Year 12 Mathematics Extension 2 **HSC ASSESSMENT TASK 1** **Term 4 Week 6 2010**	Name:____________________________ Teacher:__________________________

Friday 19th November

Set by: VUL

- Attempt **all** questions.
- All questions are of equal value.
- Marks may be deducted for insufficient, or illegible work.
- Only Board approved calculators (**excluding** graphic calculators) may be used
- Total possible mark is **40**
- **Begin each question on a new page.**
- **TIME ALLOWED :** 60 minutes

Question 1 **Marks**

(a) Let $z = 3 - 4i$.

(i) Find z^3 in the form $x + iy$ **1**

(ii) Find $z - 2\overline{z}$ in the form $x + iy$ **1**

(iii) Find $\frac{-i}{z}$ in the form $x + iy$ **2**

(b) Solve $z^2 = 5 + 12i$ for z giving your answers in the form $x + iy$ where x and y are real. **3**

(c) (i) Express $-1 + i$ in modulus-argument form. **1**

(ii) Hence evaluate $(-1+i)^{12}$ **2**

(d) On separate Argand diagrams, sketch the locus of points z such that:

(i) $\arg(z - i) = -\frac{\pi}{4}$ **2**

(ii) the inequalities $|z - 3 + i| \le 5$ and $|z+1| \le |z-1|$ both hold **3**

(iii) $|z| = \operatorname{Im}(z)$ **2**

(e) Prove that $12^n > 5^n + 7^n$ positive integers $n \ge 2$ **3**

Question 2 **Start a new page.**

(a) The points O, A, Z and C on the Argand diagram represent the complex numbers 0, 1, z and $z+1$ respectively, where $z = \cos\theta + i\sin\theta$ is any complex number of modulus 1, with $0 < \theta < \pi$.

(i) Explain why $OACZ$ is a rhombus **1**

(ii) Show that $\dfrac{z-1}{z+1}$ is purely imaginary **2**

(iii) Find the modulus and argument of $z+1$ **2**

(b) (i) By considering $z^9 - 1$ as the difference of two cubes write $1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8$ as a product of two polynomial factors with real coefficients, one of which is quadratic. **2**

(ii) Solve $z^9 - 1 = 0$ and determine the six solutions of $z^6 + z^3 + 1 = 0$ **2**

(iii) Hence show that $\cos\dfrac{2\pi}{9} + \cos\dfrac{4\pi}{9} = \cos\dfrac{\pi}{9}$ **2**

(c) Sketch the locus of the complex number $z = x + iy$ where $\arg\left[\dfrac{z-3}{z-1}\right] = \dfrac{\pi}{2}$. **3**

Describe this locus geometrically .

(d) If $z = \cos\theta + i\sin\theta$:

(i) Show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ **2**

(ii) Let $\omega = z + \dfrac{1}{z}$. Prove that $\omega^3 + \omega^2 - 2\omega - 2 = z + \dfrac{1}{z} + z^2 + \dfrac{1}{z^2} + z^3 + \dfrac{1}{z^3}$ **2**

(iii) Hence solve $\cos\theta + \cos 2\theta + \cos 3\theta = 0$ for $0 \le \theta \le 2\pi$ **2**

End of Assessment Task

2010 YEAR 12 X2 TASK 1 SOLUTIONS

Question 1:

a) $z = 3 - 4i$

i) $z^3 = 3^3 + 3(3)^2(-4i) + 3(3)(-4i)^2 + (-4i)^3$

$= 27 - 108i - 144 + 64i$

$= -117 - 44i$ ✓

ii) $z - 2\bar{z} = (3-4i) - 2(3+4i)$

$= -3 - 12i$ ✓

iii) $\dfrac{-i}{z} = \dfrac{-i}{3-4i} \times \dfrac{3+4i}{3+4i}$ ✓

$= \dfrac{4-3i}{25}$

$= \dfrac{4}{25} - \dfrac{3}{25}i$ ✓

b) let $z = x + iy$

$\therefore (x+iy)^2 = 5 + 12i$ ✓

So $x^2 - y^2 + 2xyi = 5 + 12i$

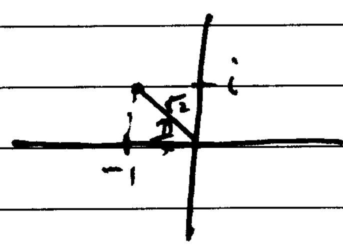
Equating real and imaginary parts.

$x^2 - y^2 = 5 \qquad xy = 6$ ✓

$\therefore x = \pm 3 \quad y = \pm 2$

So $z = 3 + 2i$ or $-3 - 2i$ ✓

c) i)

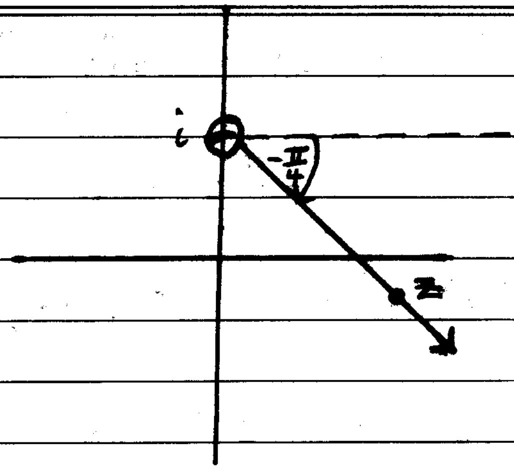


$-1 + i = \sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)$ ✓

ii) $(-1+i)^{12} = \left(\sqrt{2}\operatorname{cis}\frac{3\pi}{4}\right)^{12} = 2^6 \operatorname{cis} 9\pi = 64 \operatorname{cis} \pi$ ✓

$= -64$ ✓

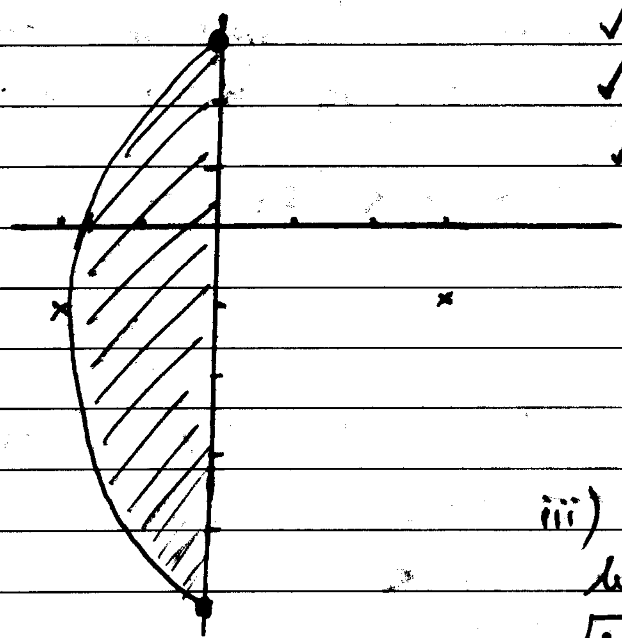
d) i)



✓ origin of ray at i not filled in

✓ direction

ii)



✓ circle centre $(3, -i)$

✓ $x \leq 0$

✓ intersection

iii) $|z| = \mathrm{Im}(z)$

let $z = x + iy$

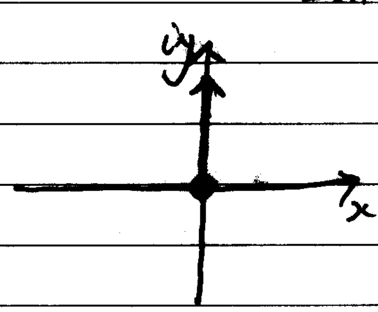
$\sqrt{x^2 + y^2} = y$

$x^2 + y^2 = y^2$

$x = 0$ ✓

but $|z| \geq 0$ ∴

$\mathrm{Im}(z) = y \geq 0$ ✓



e) To Prove $12^n > 5^n + 7^n$ $n \geq 2$

ie $12^n - (5^n + 7^n) > 0$

So when $n = 2$ $12^n - (5^n + 7^n) = 12^2 - (5^2 + 7^2)$

$= 70 > 0$ ✓

$\therefore 12^n > 5^n + 7^n$ when $n = 2$

Assume the statement is true for $n = k$ some fixed positive integer.

ie $12^k > 5^k + 7^k$

When $n = k+1$

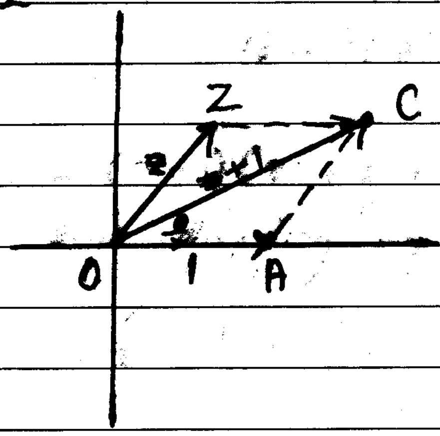
$$12^n - (5^n + 7^n) = 12^{k+1} - (5^{k+1} + 7^{k+1})$$
$$= 12 . 12^k - 5.5^k - 7.7^k$$
$$> 12(5^k + 7^k) - 5.5^k - 7.7^k$$
$$= 12.5^k - 5.5^k + 12.7^k - 7.7^k$$
$$= 7 \times 5^k + 5.7^k > 0$$

as $5^k, 7^k > 0$ for all k

If the statement is true for $n = k$
Therefore the statement is true for $n = k+1$
Since the statement is true for $n = 1$
Then the statement is true for $n = 2, 3, 4, \ldots$
$\therefore 12^n > 5^n + 7^n$ for $n \geq 2$.

Question 2

a) i)



Since $|z| = 1$, $|OZ| = |OA| = |ZC| = |AC|$

$\therefore$ OACZ is a Rhombus (All sides equal)

ii) Since OACZ is a rhombus

$OC \perp AZ$ (diagonals bisect at 90°)

$\therefore \quad \vec{AZ} = ki\,\vec{OC}$ (anticlockwise rotation of 90°)

ie $\quad z - 1 = ki(z+1)$

$\therefore \quad \frac{z-1}{z+1} = ki$ which is purely imaginary.

iii) $|z+1| = 1^2 + 1^2 - 2(1)(1)\cos(180-\theta)$

$= 2 + 2\cos\theta$

$\arg(z+1) = \frac{\theta}{2}$

b) i) $z^9 - 1 = (z^3-1)(z^6+z^3+1)$
$= (z-1)(z^2+z+1)(z^6+z^3+1)$

and $z^9 - 1 = (z-1)(z^8+z^7+z^6+z^5+z^4+z^3+z^2+z+1)$

Equating and dividing by $z-1 \neq 0$

$1+z+z^2+\dots+z^8 = (z^2+z+1)(z^6+z^3+1)$

ii) For $z^9 - 1 = 0$

$z_k = \text{cis}\,\frac{2k\pi}{9} \qquad k = 0, 1, 2, \dots 8$

$z^2+z+1 = 0$

$z = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

Solutions of $z^3 - 1 = 0$ are $1, \text{cis}\,\frac{2\pi}{3}, \text{cis}\,\frac{-2\pi}{3}$

So remaining solutions ofe solutions of

$z^6+z^3+1=0$ are $\text{cis}\,\frac{\pm 2\pi}{9}, \text{cis}\,\frac{\pm 4\pi}{9}, \text{cis}\,\frac{\pm 8\pi}{9}$

iii) Sum of roots $= 0$

$\therefore \text{cis}\,\frac{2\pi}{9} + \text{cis}\,\frac{4\pi}{9} + \text{cis}\,\frac{8\pi}{9} + \text{cis}\,\frac{-8\pi}{9} + \text{cis}\,\frac{-4\pi}{9} + \text{cis}\,\frac{-2\pi}{9} = 0$

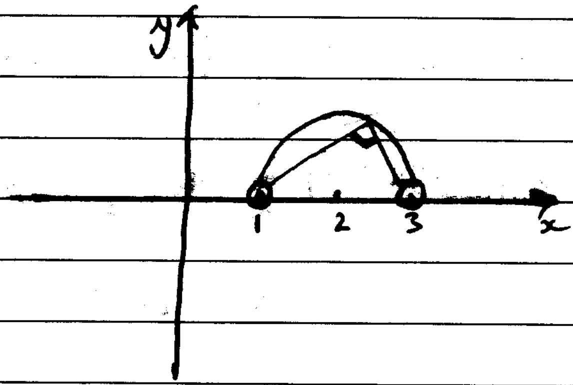
ie $2\left(\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9}\right) = 0$ (equating real parts

so $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = -\cos\frac{8\pi}{9}$

so $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$

as $\cos(\pi - \theta) = -\cos\theta$

c) Semi Circle Centre $(2,0)$ radius 1 unit. ✓ describe



✓ sketch

✓ open circles at $x=1$ or $x=3$

d) i) if
$$z = \cos\theta + i\sin\theta$$
$$z^n = \cos n\theta + i\sin n\theta$$
$$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos(n\theta) - i\sin(n\theta)$$ ✓

but $\cos(-\alpha) = \cos(\alpha)$
$\sin(-\alpha) = -\sin\alpha$

$$\therefore\ z^n + z^{-n} = 2\cos n\theta$$ ✓

ii)
$$L.H.S = w^3 + w^2 - 2(w+1)$$
$$= w^2(w+1) - 2(w+1)$$
$$= (w^2 - 2)(w+1)$$ ✓
$$= \left[\left(z + \frac{1}{z}\right)^2 - 2\right]\left[z + \frac{1}{z} + 1\right]$$
$$= \left(z^2 + \frac{1}{z^2} + 2 - 2\right)\left(z + \frac{1}{z} + 1\right)$$
$$= \left(z^2 + \frac{1}{z^2}\right)\left(z + \frac{1}{z} + 1\right)$$
$$= z^3 + z + z^2 + \frac{1}{z} + \frac{1}{z^3} + \frac{1}{z^2}$$
$$= z + \frac{1}{z} + z^2 + \frac{1}{z^2} + z^3 + \frac{1}{z^3}$$ ✓

iii) $\cos\theta + \cos 2\theta + \cos 3\theta = 0 \qquad 0 \le \theta \le 2\pi$

if $\cos n\theta = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$

then $\frac{1}{2}\left(z + \frac{1}{z}\right) + \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right) + \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right) = 0$

ie, $\frac{1}{2}\left(w^3 + w^2 - 2w - 2\right) = 0$ from part ii)

so. $w^2(w+1) - 2(w+1) = 0$

$(w^2 - 2)(w+1) = 0$

$\therefore w = -1, \pm\sqrt{2}$

but $w = z + \frac{1}{z} = 2\cos\theta$

$\therefore 2\cos\theta = -1 \qquad 2\cos\theta = \pm\sqrt{2}$

$\cos\theta = -\frac{1}{2} \qquad \cos\theta = \pm\frac{1}{\sqrt{2}}$

$\therefore \theta = \frac{\pi}{4}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{4\pi}{3}, \frac{7\pi}{4}$