Question 1. **Marks**

(a) Given $z_1 = 2-3i$ and $z_2 = 3+4i$,

(i) Find: $|z_1|$. **1**

(ii) Find; $z_1 + \bar{z}_2$. **1**

(iii) Find: $\frac{z_2}{z_1}$. **2**

(b) If $(a+3i)(7-i) = 17+bi$, where a and b are real numbers. Find the value of b. **2**

(c) Given $\Omega^2 = 35-12i$, find Ω. **2**

(d) Show that: $(1+i)^{2011} = 2^{1005}(-1+i)$. **3**

(e) Suppose $\alpha = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $\beta = 1-i$,

(i) Find $\arg\beta$. **1**

(ii) Find the smallest positive integer m such that $\frac{\alpha^m}{\beta^m}$ is purely imaginary. **2**

(iii) For this value of m, find the value of $\frac{\alpha^m}{\beta^m}$. **1**

Question 2. **[Start a new page]**

(a) Find the complex number $a+ib$ when $1+2i$ is rotated about the point $(3,1)$ by $\frac{\pi}{2}$ in an Argand plane. **2**

(b) Shade the region on an Argand plane satisfying z for $\left|\frac{1}{z}+1\right| \geq 1$. **1**

(c) It is known that $5-6i$ is a zero of the polynomial function:

$$P(z) = 2z^3 - 19z^2 + 112z + a_0, \text{ where } a_0 \text{ is real.}$$

(i) Find the other two zeros of $P(z)$. **2**

(ii) Find the value of a_0. **2**

(d) On an Argand plane, sketch the region described for z when: **4**

$$\left|\frac{1}{z}+\frac{1}{\bar{z}}\right| \geq \frac{1}{2} \text{ and } \operatorname{Im}(z) \geq 0 \text{ and } \frac{\pi}{6} \leq \arg z \leq \frac{3\pi}{4}.$$

(e) Given that ω is one of the complex cube roots of unity,

(i) Show that ω^2 can be the other complex root. **1**

(ii) What is the value of $1+\omega+\omega^2$. **1**

(iii) Find: $(\omega-1)(1+2\omega+3\omega^2)$ **2**

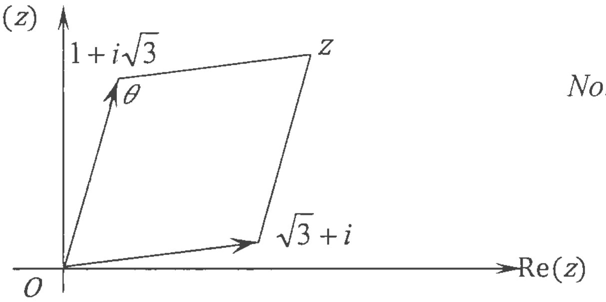
Question 3. **[Start a new page]** **Marks**

(a) Given $z = \cos\theta + i\sin\theta$, where $z \neq 0$.

(i) Show that: $\cos(n\theta) = \dfrac{z^n + z^{-n}}{2}$. **2**

(ii) Hence show that $\cos\theta.\cos 2\theta = \dfrac{1}{2}(\cos\theta + \cos 3\theta)$. **2**

(b) On the Argand diagram, the complex numbers 0, $\sqrt{3}+i$, z and $1+i\sqrt{3}$ form a rhombus as shown.



Not to scale.

(i) Find z in the form $a + ib$, where a and b are real numbers. **1**

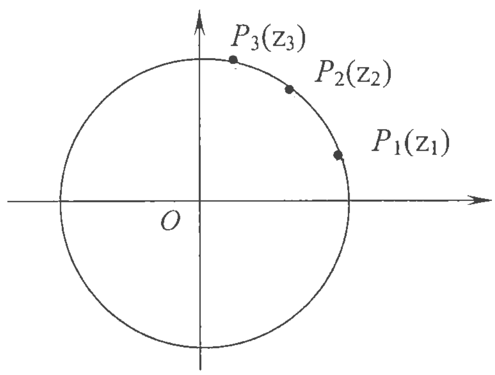
(ii) Find the value of θ, the marked interior angle of the rhombus. **2**

(c) Find the locus of z when: $\arg(z+2) + \arg(z-2) = \pi$. **3**

(d) For the three complex numbers z_1, z_2 and z_3, **3**

If $|z_1| = |z_2| = |z_3|$ such that $0 < \arg z_1 < \arg z_2 < \arg z_3 < \dfrac{\pi}{2}$,

as indicated in the diagram



On the diagram provided (see last page), explain why $\arg\left(\dfrac{z_3 - z_2}{z_3 - z_1}\right) = \dfrac{1}{2}\arg\left(\dfrac{z_2}{z_1}\right)$.

Question 3 *continued over the page*:

Question 3 *continued*

(e) Which one of the following Argand planes below could represent the position of the roots of $z^5+z^2-z+k=0$, where k is a real number. Give reasons. **2**

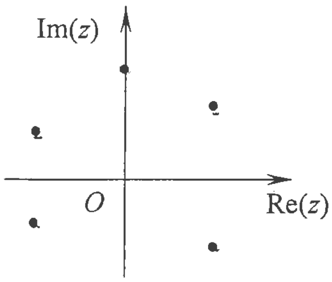


Diagram A

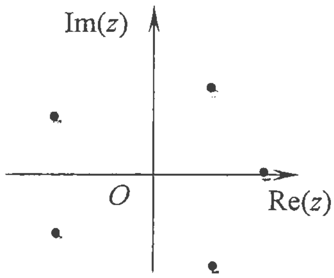


Diagram B

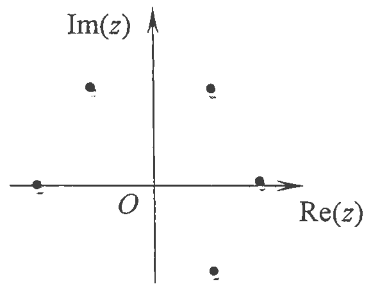


Diagram C

Question 4. **[Start a new page]** **Marks**

(a) Find the locus of z when: $\dfrac{z}{z+6}$ is purely imaginary. **2**

(b) By considering that: $\cos\theta+i\sin\theta=\cos\theta(1+i\tan\theta)$ and de Moivres' Theorem.

(i) Find the expression for $\cos 4\theta$ in terms of $\cos\theta$ and $\tan\theta$. **1**

(ii) As $\sin 4\theta=\cos^4\theta(4\tan\theta-4\tan^3\theta)$, find the result for $\tan 4\theta$ in terms of $\tan\theta$. **1**

(iii) Show that the solutions to the polynomial equation: **1**

$$x^4+4\sqrt{3}x^3-6x^2-4\sqrt{3}x+1=0$$

can be calculated from $\tan 4\theta=\dfrac{1}{\sqrt{3}}$.

(iv) Find the four solutions to this quartic equation. **2**

(v) Hence show that: $\tan\dfrac{7\pi}{24}\tan\dfrac{11\pi}{24}=\cot\dfrac{\pi}{24}\cot\dfrac{5\pi}{24}$. **2**

(c) (i) Find, in the form '$cis\theta$', the roots of the equation: **2**

$$z^{2n+1}=1,\text{ where } n=0,\ 1,\ 2,\ \ldots$$

(ii) Hence factorise $z^{2n}+z^{2n-1}+\ldots+z+1$ into quadratic factors with real coefficients. **2**

(iii) Hence, or otherwise find **2**

$$2^n\times\sin\frac{\pi}{2n+1}\times\sin\frac{2\pi}{2n+1}\times\ldots\times\sin\frac{n\pi}{2n+1}.$$

THE END

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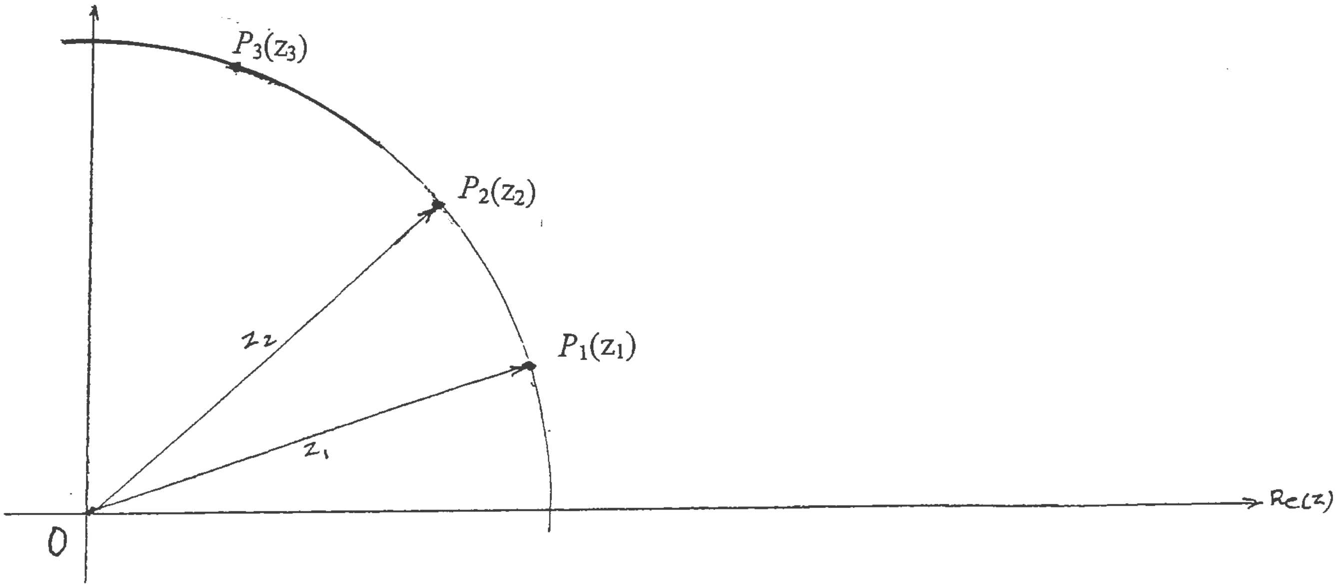
Question 3 (d)

Student id: _______

Attach this page to your Question 3 section.



MATH. EXT 2 ASSESSMENT TASK 1
TERM 4, 2011

MATHEMATICS Extension 2: Question 1

Suggested Solutions	Marks	Marker's Comments
Q1 (a) $z_1 = 2-3i \quad z_2 = 3+4i$		
(i) $\|z_1\| = \|2-3i\| = \sqrt{2^2+(-3)^2} = \sqrt{13}$	1	[1]
(ii) $z_1 + \bar{z}_2 = 2-3i + \overline{(3+4i)}$ $= 2-3i + 3-4i$ $= 5-7i$	1	[1]
(iii) $\dfrac{z_2}{z_1} = \dfrac{3+4i}{2-3i} \times \dfrac{2+3i}{2+3i} = \dfrac{6+9i+8i-12}{4+9}$ $= \dfrac{-6+17i}{13}$	1 1	[2]
(b) $(a+3i)(7-i) = 17+bi$ $7a - ai + 21i + 3 = 17+bi$ Equating Real and Imaginary parts $7a+3 = 17$ — (1) $21 - a = b$ — (2) (1) $7a = 14 \Rightarrow a = 2$ (2) $\therefore b = 21-2 = 19$	1 1	[2]
(c) Let $\Omega = x+iy$; $x, y \in \mathbb{R}$ $\Omega^2 = 35-12i$ $x^2-y^2+2xyi = 35-12i$ so $x^2-y^2 = 35$ — (1) $xy = -6$ — (2) $\therefore x^2y^2 = 36$ ---(2a) $x^4 - x^2y^2 = 35x^2$ ---(1a) $x^4 - 36 = 35x^2$ $x^4 - 35x^2 - 36 = 0$ $(x^2-36)(x^2+1) = 0$ ↳ no real solns $\therefore x = \pm 6$ $\therefore y = \dfrac{-6}{x} = \mp 1$ $\therefore \Omega = 6-i$ or $-6+i$ there are various approaches		[2]

MATHEMATICS Extension 2: Question 1

Suggested Solutions	Marks	Marker's Comments

Q1 (d) $(1+i)^{2011}$

METHOD 1 $\quad (1+i)^2 = 1-1+2i = 2i$

$\therefore (1+i)^{2011} = \left[(1+i)^2\right]^{1005} \times (1+i)$

$= (2i)^{1005}(1+i)$

$= 2^{1005} \times i^{1005}(1+i)$

$= 2^{1005}(i^4)^{250} \times i\,(1+i)$

$= 2^{1005}(i-1) = 2^{1005}(-1+i)$

METHOD 2

$1+i = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

$(1+i)^{2011} = \left[2^{\frac{1}{2}}\,\text{cis}\,\frac{\pi}{4}\right]^{2011}$

$= 2^{1005\frac{1}{2}} \cdot \text{cis}\left(\frac{2011\pi}{4}\right)$ (De Moivres Thm)

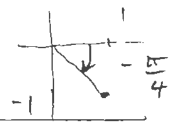
$= 2^{1005}\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$

$= 2^{1005}\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$ $\boxed{2}$

$= 2^{1005}(-1+i)$

(e) $\alpha = 2\,\text{cis}\,\frac{\pi}{3} \qquad \beta = 1-i$

(i) $\text{Arg}\,\beta = \text{Arg}(1-i) = -\frac{\pi}{4}$ (Acc $\frac{7\pi}{4}$) — 1



$\boxed{1}$

(ii) $\frac{\alpha^m}{\beta^m} = \left(\frac{\alpha}{\beta}\right)^m = \left(\frac{2\,\text{cis}\,\frac{\pi}{3}}{\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)}\right)^m$

$= \left(\sqrt{2}\,\text{cis}\left(\frac{7\pi}{12}\right)\right)^m$ — 1

$= 2^{\frac{1}{2}m}\,\text{cis}\,\frac{7m\pi}{12}$ (De M. thm)

To be purely imaginary — Real part $= 0$ $\boxed{2}$

— $\text{Arg}(\ ) = \pm\frac{\pi}{2}$

So $\frac{7m\pi}{12} = \frac{k\pi}{2}$ $\left(\cos\frac{\pi}{2} = 0\right)$ — 1 — $7m = 6k$

$\therefore$ least positive integer $m = 6$ (for $k=7$)

(iii) $\therefore \left(\frac{\alpha}{\beta}\right)^6 = 2^3\,\text{cis}\,\frac{7\pi}{2} = 2^3\,\text{cis}\left(\frac{3\pi}{2}\right)$

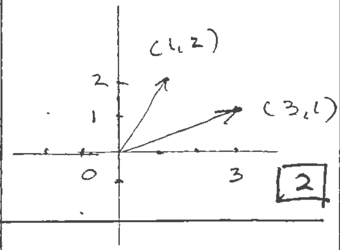
$= -8i$ — 1 $\boxed{1}$

MATHEMATICS Extension 2: Question 2

Suggested Solutions | **Marks** | **Marker's Comments**

Q2 (a)

$$a+ib = (3+i) + [1+2i - (3+i)] \times i \qquad 1$$
$$= 3+i + [-2+i]\, i$$
$$= 3+i - 2i - 1$$
$$\therefore a+ib = \underline{2-i} \qquad 1$$



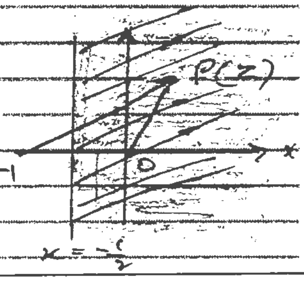
$\boxed{2}$

(b) $\left|\frac{1}{z}+1\right| \geq 1 \Rightarrow \left|\frac{1+z}{z}\right| \geq 1$

$\therefore |z+1| \geq |z|$

$(x+1)^2+y^2 \geq x^2+y^2$

$x \geq -\frac{1}{2} \qquad 1$



$\boxed{1}$

(c) (i) Since all the coefficients $(2, -19, 112, a_0)$ are real

$\overline{5-6i} = \underline{5+6i}$ is also a root $\qquad 1$

$\Sigma\alpha = s_1 = 5-6i+5+6i+\alpha = -\frac{b}{a} = \frac{19}{2}$

$\underline{\alpha = -\frac{1}{2}} \qquad 1$

$\therefore$ others $5+6i$ and $-\frac{1}{2}$

$\boxed{2}$

(ii) $s_3 = (5-6i)(5+6i)\left(-\frac{1}{2}\right) = -\frac{a_0}{2} \qquad 1$

$-\frac{1}{2}(25+36) = -\frac{a_0}{2}$

$\therefore \underline{a_0 = 61} \qquad 1$

$\boxed{2}$

(d) Let $z = x+iy,\ z \neq 0$

$\left|\frac{1}{z}+\frac{1}{\bar{z}}\right| \geq \frac{1}{2} \Rightarrow \left|\frac{z+\bar{z}}{z\bar{z}}\right| \geq \frac{1}{2}$

ie $\frac{|2x|}{x^2+y^2} \geq \frac{1}{2} \Rightarrow |4x| \geq x^2+y^2$

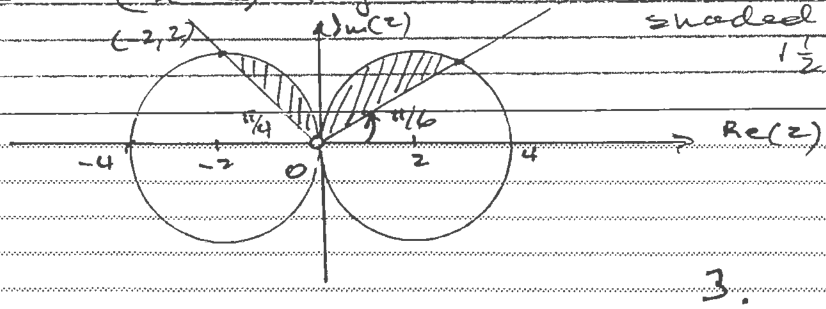
ie $x^2-|4x|+y^2 \leq 0$

$x^2-4|x|+4+y^2 \leq 4$

$x \geq 0 \quad (x-2)^2+y^2 \leq 4 \qquad 1$

$x < 0 \quad (x+2)^2+y^2 \leq 4 \qquad \frac{1}{2}$

shaded Region for $Im(z) \geq 0$, $\frac{3\pi}{4} \leq Arg\, z \leq \frac{\pi}{6}$ and $\left|\frac{1}{z}+\frac{1}{\bar{z}}\right| \geq \frac{1}{2}$ $\qquad 1\frac{1}{2}$



$\boxed{4}$

MATHEMATICS Extension 2: Question 2.

Suggested Solutions	Marks	Marker's Comments
Q 2 (a) $z^3 = 1 \quad z = 1, \omega, \bar{\omega}$		
(i) $z^3 = (\omega^2)^3 = \omega^6 = (\omega^3)^2 = 1^2 = 1$	1	$\boxed{1}$
$\therefore$ by testing / using Factor thm $z = \omega^2$ is also a root.		
(ii) $\underline{1 + \omega + \omega^2 = 0}$ (from factor $(1+z+z^2)$)	1	$\boxed{1}$
(iii) $(\omega - 1)(1 + 2\omega + 3\omega^2)$		
$= (\omega - 1)(1 + \omega + \omega^2 + \omega + 2\omega^2)$		
$= (\omega - 1)(\omega + 2\omega^2)$	1	
$= \omega^2 + 2\omega^3 - \omega - 2\omega^2$		
$= -\omega^2 - \omega + 2$		
$= -(\omega + \omega^2) + 2$		
$= -(-1) + 2$		$\boxed{2}$
$\underline{= 3}$	1	
Many paths to "3"		

Suggested Solutions	Marks	Marker's Comments

Q3(a)(i) $z = \cos\theta + i\sin\theta$

$z^n = (\text{cis}\,\theta)^n = \cos n\theta + i\sin n\theta$ — $\frac{1}{2}$ (De Moivres' thm)

$z^{-n} = (\text{cis}\,\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta)$ — (" ")

$= \cos n\theta - i\sin n\theta$ — $\frac{1}{2}$

$z^n + z^{-n} = 2\cos n\theta$ — $\frac{1}{2}$

$\therefore \frac{z^n + z^{-n}}{2} = \cos n\theta$ q.e.d. — $\boxed{2}$

(ii) $\cos\theta \cdot \cos 2\theta = \frac{1}{2}(z + z^{-1}) \times \frac{1}{2}(z^2 + z^{-2})$

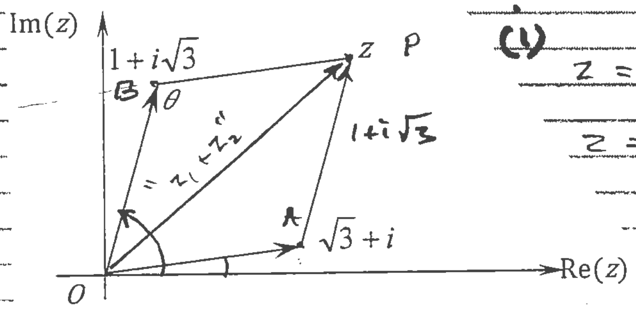
$= \frac{1}{4}[z^3 + z^{-1} + z^1 + z^{-3}]$

$= \frac{1}{4}[z + z^{-1} + z^3 + z^{-3}]$

$= \frac{1}{4}[2\cos\theta + 2\cos 3\theta]$

$= \frac{1}{2}[\cos\theta + \cos 3\theta]$ — $\boxed{2}$

(b)



(i) $z = \sqrt{3} + i + 1 + i\sqrt{3}$ — 1 (Vector addition)

$z = 1 + \sqrt{3} + i(1 + \sqrt{3})$ — $\boxed{1}$

$= (1 + \sqrt{3})(1 + i)$

(ii) $\text{Arg}(\sqrt{3} + i) = \frac{\pi}{6}$ — 1

$\text{Arg}(1 + i\sqrt{3}) = \frac{\pi}{3}$

$\therefore \angle AOB = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

$\therefore \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ (co-interior angles are supplementary as $BP \parallel OA$) — $\boxed{2}$

Many ways to arrive at $\frac{5\pi}{6}$ — 1

(c) NOTE ORDER $z^5 + z^2 - z + k = 0,\ k \in \mathbb{R}$

Degree 5 all coefficients real — 1

$\therefore$ could have 5 real roots 3 + 1 conjugate pair or 1 real + 2 pair of conjugate roots

$\therefore$ Diagram B — 1 — $\boxed{2}$

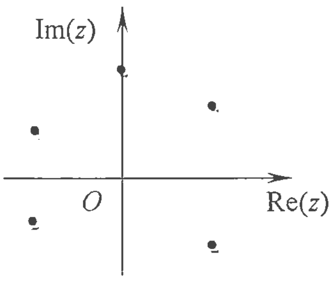


Diagram A

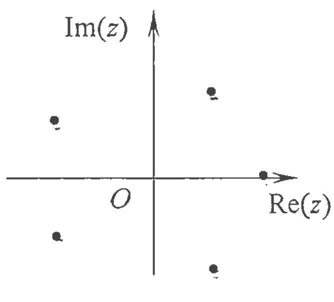


Diagram B

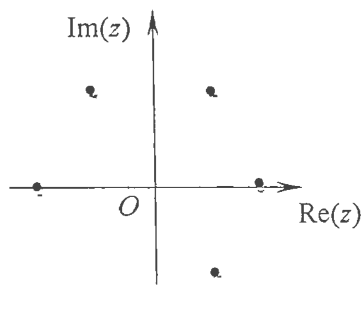


Diagram C

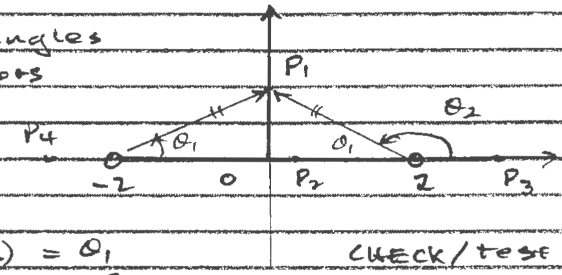
5.

Suggested Solutions	Marks	Marker's Comments

Q3(c) $\text{Arg}(z+2) + \text{Arg}(z-2) = \pi$

$*$ Locus for z: $\underline{-2 < x < 2 \text{ or } y \geq 0}$ $*$ — $1+1$

• Using Angles and Vectors



$\text{Arg}(z+2) = \theta_1$ — CHECK/test $P_1(0, ki)$

$\text{Arg}(z-2) = \theta_2$ — ✓ ✓ $P_{2,3,4}(x, 0)$

$k<0$ does not give π

$\boxed{3}$

• $\tan(\theta_1 + \theta_2) = 0$

$$\frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1\tan\theta_2} = 0$$

$\Rightarrow \tan\theta_1 + \tan\theta_2 = 0$ BUT $\tan\theta_1\tan\theta_2 \neq 1$

i.e. $\dfrac{y}{x+2} + \dfrac{y}{x-2} = 0$ | $\dfrac{y^2}{x^2-4} \neq 1$

$\therefore \dfrac{2xy}{x^2-4} = 0$ | $\Rightarrow x^2 - y^2 \neq 4$

$\therefore x = 0$ or $y = 0$ but $x \neq \pm 2$

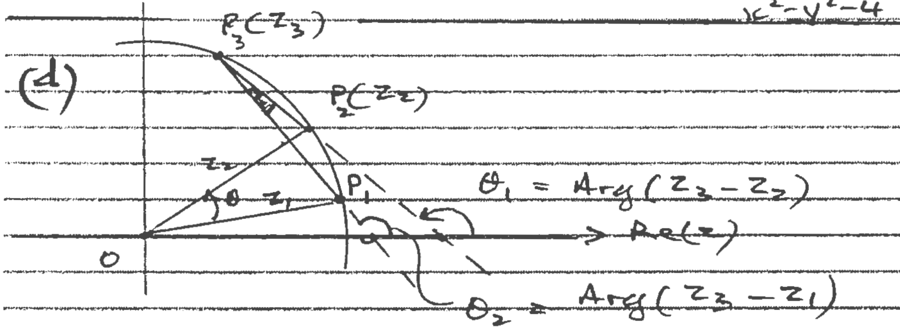
Need to test as incomplete information from this algebra

• $\text{Arg}[(z+2)(z-2)] = \pi$ if ....

$\tan \text{Arg}(z^2 - 4) = 0 \Rightarrow \dfrac{2xy}{x^2-y^2-4} = 0$

see above $\text{Im}(\text{part}) = 0$ Real part < 0.

(d)



Let $\theta_1 = \text{Arg}(z_3 - z_2)$

$\theta_2 = \text{Arg}(z_3 - z_1)$

$\theta = \text{Arg}(z_2) - \text{Arg}(z_1) = \text{Arg}\left(\dfrac{z_2}{z_1}\right)$ — 1

$\theta_1 = \theta_2 + \angle P_2P_3P_1$ (Exterior angle of triangle equals sum of interior opposite angles) — 1

$\therefore \angle P_2P_3P_1 = \theta_1 - \theta_2 = \text{Arg}(z_3 - z_2) - \text{Arg}(z_3 - z_1)$

$= \text{Arg}\left(\dfrac{z_3 - z_2}{z_3 - z_1}\right)$

But $\angle P_1OP_2 = \theta = 2\angle P_2P_3P_1$ (Angle subtended at centre is twice the angle subtended on circumference). — 1

$\therefore \angle P_2P_3P_1 = \text{Arg}\left(\dfrac{z_3 - z_2}{z_3 - z_1}\right) = \frac{1}{2}\angle P_1OP_2$

$= \frac{1}{2}\text{Arg}\left(\dfrac{z_2}{z_1}\right)$ qed

$\boxed{3}$

MATHEMATICS Extension 2: Question 4

Suggested Solutions	Marks	Marker's Comments

Q4(a) Let $z = x+iy$

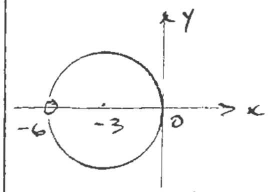
If $\frac{z}{z+6}$ is purely imaginary

$\therefore \operatorname{Re}\left(\frac{z}{z+6}\right) = 0$ or $\operatorname{Arg}\left(\frac{z}{z+6}\right) = \pm\frac{\pi}{2}$ as $\operatorname{Im}\left(\frac{z}{z+6}\right) = ki$

$$\frac{z}{z+6} = \frac{x+iy}{x+6+iy} \times \frac{(x+6)-iy}{(x+6)-iy}$$

$$= \frac{x^2+6x+y^2+i(xy+6y-xy)}{(x+6)^2+y^2} = \frac{x^2+6x+y^2+6yi}{(x+6)^2+y^2}.$$

$$\therefore \operatorname{Re}\left(\frac{z}{z+6}\right) = \frac{x^2+6x+y^2}{(x+6)^2+y^2} = 0 \quad 1$$



$\boxed{2}$

$\Rightarrow x^2+6x+y^2 = 0$ with $z \neq -6$

ie $(x+3)^2+y^2 = 9$ 1

the locus is a circle $C(-3,0)$ radius 3 excluding $(-6,0)$

(b)(i) $\cos\theta + i\sin\theta = \cos\theta(1+i\tan\theta)$

$\therefore (\cos\theta+i\sin\theta)^4 = \cos^4\theta(1+i\tan\theta)^4$

$\cos4\theta + i\sin4\theta = \cos^4\theta(1+4it+6i^2t^2+4i^3t^3+i^4t^4)$

$= \cos^4\theta(1-6\tan^2\theta+\tan^4\theta+i(4\tan\theta-4\tan^3\theta))$

equating real part

$\therefore \cos4\theta = \cos^4\theta(1-6\tan^2\theta+\tan^4\theta)$. 1

$\boxed{1}$

(ii) $$\tan4\theta = \frac{\sin4\theta}{\cos4\theta} = \frac{\cos^4\theta(4\tan\theta-4\tan^3\theta)}{\cos^4\theta(1-6\tan^2\theta+\tan^4\theta)}$$

$$= \frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta} \quad 1$$

$\boxed{1}$

(iii) $x^4+4\sqrt3x^3-6x^2-4\sqrt3x+1 = 0$

$\sqrt3(4x-4x^3) = x^4-6x^2+1$

So $$\frac{4x-4x^3}{x^4-6x^2+1} = \frac{1}{\sqrt3}$$

By letting $x = \tan\theta$ 1

So $$\frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta} = \tan4\theta = \frac{1}{\sqrt3}$$

$\boxed{1}$

$\therefore$ solutions can be found for the quartic from $\tan4\theta = \frac{1}{\sqrt3}$

(iv) $\tan4\theta = \frac{1}{\sqrt3}$ $\therefore 4\theta = n\pi+\frac{\pi}{6}$ $n\in\mathbb{Z}$

$\theta = (6n+1)\frac{\pi}{24}$

$\therefore \theta_1 = \frac{\pi}{24}$ $\theta_2 = \frac{7\pi}{24}$ $\theta_3 = \frac{13\pi}{24}$ $\theta_4 = \frac{19\pi}{24}$ 1

$\boxed{2}$

$\therefore x_1 = \tan\frac{\pi}{24}$; $x_2 = \tan\frac{7\pi}{24}$; $x_3 = \tan\frac{13\pi}{24} = -\tan\frac{11\pi}{24}$ 1

$x_4 = \tan\frac{19\pi}{24} = -\tan\frac{5\pi}{24}$.

(v) $S_4 = x_1x_2x_3x_4 = \frac{e}{a}$ 1

1 For convert to $-\tan(\ldots)$

$$\tan\frac{\pi}{24}\times\tan\frac{7\pi}{24}\times\left(-\tan\frac{11\pi}{24}\right)\times\left(-\tan\frac{5\pi}{24}\right) = 1$$

$\boxed{2}$

$$\therefore \tan\frac{\pi}{24}\cdot\tan\frac{7\pi}{24} = +\cot\frac{5\pi}{24}\cdot\cot\frac{11\pi}{24}$$ red

MATHEMATICS Extension 2: Question 4.

Suggested Solutions	Marks	Marker's Comments
Q4 (c) (i) $z^{2n+1} = 1$; $z = r\,\text{cis}\,\theta$		
$r^{2n+1}(\text{cis}\,\theta)^{2n+1} = 1 = 1.\text{cis}(2k\pi)$ $k \in \mathbb{Z}$		solutions seperated by $\theta = \frac{2\pi k}{2n+1}$
$r^{2n+1}\,\text{cis}((2n+1)\theta) = 1\,\text{cis}(2k\pi)$		
$\therefore r = 1$ $(2n+1)\theta = 2k\pi$		
$\theta = \frac{2k\pi}{2n+1}$ ✓		2
$\therefore$ roots are going to be		
$z_k = 1\,\text{cis}\left(\frac{2k\pi}{2n+1}\right)$ $k = 0, \pm1, \pm2, \ldots, n$		for $n = 0, 1, 2, \ldots$
ie $\text{cis}\,0$, $\text{cis}\frac{2\pi}{2n+1}$, $\text{cis}\frac{4\pi}{2n+1}$, $\text{cis}\frac{6\pi}{2n+1}$, ..., $\text{cis}\frac{2n\pi}{2n+1}$		
(ii) $z^{2n+1} - 1 = (z-1)[z^{2n} + z^{2n-1} + \ldots + z + 1]$		
$\therefore z = 1 = \text{cis}\,0$ is a root (for $k=0$).		
All other roots come in conjugate pairs α and $\bar{\alpha}$ since all coeffts real.		
So $(z-\alpha_k)(z-\bar{\alpha}_k) = z^2 - 2Re(\alpha_k)z + \lvert\alpha_k\rvert^2$, where $\alpha_k = \text{cis}\frac{2k\pi}{2n+1}$		
$= z^2 - 2\cos\theta_k z + 1$.		
So $z^{2n} + z^{2n-1} + \ldots + z + 1$		2
$= \left[z^2 - 2\cos\frac{2\pi}{2n+1}z + 1\right]\left[z^2 - 2\cos\frac{4\pi}{2n+1}z+1\right]\ldots\left[z^2 - 2\cos\frac{2n\pi}{2n+1}z + 1\right]$		
factorised over the reals.		
$\sum_{m=0}^{2n} z^m = \prod_{k=1}^{n}\left[z^2 - 2\cos\frac{2k\pi}{2n+1}z + 1\right]$		
(iii) Let $z = 1$ in (ii)		
$1+1+\ldots+1+1 = \left[2 - 2\cos\frac{2\pi}{2n+1}\right]\left[2 - 2\cos\frac{4\pi}{2n+1}\right]\ldots\left[2 - 2\cos\frac{2n\pi}{2n+1}\right]$		
$(2n+1)\times 1 = 2^n\left(1-\cos\frac{2\pi}{2n+1}\right)\left(1-\cos\frac{4\pi}{2n+1}\right)\ldots\left(1-\cos\frac{2n\pi}{2n+1}\right)$ n-pairs		
$2n+1 = 2^n \times 2\sin^2\frac{\pi}{2n+1} \times 2\sin^2\frac{2\pi}{2n+1} \times \ldots \times 2\sin^2\frac{n\pi}{2n+1}$.		
$= 2^n \times 2^n \times \sin^2\frac{\pi}{2n+1} \times \sin^2\frac{2\pi}{2n+1} \times \ldots \times \sin^2\frac{n\pi}{2n+1}$.		2
$2n+1 = \left(2^n \sin\frac{\pi}{2n+1} \times \sin\frac{2\pi}{2n+1} \times \ldots \times \sin\frac{n\pi}{2n+1}\right)^2$		
So $2^n \sin\frac{\pi}{2n+1} \times \sin\frac{2\pi}{2n+1} \times \ldots \times \sin\frac{n\pi}{2n+1} = \sqrt{2n+1}$		