac{3\pi}{8}\right)=1+\sqrt{2}$ **2**

$$z=\frac{1+i}{1-i}=cis\frac{\pi}{2}=i$$

$$w=\frac{\sqrt{2}}{1-i}=cis\frac{\pi}{4}=\frac{1}{\sqrt{2}}(1+i)$$



$\angle POR = \frac{\pi}{2}$ and $\angle QOR = \frac{\pi}{4} \Rightarrow \angle POQ = \frac{\pi}{4}$

$\angle SOQ = \frac{\pi}{8}$ $\left(POQS \text{ is a rhombus and diagonals bisect opposite angles}\right)$

$\angle SOR = \frac{3\pi}{8}$

$\tan\left(\angle SOR\right) = \frac{ST}{OT}$

$OT = QT$ $\left(\Delta OTQ \text{ is isosceles}\right)$

$\frac{QT}{1} = \sin\frac{\pi}{4}$

$QT = OT = \frac{1}{\sqrt{2}}$

$ST = 1 + \frac{1}{\sqrt{2}}$

$$\tan\frac{3\pi}{8} = \frac{1+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$$

$$= 1 + \sqrt{2}$$

Question 3: (15 Marks) (START A NEW PAGE) **Marks**

(a) (i) Express $1+i\sqrt{3}$ in mod-arg form. **2**

$$1+i\sqrt{3}=2cis\frac{\pi}{3}$$

(ii) Hence evaluate $\left(1+i\sqrt{3}\right)^8$, expressing your answer in the form $x+iy$ where x and y are real numbers and not trigonometric expressions. **2**

$$\begin{aligned}\left(1+i\sqrt{3}\right)^8 &= \left(2cis\frac{\pi}{3}\right)^8\\ &= 256cis\frac{8\pi}{3}\\ &= 256\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\\ &= -128+128i\sqrt{3}\end{aligned}$$

(b) Solve $\bar{z}-\dfrac{12}{z}=1-i$, expressing your answer in the form $a+ib$ where a an b are real numbers. **4**

Le t $z=a+ib$ where a and b are real

$a-ib-\dfrac{12}{a+ib}=1-i$

$(a+ib)(a-ib)-12=(1-i)(a+ib)$

$a^2+b^2-12=a+ib-ia+b$

$a^2+b^2-12=a+b+i(b-a)$

on equating real and imaginary parts

$a^2+b^2-12=a+b$ ....(1)

$b-a=0$ ....(2)

substitute (2) into (1)

$2a^2-12=2a$

$a^2-a-6=0$

$(a-3)(a+2)=0$

$a=-2 \text{ or } 3$

$a=-2\Rightarrow b=2$

$a=3\Rightarrow b=-3$

$z=-2-2i \quad or \quad 3+3i$

(c) The point Z represents the complex number z in the complex plane. If $\text{Re}\left(\frac{z-ib}{z-a}\right)=0$, where a and b are real, prove that the locus of Z is a circle and give its centre and radius. **4**

$$\frac{z-ib}{z-a}=\frac{x+iy-ib}{x+iy+a}$$
$$=\frac{x+i(y-b)}{(x+a)+iy}\times\frac{(x+a)-iy}{(x+a)-iy}$$
$$=\frac{[x(x-a)+y(y-b)]+i[-xy+(x-a)(y-b)]}{(x+a)^2+y^2}$$

If $\text{Re}\left(\frac{z-ib}{z-a}\right)=0$ then $x(x-a)+y(y-b)=0$

$$x^2-ax+y^2-yb=0$$
$$\left(x-\frac{a}{2}\right)^2+\left(y-\frac{b}{2}\right)^2=\frac{a^2+b^2}{4}$$

Therefore locus is a circle with centre $\left(\frac{a}{2},\frac{b}{2}\right)$ and radius $\frac{\sqrt{a^2+b^2}}{2}$

(d) If $z=r(\cos\theta+i\sin\theta)$, prove that $\frac{z}{z^2+r^2}$ is real and express its value in terms of θ. **3**

$$\frac{z}{z^2+r^2}=\frac{r(\cos\theta+i\sin\theta)}{\{r(\cos\theta+i\sin\theta)\}^2+r^2} \qquad (z\neq ir)$$
$$=\frac{r(\cos\theta+i\sin\theta)}{r^2(\cos 2\theta+i\sin 2\theta)+r^2}$$
$$=\frac{(\cos\theta+i\sin\theta)}{r(\cos 2\theta+i\sin 2\theta+1)}$$
$$=\frac{1}{r}\left(\frac{\cos\theta+i\sin\theta}{2\cos^2\theta+2i\sin\theta\cos\theta}\right)$$
$$=\frac{1}{2r\cos\theta}\left(\frac{\cos\theta+i\sin\theta}{\cos\theta+i\sin\theta}\right)$$
$$\frac{z}{z^2+r^2}=\frac{1}{2r\cos\theta}$$

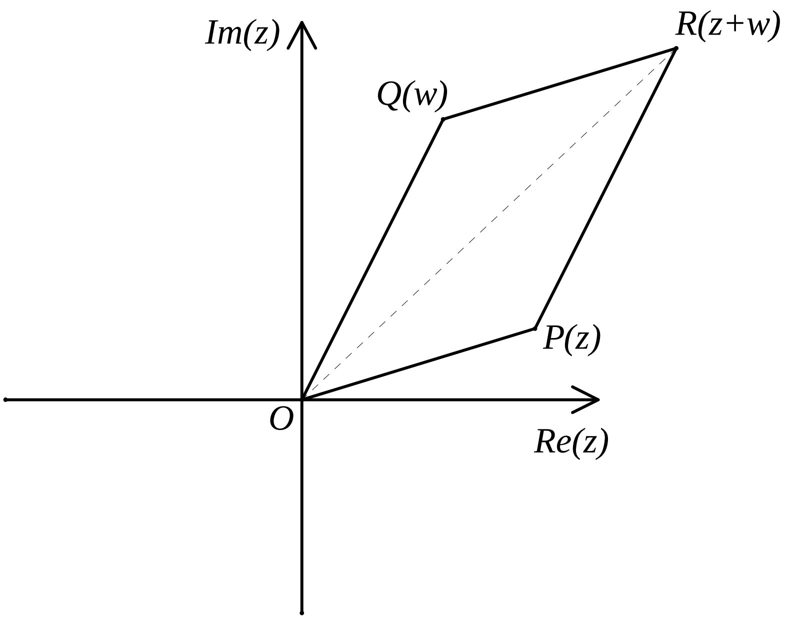
Therefore $\frac{z}{z^2+r^2}$ is real and its value is $\frac{1}{2r\cos\theta}$

Question 4: (15 Marks) (START A NEW PAGE) **Marks**

(a) If α and β are two complex numbers and $|\alpha| = 1$, show that $\left|\dfrac{\alpha\overline{\beta} - 1}{\alpha - \beta}\right| = 1$. **3**

$$\begin{aligned}
\left|\frac{\alpha\overline{\beta} - 1}{\alpha - \beta}\right| &= \frac{|\alpha\overline{\beta} - 1|}{|\alpha - \beta|} \\
&= \frac{|\alpha\overline{\beta} - 1|}{|\overline{\alpha} - \overline{\beta}|} \quad \left(\text{since } |z| = |\overline{z}|\right) \\
&= \frac{1}{|\alpha|}\frac{|\alpha\overline{\beta} - 1|}{|\overline{\alpha} - \overline{\beta}|} \quad \left(\text{since } |\alpha| = 1\right) \\
&= \frac{|\alpha\overline{\beta} - 1|}{|\alpha\overline{\alpha} - \alpha\overline{\beta}|} \\
&= \frac{|\alpha\overline{\beta} - 1|}{|1 - \alpha\overline{\beta}|} \quad \left(\text{since } |\alpha\overline{\alpha}| = 1\right) \\
&= 1
\end{aligned}$$

(b) (i) The points P, Q and R represent the complex numbers z, w and $z+w$ respectively on an Argand diagram, where arg(w) > arg(z) and $0 < \arg(z) < \dfrac{\pi}{2}$. Plot the positions of points P, Q and R on an Argand diagram. **1**



(ii) If O is the origin, what type of quadrilateral is $OPRQ$ given that $\left|\frac{z-w}{z+w}\right|=1$? **1**

$$\left|\frac{z-w}{z+w}\right|=1$$
$$|z-w|=|z+w|$$
$$\therefore OR=PQ$$

Since $OPRQ$ is a parallelogram by construction of $z+w$ then $OPRQ$ is a rectangle (parallelogram with equal diagonals) or possibly a square if $|z|=|w|$.

(iii) Hence, what can be stated about the expression $\frac{w}{z}$? **2**

Since $OPRQ$ is a rectangle $\angle QOP=\frac{\pi}{2}$ (angles at vertex in rectangle = $\frac{\pi}{2}$)

Now $\arg\left(\frac{w}{z}\right)=\arg(w)-\arg(z)$

But $\arg(w)-\arg(z)=\angle QOP$

$\therefore \arg\left(\frac{w}{z}\right)=\frac{\pi}{2}$

$\therefore \frac{w}{z}$ is a positive pure imaginary number

(c) (i) Write -1 in mod-arg form and hence express both complex roots of $z^3=-1$ in mod-arg form. **3**

$$-1=cis\pi$$

Let $z=rcis\theta$

$$z^3=r^3cis3\theta$$
$$r^3cis3\theta=cis(\pi+2k\pi) \quad k=0,1,2$$
$$r=1, 3\theta=\pi+2k\pi$$
$$\theta=\frac{(2k+1)\pi}{3}$$
$$\theta=\frac{\pi}{3},\pi,\frac{5\pi}{3}$$

Complex roots are $z=cis\frac{\pi}{3}$ and $cis\frac{5\pi}{3}$

(ii) If $z = \lambda$ is either of the two complex roots of $z^3 = -1$, prove that $\lambda^2 - \lambda + 1 = 0$. **2**

$z^3 = -1$

$z^3 + 1 = 0$

$(z+1)(z^2 - z + 1) = 0$

when $z = \lambda \quad (\lambda \in C)$

$(\lambda + 1)(\lambda^2 - \lambda + 1) = 0$

but $\lambda \neq -1$ since λ is complex

$\therefore \lambda^2 - \lambda + 1 = 0$

(iii) Hence evaluate $(\lambda^2 + \lambda + 1)^8 + (\lambda^2 + \lambda - 1)^8$. **3**

$$\begin{aligned}(\lambda^2 + \lambda + 1)^8 + (\lambda^2 + \lambda - 1)^8 &= (2\lambda)^8 + (2\lambda^2)^8 \text{ since } \lambda^2 + 1 = \lambda \text{ and } \lambda - 1 = \lambda^2 \\ &= 256(\lambda^8 + \lambda^{16}) \\ &= 256\{(\lambda^3)^2 \lambda^2 + (\lambda^3)^5 \lambda\} \\ &= 256(\lambda^2 - \lambda) \text{ since } \lambda^3 = -1 \\ &= 256(-1) \\ &= -256\end{aligned}$$

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Alternative

Question 4: (15 Marks) (START A NEW PAGE) **Marks**

(a) Find the greatest value of $|z|$ if the complex number z satisfies the condition $\left|z-\frac{4}{z}\right|=8$. **4**

(b) (i) Find the two complex cube roots of unity expressing them in mod-arg form. **3**

(ii) If ω is one of the complex cube roots of unity prove that $1+\omega+\omega^2=0$. **2**

(iii) Evaluate $\left(1+\omega-\omega^2\right)^8+\left(1-\omega+\omega^2\right)^8$. **3**

(iv) Prove that if *n* is an integer, then $1+\omega^n+\omega^{2n}=3$ or 0, depending on whether *n* is or is not a multiple of 3. **3**

Extra Questions

(a) On the Argand Diagram provided and with the aid of ruler, protractor and a pair of compasses, show the position of the complex numbers represented by

(i) $\bar{z}_1$ **1**

(ii) $-z_2$ **1**

(iii) z_1-z_2 **1**

(iv) iz_1 **1**

(v) iz_1+z_2 **1**

Clearly show all relevant lengths and angles.

If ω is a complex root of $\omega^8=1$, prove that $\omega+\omega^7$ is real.