Question 1 (15 marks) **Marks**

(a) Given the complex numbers $\alpha = 2 + 3i$ and $\beta = 5 + i$, simplify the following, leaving your answers in $x + iy$ form.

(i) $\alpha\beta$ **1**

(ii) $\overline{\alpha + \beta}$ **1**

(iii) $\dfrac{1}{\overline{\alpha} - \beta}$ **2**

(b) Write the following complex numbers in modulus – argument form.

(i) $z_1 = -9$ **1**

(ii) $z_2 = 12i$ **1**

(iii) $z_3 = (3 + 3i)^5$ **2**

(c) Solve $z^2 - (2 + i)z + (1+i) = 0$ over the complex field. **2**

(d) A complex number z is such that $arg\,(z + 2) = \dfrac{\pi}{6}$ and $arg(z - 2) = \dfrac{2\pi}{3}$. **3**
Find z, in form of $a + ib$, where a and b are real numbers.

(e) Use De Moivre's theorem to prove that $\sin 3t = 3\cos^2 t \sin t - \sin^3 t$. **2**

Question 2 (15 Marks) **START A NEW PAGE**

(a) P_1 and P_2 are points representing the complex numbers z_1 and z_2 as shown on the Argand diagram. **2**

If OP_1P_2 is an isosceles triangle with $\angle P_1OP_2 = 90°$, show that $z_1^2 + z_2^2 = 0$.

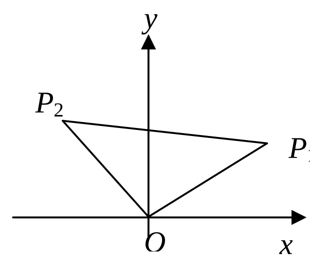


Diagram not to scale

(b) If the point P, represents the complex number z, which lies on the unit circle about the Origin, by using the triangle inequality, or otherwise, show that: **2**

$$|z^2 + z + 1| \le 3.$$

(c) Find $arg(z)$ if $z = \dfrac{-2}{1 + i\sqrt{2}}$. Give your answer correct to 4 decimal places. **3**

Question 2 continues on the next page.

Question 2 continued **Marks**

(d) $z = (1 + i)$ is one root of the equation $z^3 + pz^2 + qz + 6 = 0$, where p and q are real numbers.

(i) Find the other 2 roots of the equation. **2**

(ii) Hence, or otherwise find the values of p and q. **2**

(e) ***WRITE YOUR ANSWERS TO THIS PART OF QUESTION 2 ON THE PAGE PROVIDED AT THE END OF THE EXAM BOOKLET.***

Given $z_1 = -3 + 2i$ and $z_2 = 1 + 4i$,

(i) Draw neatly on the Argand diagram provided, the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ if A and B are the points representing the complex numbers z_1 and z_2. **1**

(ii) On this Argand diagram, indicate the point C representing $z_1 - z_2$. **1**

(iii) Find the $|z_1 - z_2|$ and $arg(z_1 - z_2)$. **2**

Question 3 (15 Marks) **START A NEW PAGE**

(a) If $|z| = 3$ and $arg\, z = \theta$ determine

(i) $\left|\dfrac{i}{z^2}\right|$ **1**

(ii) $arg\left(\dfrac{i}{z^2}\right)$ **2**

(b) On an Argand diagram, neatly shade the region that holds simultaneously for $|z - (2 + i)| \leq \sqrt{5}$ and $Arg\, z < \dfrac{\pi}{12}$. **3**

(c) (i) Show that the solutions of $z^6 + z^3 + 1 = 0$ are contained in the solutions of $z^9 - 1 = 0$. **1**

(ii) Neatly sketch the **nine** solutions of $z^9 - 1 = 0$ on an Argand diagram, showing all important features. **3**

(iii) Mark **clearly** on your diagram the **six** roots: z_1, z_2, z_3, z_4, z_5 and z_6 of $z^6 + z^3 + 1 = 0$. **2**

(iv) Hence show that the sum of the six roots of $z^6 + z^3 + 1 = 0$ is given by $2\left(\cos\dfrac{2\pi}{9} + \cos\dfrac{4\pi}{9} - \cos\dfrac{\pi}{9}\right)$. **3**

Question 4 (15 Marks) **START A NEW PAGE** **Marks**

(a) For which values of c does $x^2 + 4x + c$ have two complex conjugate roots? **2**

(b) If ω is one of the complex cube roots of unity, evaluate by simplifying **3**

$$(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8).$$

(c) The points A_1, A_2 and A_3 represent the complex numbers α_1, α_2 and α_3 respectively where $\alpha_1\alpha_3 = \alpha_2^2$ **3**

Show that OA_2 bisects the angle A_1OA_3, where O is the origin.

(d) Triangle OAB is scalene. External equilateral triangles ABF, BOD and OAE are constructed on the sides of ΔOAB. The triangles are positioned on the Argand diagram as shown.

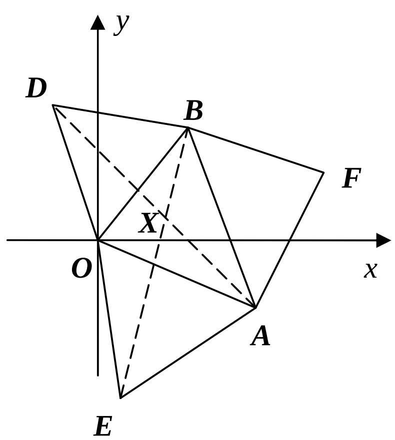


Diagram not to scale

The points A, B, D and E represent the complex numbers α, β, δ and ε respectively.

Let $w = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$.

(i) Show that $1 - w = -w^2$. **1**

(ii) Explain why $\delta = w\beta$. **1**

(iii) State the complex number α, represented by the point A, in terms of ε. **1**

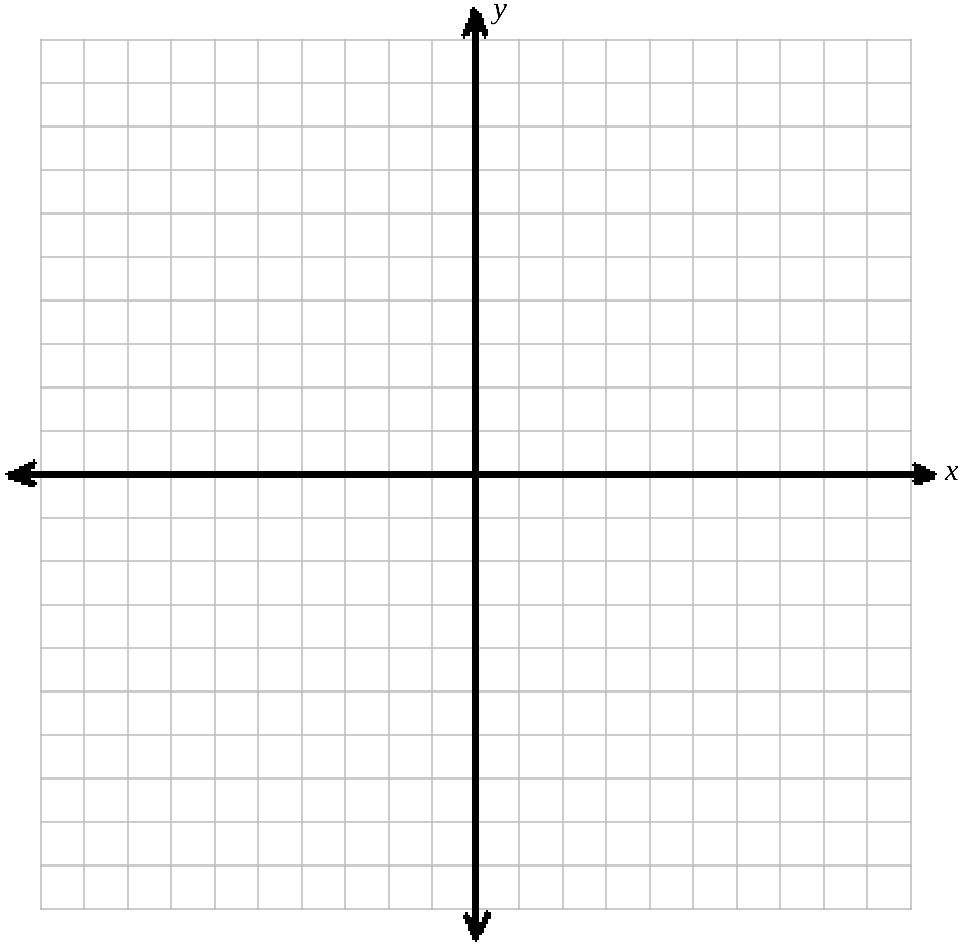
(iv) Hence, show that the complex number represented by the point F is $-w^2(\beta - \varepsilon)$. **2**

(v) Hence, show that $AD = BE = OF$. **2**

Student Number: ______________

Question 2 (e) ~ STAPLE TO THE BACK OF YOUR ANSWER SHEET FOR QUESTION 2.

(i) and (ii)



(iii) ____

____

____

____

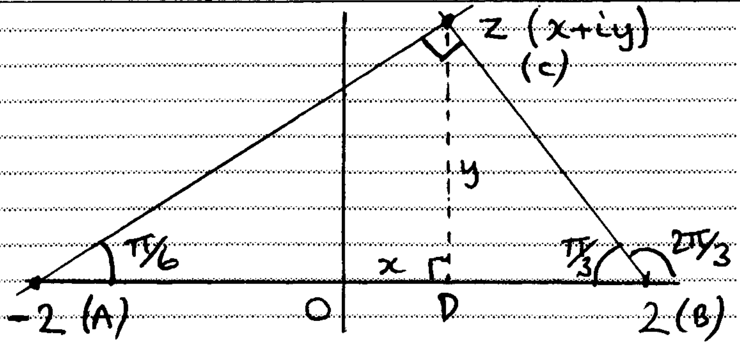
____

Y12 M.EXT 2 ASSESS TASK 1
TERM 4, 2010

TERM 4 2010 **MATHEMATICS Extension 2: Question 1**

Suggested Solutions	Marks	Marker's Comments
a) i) $\alpha\beta = (2+3i)(5+i)$		
$= \underline{\underline{7+17i}}$	1	
ii) $\alpha+\beta = 7+4i$		
$\therefore \overline{\alpha+\beta} = \underline{\underline{7-4i}}$	1	
iii) $\dfrac{1}{\alpha-\beta} = \dfrac{1}{(2-3i)-(5+i)}$		
$= \dfrac{-1}{3+4i}$	1	
$= \dfrac{-(3-4i)}{(3+4i)(3-4i)}$		
$= \dfrac{-(3-4i)}{25}$		½ mark deducted if left with long vincula.
$= \underline{\underline{-\dfrac{3}{25} + \dfrac{4}{25}i}}$	1	
b) i) $z_1 = \underline{\underline{9(\cos\pi + i\sin\pi)}}$	1	
ii) $z_2 = \underline{\underline{12(\cos\pi/2 + i\sin\pi/2)}}$	1	½ mark docked (once) if cis form left in final answer.
iii) $3+3i = 3\sqrt{2}(\text{cis}\,\pi/4)$		
$\therefore (3+3i)^5 = (3\sqrt{2})^5(\text{cis}\,5\pi/4)$ (de Moivre)	1	
$= \underline{\underline{972\sqrt{2}\left(\cos\dfrac{5\pi}{4} + i\sin\dfrac{5\pi}{4}\right)}}$	1	½ mark docked if modulus not expanded.
c) $z = \dfrac{(2+i) \pm \sqrt{(2+i)^2 - 4(1+i)}}{2}$	1	
$= \dfrac{(2+i) \pm \sqrt{-1}}{2}$		
$= \dfrac{(2+i) \pm i}{2}$		
$= \underline{(1+i) \text{ or } 1}$	1	

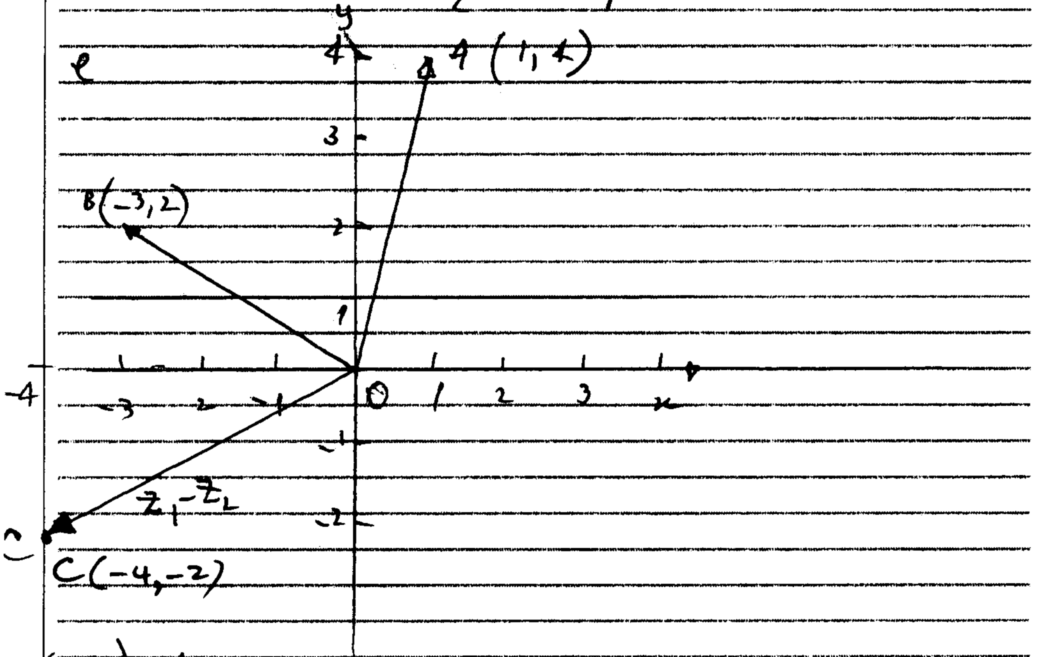
Suggested Solutions	Marks	Marker's Comments
d)		Gave ½ for a good diagram if nothing else.



Suggested Solutions	Marks	Marker's Comments
On a clear diagram, it is simple to show $\angle OBC = \pi/3$, $\angle AZB = \pi/2$ (Angles of ΔABC).		
Using simple trig.		
$\frac{y}{x+2} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ (triangle ADZ)		
$\Rightarrow y = \frac{x}{\sqrt{3}} + \frac{2}{\sqrt{3}}$ — ①	1	
$\frac{y}{2-x} = \tan\pi/3 = \sqrt{3}$ (triangle BDZ)		
$\Rightarrow y = 2\sqrt{3} - x\sqrt{3}$ — ②	1	
Solving ① and ②		
$x = 1,\ y = \sqrt{3}$ $\therefore z = 1 + i\sqrt{3}$	1	
e) $(\cos t + i\sin t)^3 = (\cos 3t + i\sin 3t)$ (de Moivre)	1	½ off if no mention of de Moivre
Expanding left hand side		
$\cos^3 t + 3i\cos^2 t\sin t - 3\cos t\sin^2 t - i\sin^3 t$		½ off if reason not given for plucking out right answer.
$= \cos 3t + i\sin 3t$		
Equating imaginary parts		
$\sin 3t = 3\cos^2 t\sin t - \sin^3 t$	1	

Y11 – 2010 – T4 **MATHEMATICS Extension 2: Question** 2

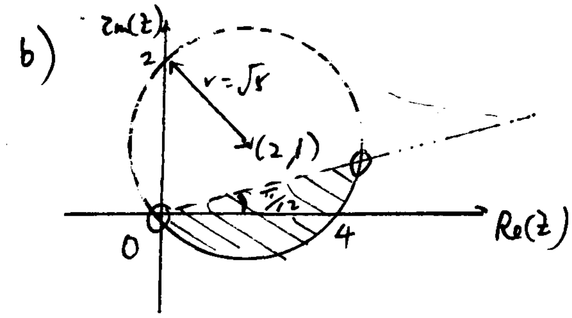
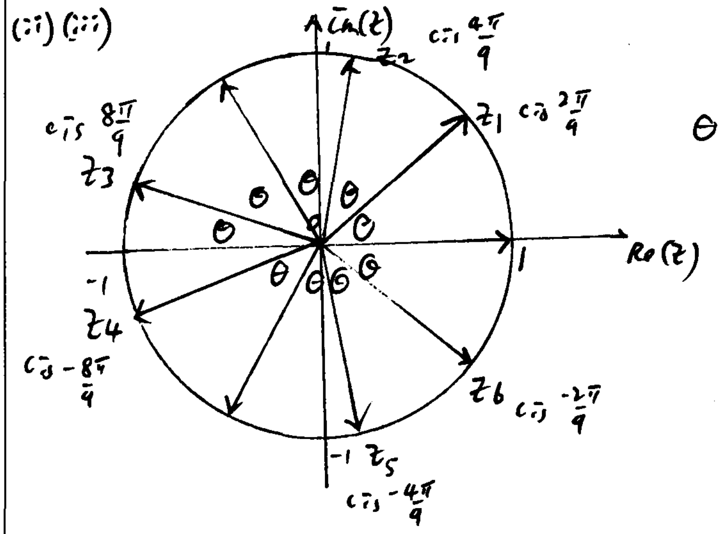
Suggested Solutions	**Marks**	**Marker's Comments**
2 (a) $\angle P_1OP_2 = \frac{\pi}{2}$		
$\lvert z_1\rvert = \lvert z_2\rvert$	$\frac{1}{2}$	
$\Rightarrow \quad z_2 = iz_1$	$\frac{1}{2}$	
$z_2^2 = -z_1^2$	1	
$z_2^2 + z_1^2 = 0$		
(b) $\lvert z^2+z+1\rvert \le \lvert z^2\rvert + \lvert z\rvert + \lvert 1\rvert$ (Δ Inequality)	1	
$\le 1 + 1 + 1 \qquad \lvert z\rvert = 1$	1	
≤ 3.		
(c) $\text{Arg}\left(\frac{-2}{1+i\sqrt{2}}\right) = \text{Arg}\,{-2} - \text{Arg}\,(1+i\sqrt{2})$	1	Some students get incorrect $\angle$ since No diagram
$= \pi - \tan^{-1}\sqrt{2}$	1	
$= 2.1863$	1	No need to rationalize
(d)(i) $z_1 = 1+i$		
$z_2 = 1-i$ Roots conjugate pairs real coeff.	1	
$z_3 = -3$ Product roots	1	
(ii) $\Sigma z_1 + z_2 + z_3 = 1+i + 1-i - 3$		
$-p = -1$	1	
$p = 1$		
$\Sigma z_1z_2 + z_1z_3 + z_2z_3 = (1+i)(1-i) - 3(1+i) - 3(1-i)$	1	
$q = -4$		



(e)(i)	1	
	1	show direction $z_1 - z_2$.
(iii) $z_1 - z_2 = -4 - 2i$		
$\lvert z_1 - z_2\rvert = 2\sqrt{5}$	1	
$\text{Arg}(z_1 - z_2) = -\pi + \tan^{-1}\left(\frac{1}{2}\right) = -2.6779$	1	

2010, T4 → Y11

MATHEMATICS Extension 2 : Question 3

Suggested Solutions	Marks	Marker's Comments
a i) $\frac{1}{9}$	1 m	no half mark Well done
ii) $\arg z - 2\arg z$	1 m	
$= \frac{\pi}{2} - 2\theta$	1 m	
b) 	3m	1m circle with centre (2,1) radius $\sqrt{5}$, must pass origin 1m dotted line : $\text{Arg } z = \frac{\pi}{12}$ (15° using protractor) 1m correct region hole at (0,0) includes: most students forgot dotted upper circle $-\frac{1}{2}$m
c i) $(z^3-1)(z^6+z^3+1) = z^9-1 = 0$ z^6+z^3+1 is a factor of z^9-1 $\therefore$ solns of $z^6+z^3+1=0$ are contained in solns of $z^9-1=0$	1m	
(ii) (iii)  $\theta = \frac{2\pi}{9}$	3m	1m unit circle with intercepts 1m 9 points marked with vectors with $\theta = \frac{2\pi}{9}$ at centre of circle (many forgot to write $\theta = \frac{2\pi}{9}$ $-\frac{1}{2}$m) 1m measure $\frac{2\pi}{9}$ (40°) with protractor. (some v. poor measurement)
	2 m	6 roots labelled $z_1, z_2, \dots z_6$ 1-2 wrong root −1m
(iv) Sum of roots $= \left(\text{cis}\frac{2\pi}{9} + \text{cis}\frac{-2\pi}{9}\right) + \left(\text{cis}\frac{4\pi}{9} + \text{cis}\frac{-4\pi}{9}\right) + \left(\text{cis}\frac{8\pi}{9} + \text{cis}\frac{-8\pi}{9}\right)$	1m	roots in conjugate pairs
$= 2\left(\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9}\right)$	1m	Generally ii, iii, iv are well done.
$= 2\left(\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} - \cos\frac{\pi}{9}\right)$ since $\cos\frac{8\pi}{9} = -\cos\frac{\pi}{9}$	1m	

Y11 M. EXT 2 ASSESSMENT TASK 1
TERM 4, 2010

MATHEMATICS Extension 2: Question 4

Suggested Solutions	Marks	Marker's Comments

Q 4(a) x^2+4x+c

METHOD I: For complex/unreal roots $\Delta < 0$ — $\frac{1}{2}$

i.e. $4^2 - 4\times1\times c < 0$ — $\frac{1}{2}$

$\underline{c > 4}$ — 1 — **[2]**

METHOD II: $x^2+4x+c = (x+2)^2 + c - 4 = 0$

As $(x+2)^2 \geq 0 \ \forall x$

to be complex $x^2+4x+c > 0 \ \forall x$

$\therefore c - 4 > 0$ only

i.e. $\underline{c > 4}$.

METHOD III: Let α be the complex root: $\alpha = a+ib$; $a, b \in \mathbb{R}$ $b \neq 0$

$\alpha + \bar{\alpha} = 2a = \text{"}-\frac{b}{a}\text{"} = -4 \quad \therefore a = -2$

$\alpha\bar{\alpha} = a^2+b^2 = \text{"}\frac{c}{a}\text{"} = c$

$4 + b^2 = c \quad$ so $b^2 = c-4$

as $b^2 > 0 \quad \therefore c - 4 > 0 \Rightarrow \underline{c > 4}$

(b) ω is a complex root

$\therefore \omega^3 = 1 \ ; \quad \omega^2+\omega+1 = 0$ — $\frac{1}{2}+\frac{1}{2}$ if use both

so $\omega^4 = \omega$; $\omega^8 = (\omega^3)^2 \times \omega^2 = \omega^2$ — $\frac{1}{2}$

so $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$

$= (1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$

$= [(1-\omega)(1-\omega^2)]^2$ — $\frac{1}{2}$

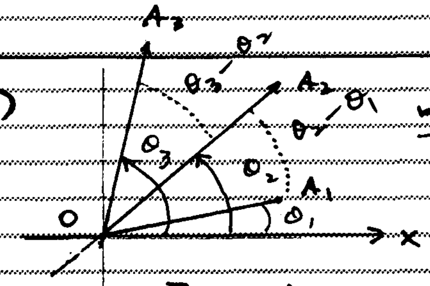
$= [1-\omega-\omega^2+\omega^3]^2 = [2-\omega-\omega^2]^2$

$= [2-(\omega+\omega^2)]^2$ — **[3]**

$= [2-(-1)]^2 = 3^2$

$\underline{= 9}$ — 1

(c)



WLOG! $\operatorname{Arg}\alpha_1 = \theta_1 \quad \alpha_1 = r_1 \operatorname{cis}\theta_1$ — case 1 "$\theta_3 > \theta_2 > \theta_1$"

$\operatorname{Arg}\alpha_2 = \theta_2 \quad \alpha_2 = r_2 \operatorname{cis}\theta_2$

$\operatorname{Arg}\alpha_3 = \theta_3 \quad \alpha_3 = r_3 \operatorname{cis}\theta_3$

($\frac{1}{2}$) Diagram only! — **[3]**

APPROACH I: As $\alpha_1\alpha_3 = \alpha_2^2$

A_2 ???

take Arg of both sides !!

$\operatorname{Arg}\alpha_1 + \operatorname{Arg}\alpha_3 = \operatorname{Arg}\alpha_2^2 = 2\operatorname{Arg}\alpha_2$ — ✓ 1

i.e $\theta_1 + \theta_3 = 2\theta_2$

i.e $\theta_3 - \theta_2 = \theta_2 - \theta_1$ — ✓ 1

i.e. $\angle XOA_3 - \angle XOA_2 = \angle XOA_2 - \angle XOA_1$ — ✓ 1

i.e. $\angle A_2OA_3 = \angle A_1OA_2$

$\therefore OA_2$ bisects $\angle A_1OA_3$ qed.

Suggested Solutions	Marks	Marker's Comments

(c) continued

APPROACH II : see I

$$\therefore \theta_2 = \frac{\theta_1+\theta_2}{2}$$

$$\therefore \angle A_2OA_3 = \theta_3 - \theta_2 = \theta_3 - \tfrac{1}{2}(\theta_1+\theta_3) = \tfrac{1}{2}(\theta_3-\theta_1)$$

and $\angle A_1OA_2 = \theta_2 - \theta_1 = \tfrac{1}{2}(\theta_1+\theta_3) - \theta_1 = \tfrac{1}{2}(\theta_3-\theta_1)$

$\therefore \angle A_2OA_3 = \angle A_1OA_2$

$\therefore OA_2$ bisects $\angle A_1OA_3$

watch this approach

APPROACH III : As $\alpha_1\alpha_3 = \alpha_2^2$

$$\therefore \frac{\alpha_3}{\alpha_2} = \frac{\alpha_2}{\alpha_1} \qquad \alpha_1, \alpha_2 \neq 0+0i$$

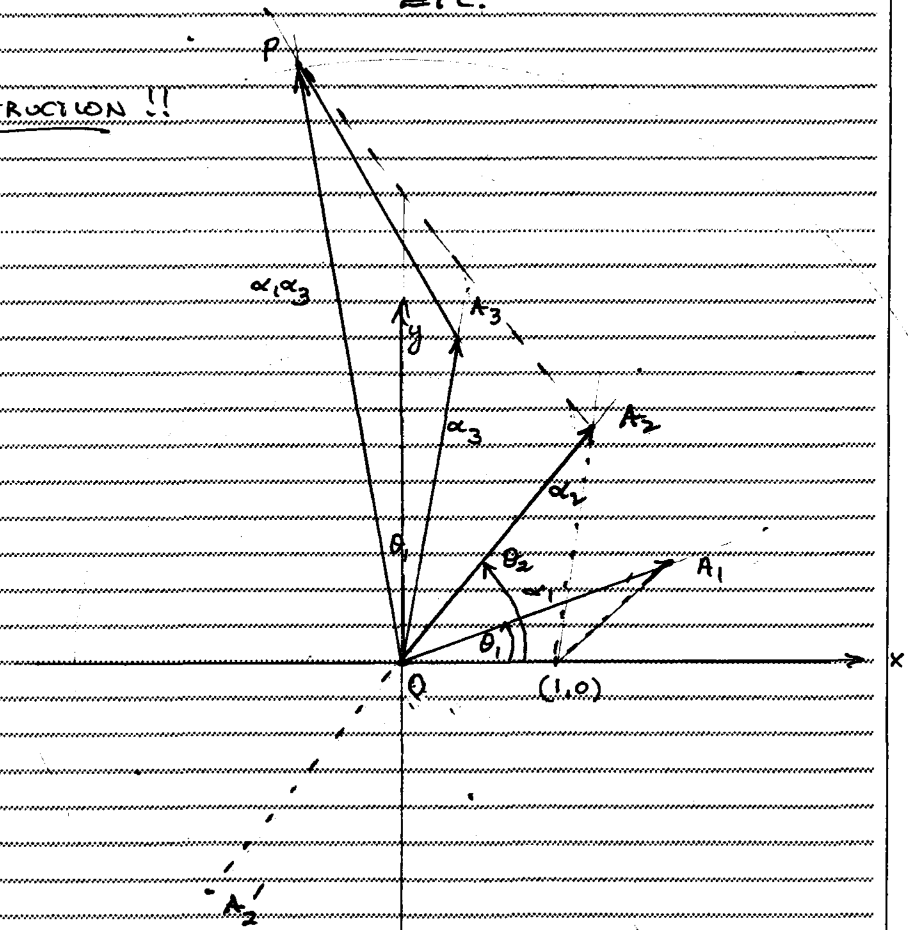
$$\therefore \operatorname{Arg}\frac{\alpha_3}{\alpha_2} = \operatorname{Arg}\frac{\alpha_2}{\alpha_1}$$

ie $\operatorname{Arg}\alpha_3 - \operatorname{Arg}\alpha_2 = \operatorname{Arg}\alpha_2 - \operatorname{Arg}\alpha_1$

ie $\theta_3 - \theta_2 = \theta_2 - \theta_1$

ETC.

CONSTRUCTION !!



Marker's comments:

If $\alpha_2^2 = \alpha_1\alpha_3$

$\alpha_2 = $ "$\sqrt{\alpha_1\alpha_3}$ or $-\sqrt{\alpha_1\alpha_3}$"

Let $\operatorname{Arg}\alpha_1\alpha_3 = \theta$

$\therefore \operatorname{Arg}\alpha_2 = \tfrac{1}{2}\theta$ or $\pi - \tfrac{1}{2}\theta$

by construction of ||| Δs

$\theta \equiv \theta_3 + \theta_1$

so $\operatorname{Arg}\alpha_2 = \tfrac{1}{2}(\theta_3+\theta_1)$ or $\pi - \tfrac{1}{2}(\theta_3+\theta_1)$

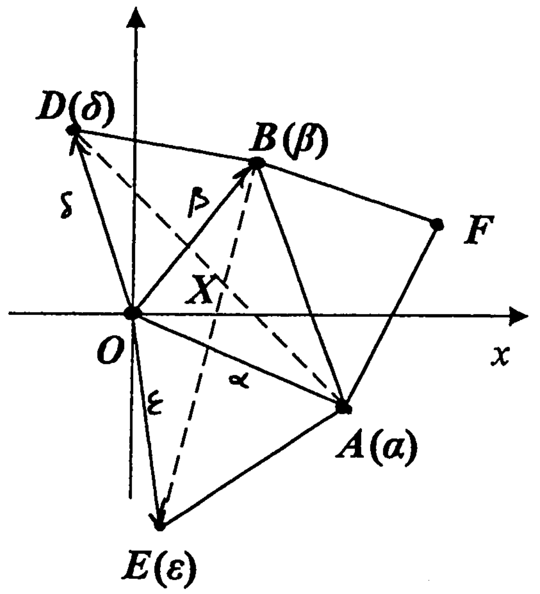
ie. $\theta_2 = \tfrac{1}{2}(\theta_3+\theta_1)$ or $\pi - \tfrac{1}{2}(\theta_3+\theta_1)$

Now to show OA_2 bisects !!?

MATHEMATICS Extension 2: Question 4 TERM 4, 2010

Suggested Solutions	Marks	Marker's Comments

(d)



(i) METHOD I

$w = \text{cis}\frac{\pi}{3}$

$\therefore w^3 = \text{cis}\,\pi$ (De Moivres Thm)

$= -1$

ie $w^3 + 1 = 0$

$(w+1)(w^2 - w + 1) = 0$

as $w \neq -1$ $\quad w^2 - w + 1 = 0$

i.e $1 - w = -w^2$. q.e.d.

METHOD II: LHS: $1 - w = 1 - \text{cis}\frac{\pi}{3} = 1 - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{2} - i\frac{\sqrt{3}}{2}$ — $\frac{1}{2}$

RHS: $-w^2 = -\left(\text{cis}\frac{\pi}{3}\right)^2 = -\text{cis}\frac{2\pi}{3}$ (De Moivre's thm)

$= -\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$

$= -\left[-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$ ASTC

$= \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$

$= \frac{1}{2} - i\frac{\sqrt{3}}{2}$ — $\frac{1}{2}$

$\therefore 1 - w = -w^2 \quad \left(= \overline{w} = \frac{1}{w}\right)$!!!

Marker's comments: Many ways to show (i)

Marks: $\boxed{1}$

(ii) As Δ OBD is equilateral Δ where each angle is $\frac{\pi}{3}$ ✓ $\frac{1}{2}$

and OB = OD i.e $|\beta| = |\delta|$

$\therefore \overrightarrow{OD} = \delta$ is the anticlockwise rotation by $\frac{\pi}{3}$ from $\overrightarrow{OB}$ i.e β — $\frac{1}{2}$

$\therefore \delta = \beta\,\text{cis}\frac{\pi}{3} = \beta w = w\beta$ q.e.d.

Marks: $\boxed{1}$

(iii) Similar argument: $\alpha = \varepsilon\,\text{cis}\frac{\pi}{3} = w\varepsilon$ ✓

Marks: $\boxed{1}$

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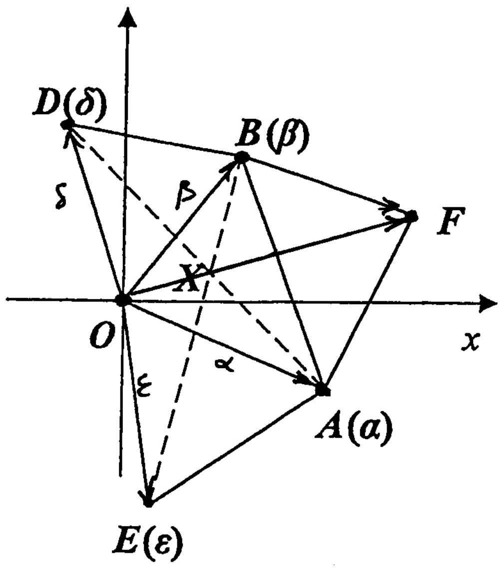
MATHEMATICS Extension 2: Question 4 TERM 4, 2010

Suggested Solutions	Marks	Marker's Comments

(d) (iv)



$\overrightarrow{OF}$

METHOD I : $\overrightarrow{OF} = \overrightarrow{OB} + \overrightarrow{BF}$ $\frac{1}{2}$

$= \beta + w\,\overrightarrow{BA}$

$= \beta + w(\alpha - \beta)$ $\frac{1}{2}$

$= (1-w)\beta + w\alpha$

$= (1-w)\beta + w.w\varepsilon$ $\frac{1}{2}$

$= -w^2\beta + w^2\varepsilon$ $\frac{1}{2}$

$= -w^2(\beta - \varepsilon)$

METHOD II $\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF}$

$= \alpha + \bar{w}\,\overrightarrow{AB}$

$= w\varepsilon + \bar{w}(\beta - \alpha)$

$= w\varepsilon + \bar{w}\beta - \bar{w}\alpha$

$= w\varepsilon + \bar{w}\beta - \bar{w}w\varepsilon$

$= w\varepsilon + \bar{w}\beta - \varepsilon$

$= \bar{w}\beta + (w-1)\varepsilon$

$= -w^2\beta + w^2\varepsilon$

$= -w^2(\beta - \varepsilon)$

$\bar{w} = \text{cis}(-\frac{\pi}{3})$

$1 - w = -w^2$!!!

$\bar{w} \equiv -w^2$ ∴

$\boxed{2}$

(v) $\overrightarrow{EB} = \beta - \varepsilon$ ∴ $EB = |\beta - \varepsilon|$ $\frac{1}{2}$

$\overrightarrow{AD} = \delta - \alpha$

$= w\beta - w\varepsilon$ ✓ $\frac{1}{2}$

$= w(\beta - \varepsilon)$ ∴ $AD = |w(\beta - \varepsilon)|$

$= |w|.|\beta - \varepsilon|$

$= |\beta - \varepsilon|$ as

stating that: $|w| = |\text{cis}\frac{\pi}{3}| \equiv 1$ $\frac{1}{2}$

$\overrightarrow{OF} = -w^2(\beta - \varepsilon)$ ∴ $OF = |-w^2|.|\beta - \varepsilon|$ $\frac{1}{2}$

$= |\beta - \varepsilon|$ as $|w^2| = |w|^2 = 1$

$|w^k| = |w|^k \equiv 1.$

∴ $BE = AD = OF \equiv |\beta - \varepsilon|$

OR Prove !!! $\Delta OEB \equiv \Delta OAD \equiv \Delta OAF$

$\boxed{2}$

Research : : Napoleon thm; Fermat point X, ...

- Prove OF, AD and EB are concurrent