VIRILE AGITUR

Year 12 Mathematics Extension 2 (4U)
HSC ASSESSMENT TASK 1
TERM 4, Week 6, 2006

Name: ____________________

Teacher: ____________________

Set By: DS

Tuesday 21 November 2006

- Attempt **ALL** questions.
- Marks may be deducted for careless, insufficient, or illegible work.
- Only Board approved calculators (**excluding** graphic calculators) may be used.
- Total possible mark is **40**.
- Begin each question on a new sheet of paper.
- **TIME ALLOWED: 60 minutes**

Complex Numbers and Mathematical Induction

Question 1 **(12 marks)** ***(Start a new page)***

(a) If $z = 3 - 2i$, and $w = -4 + i$, evaluate in the form of $x + iy$ the following:

(i) $2z - iw$ (ii) w^2 (iii) $\frac{w}{z}$. **1,1,1**

(b) Simplify $\frac{1}{i} - \frac{3i}{1-i}$, expressing your answer in the form $a+ib$. **3**

(c) (i) Write down the moduli and arguments of $-\sqrt{3} + i$ and $4 + 4i$. **2**

(ii) Hence express in modulus/argument form $\dfrac{-\sqrt{3}+i}{4+4i}$ where $-\pi \le \arg z \le \pi$. **2**

(iii) Evaluate $\left(-\sqrt{3}+i\right)^8$, expressing your answer in the form $a+ib$. **2**

Question 2 **(9 marks)** ***(Start a new page)***

(a) Solve the following for x and y: **3**

$$\left(\frac{1+i}{1-i}\right)^2 + \frac{1}{x+iy} = 1 + i.$$

(b) (i) Find the complex square roots of $-8+6i$. **3**

(ii) Hence or otherwise solve $2z^2 + (1-i)z + (1-i) = 0$. **3**

Question 3 **(10 marks)** ***(Start a new page)***

(a) Sketch on the Argand diagram the region which satisfies these expressions, **4**

$$2 \leq |z| \leq 3 \quad \text{and} \quad \frac{\pi}{4} < \arg(z-i) < \frac{3\pi}{4}.$$

(b) Sketch on the Argand diagram and algebraically describe the locus of the point P representing z, given that $|z|^2 = z + \bar{z} + 1$. **3**

(c) $z = x + iy$ is such that $\dfrac{z-i}{z+1}$ is purely imaginary. Find the equation of the locus of the point P representing z. **3**

Question 4 **(9 marks)** ***(Start a new page)***

(a) Prove by the method of Mathematical Induction that $5^n \geq 1 + 4n$ for $n \geq 1$. **5**

(b) Prove by the method of Mathematical Induction that $\arg z^n = n \arg z$ for $n \geq 1$. **4**

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1(a) $z = 3 - 2i \quad w = -4 + i$

(i) $2z - iw = 2(3-2i) - i(-4+i)$

$= 6 - 4i + 4i - i^2$

$= 6 + 1 = 7$ ①

(ii) $w^2 = (-4+i)^2$

$= 16 - 8i + i^2$ ①

$= 15 - 8i$

(iii) $\frac{w}{z} = \frac{-4+i}{3-2i} = \frac{(-4+i)(3+2i)}{13}$

$= \frac{-14}{13} - \frac{5i}{13}$ ①

(b) $\frac{1}{i} - \frac{3i}{1-i} = \frac{i}{-1} - \frac{3i(1+i)}{2}$

$= \frac{-2i - 3i - 3i^2}{2}$

$= \frac{-5i + 3}{2}$

$= \frac{3}{2} - \frac{5i}{2}$ ③

(c)(i) $-\sqrt{3} + i = 2 \operatorname{cis} \frac{5\pi}{6}$ ①

$4 + 4i = 4\sqrt{2} \operatorname{cis} \frac{\pi}{4}$ ①

(ii) $\frac{-\sqrt{3}+i}{4+4i} = \frac{2 \operatorname{cis} 5\pi/6}{4\sqrt{2} \operatorname{cis} \pi/4}$

$= \frac{1}{2\sqrt{2}} \operatorname{cis}\left(\frac{5\pi}{6} - \frac{\pi}{4}\right)$

$= \frac{\sqrt{2}}{4} \operatorname{cis} \frac{7\pi}{12}$ ②

1c (iii) $(-\sqrt{3}+i)^8 = 2^8\left(\operatorname{cis} \frac{5\pi}{6}\right)^8$

$= 256 \operatorname{cis} \frac{40\pi}{6}$

$= 256 \operatorname{cis} \frac{2\pi}{3}$

$= 256\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$

$= 256\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= -128 + 128\sqrt{3}\, i$ ②

2(a) $\left(\frac{1+i}{1-i}\right)^2 + \frac{1}{x+iy} = 1 + i$

$\frac{1 + 2i - 1}{1 - 2i - 1} + \frac{1}{x+iy} = 1+i$

$\frac{2i}{-2i} + \frac{1}{x+iy} = 1+i$

$-1 + \frac{1}{x+iy} = 1+i$

$\frac{1}{x+iy} = 2 + i$

$\frac{1}{x+iy} = \frac{2+i}{1}$

$x + iy = \frac{1}{2+i} = \frac{2-i}{5}$

Hence $x = \frac{2}{5}$, $y = \frac{-1}{5}$ ③

b(i) Let $-8+6i = (a+ib)^2$

Hence $a^2-b^2=-8$, $2ab=6$

$ab=3$.

$b=\frac{3}{a}$.

Hence $a^2-\frac{9}{a^2}=-8$

$a^4+8a^2-9=0$

$(a^2+9)(a^2-1)=0$

Hence $a=\pm1$ whence $b=\pm3$

(3)

Hence $\sqrt{-8+6i}=1+3i,\ -1-3i$

(ii) $2z^2+(1-i)z+(1-i)=0$

$$z=\frac{-(1-i)\pm\sqrt{(1-i)^2-4(2)(1-i)}}{4}$$

$$=\frac{-(1-i)\pm\sqrt{(-8+6i)}}{4}$$

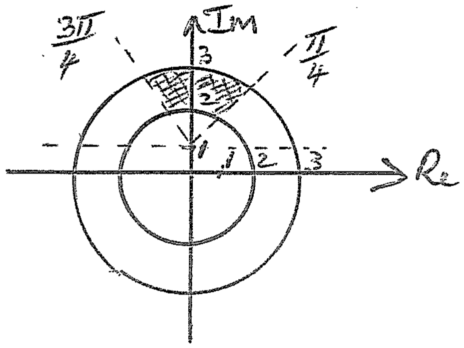
$$z_1=\frac{-1+i+(1+3i)}{4}=i$$

AND

$$z_2=\frac{-1+i+(-1-3i)}{4}=-\frac{1}{2}-\frac{i}{2}$$

(3)

3(a)



(4)

$2\le|z|\le3$ and

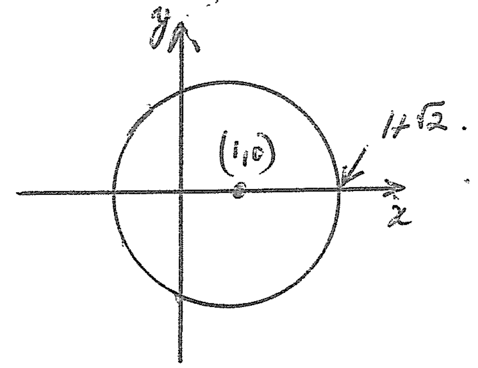
$\frac{\pi}{4}<\arg(z-i)<\frac{3\pi}{4}$.

3(b) $|z|^2=z+\bar{z}+1$ Let $z=x+iy$

$x^2+y^2=2x+1$.

$x^2-2x+y^2=1$

$(x-1)^2+y^2=2$ which is a circle centre $(1,0)$ radius $\sqrt{2}$.



(3)

3(c) $$\frac{z-i}{z+1}=\frac{x+i(y-1)}{(x+1+iy)}\times\frac{x+1-iy}{(x+1-iy)}$$

$$=\frac{x(x+1)+y(y-1)+i(y-1)(x+1)-ixy}{(x+1)^2+y^2}.$$

Now $Re\left(\frac{z-i}{z+1}\right)=0$ since $\frac{z-i}{z+1}$ is imaginary

Hence $x(x+1)+y(y-1)=0$

$x^2+x+y^2-y=0$

$\left(x+\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2}$

Which represents a circle centre $\left(-\frac{1}{2},\frac{1}{2}\right)$ with radius $=\frac{1}{\sqrt{2}}$.

(3)

4(a) Prove by induction $5^n \geq 1+4n$ $n \geq 1$, $n \in J^+$

Let S_n be the statement $5^n \geq 1+4n$

step 1 for $n=1$

$LHS = 5^1$ $RHS = 1+4 = 5$

S_n is true for $n=1$

step 2. Assume S_n is true for $n=k$.

ie $S_k = 5^k \geq 1+4k$, $k \in J^+$

step 3. Prove S_n is true for $n=k+1$ $k \in J^+$

RTP $5^{k+1} \geq 1+4(k+1)$

ie $5^{k+1} \geq 5+4k$.

Consider $5^k \geq 1+4k$ (by assumption)

$5 \times 5^k \geq 5(1+4k)$ $\times$ by 5.

$5^{k+1} \geq 5+20k \geq 5+4K$.

Step 4. If $5^n \geq 1+4n$ for $n=k$

then $5^n \geq 1+4n$ for $n=k+1$

since $5^n \geq 1+4n$ for $n=1$

then $5^n \geq 1+4n$ for $n=2,3,4\ldots$

thus $5^n \geq 1+4n$ for all $n \geq 1$.

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4(b) Prove by induction $\arg z^n = n \arg z$

$n \geq 1$.

Let S_n be the statement $\arg z^n = n \arg z$

step 1 for $n=1$

$\arg z^1 = 1 \arg z$.

S_n is true for $n=1$

Step 2 Assume S_n is true for $n=k$.

$\arg z^k = k \arg z$.

step 3. Prove S_n is true for $n=k+1$

$\arg z^k = k \arg z$ by assumption

$\arg z^{k+1} = \arg z^k \cdot z$.

$= \arg z^k + \arg z$.

$= k \arg z + \arg z$.

$= (k+1) \arg z$

Step 4 If $\arg z^n = n \arg z$ for $n=k$.

then " " " for $n=k+1$

since " " " for $n=1$

then " " " for $n=2,3,4$

thus $\arg z^n = n \arg z$ for all

$n \geq 1$

$n \in J^+$

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