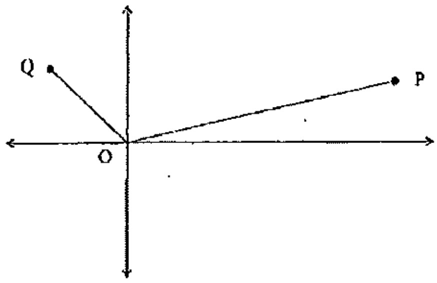
Question 2

Marks

(a) The diagram shows a complex plane with origin O.



Points P and Q represent non-zero complex numbers z and w respectively.

(i) Write down the length of PQ in terms of z and w. **1**

(ii) Copy the diagram into your booklet. Construct point R that represents $z+w$. **2**
What type of quadrilateral is **OPRQ?**

(iii) Prove that if $|z+w|=|z-w|$, the complex number $\frac{w}{z}$ is imaginary. **2**

(b) Given ω is a complex root of $z^3-1=0$;

(i) Explain carefully why ω^2 is also a complex root of $z^3-1=0$. **2**

(ii) Prove that $1+\omega+\omega^2=0$ **1**

(iii) Show the roots of $z^3-1=0$ on an Argand diagram. **1**

(iv) Simplify $\left(1+\omega^2\right)^{2012}$ completely. **1**

(c) Suppose that $H(n)=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, for $n=1, 2, 3, \ldots$. **5**

So $H(1)=1$, $H(2)=1+\frac{1}{2}$, $H(3)=1+\frac{1}{2}+\frac{1}{3}$, and so on.

Prove by mathematical induction that

$$n+H(1)+H(2)+H(3)+\cdots+H(n-1)=nH(n)$$

for $n=2, 3, 4, \ldots$.

End of Assessment

Year 12 Mathematics Extension 2
HSC ASSESSMENT TASK 1
Term 4 Week 8 2011

Name:________________________

Teacher:______________________

Wednesday 30th November 2011

Set by: VUL

- Attempt **both** questions.
- Both questions are of equal value.
- Marks may be deducted for insufficient, or illegible work.
- Only Board approved calculators (**excluding** graphic calculators) may be used
- Total possible mark is **30**
- **Begin each question on a new page.**
- **TIME ALLOWED :** 45 minutes + 2 min reading time

Question 1

Marks

(a) Express $\dfrac{23-14i}{3-4i}$ in the form $a+bi$, where a and b are real. **2**

(b) Find the two square roots of $-16+30i$. **2**

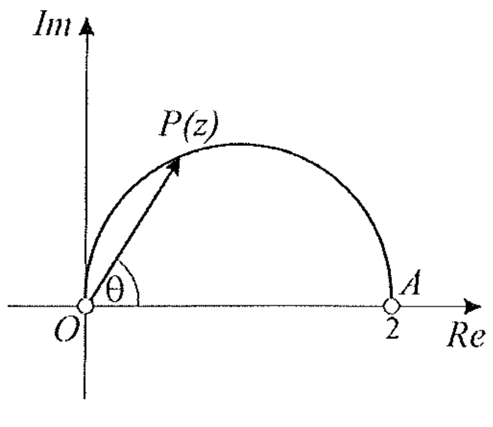
(c) Let $w=-\sqrt{3}+i$.

(i) Express w in modulus–argument form. **2**

(ii) Show that $w^9+512i=0$. **2**

(d) Shade the region in the complex plane where $|z+2|\leq 2$ and $-\frac{\pi}{6}\leq \arg(z+3)\leq \frac{\pi}{3}$ are simultaneously satisfied. **3**

(e)



The diagram above shows the semicircular locus of the point P that represents the complex number z.
Let $\arg z=\theta$, as shown on the diagram.

(i) Copy the diagram and on it show a vector representing $z-2$. **1**

(ii) Explain why $\left|\dfrac{z-2}{z}\right|=\tan\theta$. **1**

(iii) Show that $\arg\left(\dfrac{z-2}{z}\right)=\dfrac{\pi}{2}$. **2**

Continued overleaf....

2012 Year 12 Mathematics Extension 2 HSC Task 1 SOLUTIONS

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

1)(a) ③
$$\frac{23-14i}{3-4i}\cdot\frac{3+4i}{3+4i}=\frac{69+56-42i+92i}{9-16i^2} ✓$$
$$=\frac{125+50i}{25}$$
$$=5+2i ✓$$

(b) Let $-16+30i=(a+ib)^2$.

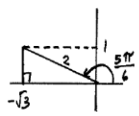
$\therefore\ a^2-b^2=-16$ and $ab=15$ ✓

By inspection, $(a,b)=(3,5)$ or $(-3,-5)$. ✓

So the two square roots are $3+5i$ and $-3-5i$.

(c)(i) $w=-\sqrt{3}+i$

$=2\operatorname{cis}\frac{5\pi}{6}$ ✓ ✓



(ii) $w^9=\left(2\operatorname{cis}\frac{5\pi}{6}\right)^9$

$=2^9\operatorname{cis}\frac{15\pi}{2}$

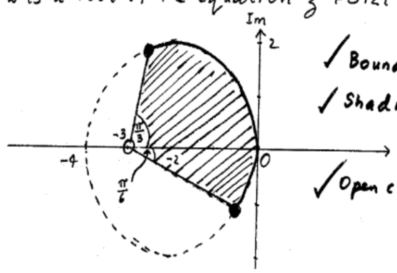
$=512\operatorname{cis}\frac{3\pi}{2}$ ✓

$=512(0-i)$

$=-512i$, so $w^9+512i=0$.

So w is a root of the equation $z^9+512i=0$. ✓

(d)

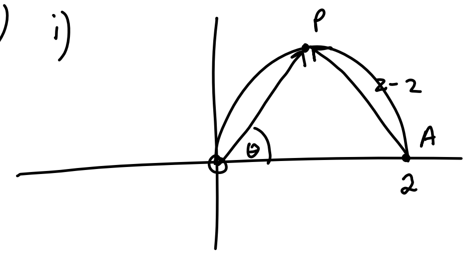


✓ Boundaries

✓ Shading

✓ Open circle

(e) i)



✓

ii) Since $\angle OPA=90^\circ$ ($\angle$ in a semi circle) ✓

$$\tan\theta=\frac{|AP|}{|OP|}$$
$$=\frac{|z-2|}{|z|}$$
$$=\left|\frac{z-2}{z}\right|$$

iii) $\arg\left(\frac{z-2}{z}\right)$

$=\arg(z-2)-\arg(z)$ ✓

$=\angle PAx-\angle POA$ $\quad$ x is positive x-axis

$=(\angle POA+\angle OPA)-\angle POA$ $\quad$ (Ext $\angle$ of Δ equals sum of opp. int $\angle$s) ✓

$=\angle OPA$

$=\frac{\pi}{2}$

2012 Year 12 Mathematics Extension 2 HSC Task 1 SOL UTIONS

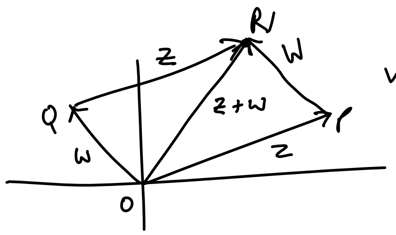
Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Question 2

a) i) $|\vec{PQ}| = |\vec{OQ} - \vec{OP}|$

$= |w - z|$

ii)



OPQR is a parallelogram

iii) $z+w$ and $z-w$ represent the diagonals of quadrilateral OPRQ. If $|z+w| = |z-w|$ these diagonals are equal in length and OPRQ is a rectangle

$\therefore \vec{OQ} = ki \times \vec{OP}$ (rotation of $90°$ in a anticlockwise direction)

so $w = kiz$

and $\frac{w}{z} = ki$ which is purely imaginary.

c) when $n = 2$

LHS $= 2 + H(1)$ RHS $= 2H(2)$

$= 2 + 1$ $= 2 \times (1 + \frac{1}{2})$

$= 3$ $= 2 \times \frac{3}{2} = 3$

$\therefore$ Statement true when $n = 2$

Assume statement true for $n = k$

ie, $k + H(1) + H(2) + H(3) + \ldots + H(k-1) = kH(k)$

when $n = k+1$

L.H.S $= k+1 + H(1) + H(2) + \ldots + H(k-1) + H(k)$

$= 1 + kH(k) + H(k)$ by assumption

R.H.S $= (k+1)H(k+1)$

$= (k+1) \times \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} + \frac{1}{k+1}\right)$

$= (k+1) \times \left[H(k) + \frac{1}{k+1}\right]$

$= (k+1)H(k) + 1$

$= kH(k) + H(k) + 1$

$\therefore$ If true for $n = k$, true for $n = k+1$

Since true for $n = 2$

true for $n = 3, 4, 5 \ldots$

$\therefore$ True for all $n \geq 2 \in \mathbb{Z}$

b) i) The complex cube roots of unity (ie, roots of $z^3 = 1$) are equaly spaced $\frac{2\pi}{3}$ radians apart on the unit circle starting with $z = 1$. So if roots $z_0 = 1$

then $z_1 = \text{cis}\frac{2\pi}{3} = w$

$z_2 = \text{cis}\frac{4\pi}{3} = \left(\text{cis}\frac{2\pi}{3}\right)^2 = w^2$

or alternatively

if $z^3 = 1$

$z^3 - 1 = 0$

$(z-1)(z^2 + z + 1) = 0$

$\therefore z = 1$ or $z = \frac{-1 \pm \sqrt{-3}}{2}$

$= \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i$

if $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = \text{cis}\frac{2\pi}{3}$

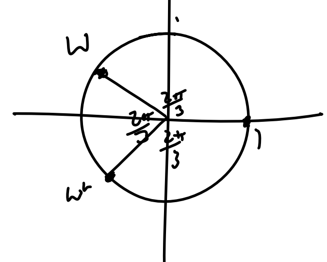
then $-\frac{1}{2} - \frac{\sqrt{3}}{2}i = \text{cis}\frac{4\pi}{3}$

$= \left(\text{cis}\frac{2\pi}{3}\right)^2$

$= w^2$

ii) $1 + w + w^2 = \text{sum of roots} = \frac{-b}{a} = 0$

iii)



iv) $(1 + w^2)^{2012} = (-w)^{2012}$

$= w^{2012}$

$= (w^3)^{670} \times w^2$

$= 1^{670} \times w^2$ as $w^3 = 1$

$= w^2$